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# Parameterization of Chaotic Particle Dynamics in a Simple Time-Dependent Field Reversal

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We investigate single charged particle dynamics in the Earth's magnetotail using a simple, scale free magnetic field model which is explicitly time dependent, with a corresponding induction electric field. The time-dependent Hamiltonian of particle motion in the simple model describes a system of two coupled oscillators which is driven. When the (time-dependent) ratio of the oscillation frequencies is different from unity, the motion is regular, but if it approaches unity at some point on the trajectory, the motion becomes chaotic. A parameter $\alpha$ is found which characterizes the adiabaticity of the system, and the transition time for behavior in the system is given by $\alpha t \rightarrow 1$. The condition $\alpha t \approx 1$ is equivalent to $\kappa \approx 1$ in the static parabolic field model discussed by previous authors (Buchner and Zelenyi, 1989). The explicit time dependence produces two possible classes of motion, ordered by $\alpha$. If the reversal is thick and is folding slowly, so that $\alpha \ll 1$, the motion is a transition in behavior, from regular $\mu$ conserving to chaotic "cucumberlike" trajectories, when $t \approx 1/\alpha$. If on the other hand, the reversal is thin and folds quickly, so that $\alpha \gg 1$, the particles execute regular "ring type" trajectories once $t > 1/\alpha$. Simple estimates of presubstorm magnetotail parameters indicate that electrons in a slowly thinning (15-min time scale), thick ($1R_E$) sheet have $\alpha \ll 1$, whereas protons in a thin (several hundreds of kilometers) sheet which thins on a 5-min time scale have $\alpha \gg 1$. Hence the behavior of the particles, and by implication, the field reversal which they support, will depend upon the adiabaticity of the system $\alpha$, as well as the "chaotization" parameter $\alpha t$; this is shown only to be the case in a model which is explicitly time dependent and which includes the induction electric field.

## 1. INTRODUCTION

Since the early 1960s [e.g., *Speiser*, 1965; *Sonnerup*, 1971; *Lyons and Speiser*, 1982; *Chen and Palmadesso*, 1986] single charged particle dynamics have been studied in simple static models of the magnetic field reversal in the Earth's magnetotail. As well as giving an insight into the local plasma populations, the dynamics of these particles, essentially the current carriers supporting the tail field reversal, should to some extent dictate the global structure and stability of the magnetotail. Changes in the trajectories may be related to disruptions of the field geometry, either indirectly (e.g., via tearing mode instability [*Buchner and Zelenyi*, 1987]), or directly (by scattering the adiabatic current carriers, causing the plasmasheet to thin [e.g, *Mitchell et al.*, 1990, and references therein]). Alternatively changes in the dynamics of some particles may be regarded as providing populations which destabilise the field plasma configuration close to substorm onset [*Galeev*, 1982] and references therein.

In particular, a static field reversal model (with parabolic magnetic field lines and no induction electric field) which is scale free has been used to characterize the particle dynamics [*Buchner*, 1986; *Buchner and Zelenyi*, 1989]. As will be demonstrated here, models with no intrinsic scale can allow the particle dynamics to be parameterized analytically; in the case of the parabolic static field reversal a single controlling parameter, $\kappa$ [*Buchner and Zelenyi*, 1986, 1989], a function of the field strength and thickness of the reversal and the particle gyroradius, determines whether the particle motion is (for all $t$) adiabatic, or chaotic. In this steady state system [see *Chen and Palmadesso*, 1986] two distinct classes of trajectory are possible: adiabatic trajectories which are always regular (integrable) and which conserve an adiabatic invariant (such as $\mu$), and chaotic or stochastic trajectories which are not regular and do not conserve a single invariant, the two types being confined to separate regions in phase space. The steady state parameter $\kappa$ has been used in time-dependent problems, e.g., as a substorm indicator in the WKB limit [*Buchner and Zelenyi*, 1987], and to discuss substorm associated phenomena [*Pulkkinen et al.* 1991]. However, the parameter $\kappa$, being derived from a steady state model, does not contain any information about the time dependence of the field. Temporal changes in the field are accommodated by treating the time-dependent system as a series of steady state magnetic field models, each characterized by a slightly different value of $\kappa$.

This paper is an attempt to introduce explicit time dependence into the system by examining a scale free model which is both time dependent and parameterizable. We examine particle dynamics in a fully time dependent scale free model for the magnetic field reversal which "folds up" with increasing time and which includes the induction electric field associated with this time dependence, as well as the constant cross-tail convective electric field by frame transformation. A new parameter $\alpha$ emerges which, as well as being a function of the field strength and thickness of the reversal and the particle gyroradius, is now also a function of the time scale over which the field changes and the particle gyroperiod. We show that the condition $\kappa \approx 1$, which identifies a steady state system which is chaotic for all $t$, is equivalent to the single time $\alpha t \approx 1$ in the time-dependent model. Hence in this model we find that, unlike in a steady state field reversal, there are no trajectories which are regular (i.e., adiabatic) for all $t$. In other words, there are no integrable trajectories which can be confined to a region in phase space for all $t$ in the time-dependent system. The explicit time dependence produces two possible classes



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of motion, depending on whether the reversal "folds" on a normalized time scale which is slow, or fast, compared with the (normalized) chaotization timescale $t \approx 1/\alpha$. The rate at which the reversal changes, and its spatial scale, as compared with the spatial and temporal scales of the particle motion in the reversal, are specified by the parameter $\alpha$. If $\alpha \ll 1$, representing a thick reversal which is folding slowly, the particle experiences a transition in behavior; when $t < 1/\alpha$ it is regular, conserving an invariant $\mu$, once $t \approx 1/\alpha$, the particle is pitch angle scattered, and the motion becomes chaotic and "cucumberlike" [*Buchner and Zelenyi*, 1989]. If on the other hand, the reversal is thin and folds quickly, so that $\alpha \gg 1$, the motion becomes regular within a Larmor period, exhibiting no extended interval of chaotic motion. The parameter $\alpha$, which essentially specifies the adiabaticity of the system (rather than of the particle), is shown to have no counterpart in the steady state model. Explicit time dependence and the inclusion of the induction electric field are required to allow the parameterization of the time scale over which the system evolves as compared with the characteristic frequencies of the particle motion.

Hence we show that the condition $\kappa \approx 1$ which has been implied to indicate the onset of chaotic behavior in initially $\mu$-conserving particles in time dependent global models of the magnetotail [*Buchner and Zelenyi*, 1987, *Pulkkinen et al.*, 1991] will only do so if the system changes sufficiently slowly and is sufficiently slowly spatially varying. In general, both the time scale, and the nature of the transition are always given by the $\alpha$ of the system.

In principle our approach can be applied to folding of thin current sheets on either short time scales (of the order of a few minutes), implied by field and plasma in-situ observations [e.g. *Ohtani et al.*, 1991; *Mitchell et al.*, 1990] in which the inductive electric field will play a substantial role in particle dynamics, or the "global" time scale (of the order of 15-30 min) over which the magnetotail as a whole gains energy (i.e., the "growth phase" [*Hones*, 1977]). In practice, the constraints required for parameterization result in a simple time-dependent model with application restricted to a "local" region in the central plasma sheet. The spatial validity of the time dependent model will be just that of the steady state parabolic field model which is appropriate for spatial regions sufficiently close to the tail center plane. In addition, the time dependent model will only be appropriate to the magnetotail for a finite range of time. Since we will show that the chaotization condition $\alpha t \approx 1$ is satisfied within a finite time, and as the particle crosses the tail centre plane, this simple model is sufficient for a determination of the "local" conditions in the plasma sheet required for chaotization.

By examining a model which is scale free and hence parameterizable, we are also able in principle to obtain the time scale for chaotization $\alpha t \approx 1$, and the adiabaticity parameter $\alpha$ for particles of any mass, charge, and gyroscales, in the model field reversal with any given radius of curvature at the center plane and time scale for variation of the field. This is generally not the case for phenomenological field models which more accurately represent the magnetotail where the details of the model obscure the underlying trends in particle behavior. In this case the dependence of the particle behavior on each of the model parameters can only be determined by numerical integration of the particle equations of motion in the model. The analysis in this paper should therefore be regarded as examining the underlying nature of single particle dynamics in time dependent field reversals, rather than an attempt to accurately model the presubstorm magnetotail.

The paper is organized as follows. In the next section the field model will be presented, with a discussion of the constraints required for parameterization. In section 3 the Hamiltonian equation of motion will be parameterized to obtain the control parameters $\alpha$ and $\alpha t$. We will then show that in the time-dependent model, a transition in behavior will occur as $\alpha t \approx 1$. The role of $\alpha$ as the adiabaticity parameter of the system will be established, and the behavior for different values of $\alpha$ will be discussed. Numerically integrated trajectories are presented in section 4, and a summary of the analytical and numerical results is given in section 5. In section 6 we will establish the relationship between the role played by $\kappa$ in the static model, and that played by $\alpha t$ in the time-dependent model, and show that $\alpha$ has no analogue in the static model. Estimates of $\alpha$ for the Earth's magnetotail are given in section 7.

## 2. THE SCALE FREE, PARAMETERIZABLE FIELD MODEL

### 2.1. *Scale Free Models*

We shall use a simple scale free model as opposed to a phenomenological geotail model as in general the latter introduces intrinsic length and time scales which obscure the parameters that characterize the trajectory [*Chapman and Cowley*, 1985]. The scale free model is such that the particle equations of motion, with suitable (scale free) normalization, hereafter denoted by an asterisk, have the following property:

$$\frac{d\mathbf{v}^*}{dt^*} = f(\mathbf{r}^*, \mathbf{v}^*, t^*) \qquad (1)$$

with

$$\frac{d\mathbf{r}^*}{dt^*} = \mathbf{v}^* \qquad (2)$$

A particle trajectory in this system is uniquely specified by three independent parameters, namely the velocity components at some point on the trajectory $\mathbf{r}^*(t^*)$ (or the speed, pitch angle, and gyrophase) at time $t^*$ [*Chapman and Cowley*, 1985].

The transition from one type of particle behavior to another, and the violation of an adiabatic invariant associated with a particular type of behavior as the transition takes place are therefore also characterized by no more than three independent parameters. A scale free field model, in which the suitably normalized Lorentz force law has the form of equation (1), is required if we are to determine any of the three independent parameters analytically. If a field model with some intrinsic spatial or temporal scale is used, the right hand side of (1) will also be a function of that intrinsic scale, and more than three parameters will be required to specify each trajectory in the field model, although the additional parameter will not be independent of the first three.

### 2.2. *Time Dependent Field Model*

With $x$, $y$ and $z$ corresponding to GSE coordinates (Geocentric Sun Earth, with $x$ toward the Earth, $z$ northward), the scale free field model to be used here,

$$\mathbf{B} = (\frac{B_x z t}{h\tau}, 0, B_z) \quad (3)$$

$$\mathbf{E} = (0, \frac{B_x}{2h\tau} z^2, 0) \quad (4)$$

represents parabolic magnetic field lines which fold toward the center plane of the magnetotail ($z = 0$) with increasing time $t$ on time scale $\tau$. The field reversal has half thickness $h$ such that at this scale height and at time $t = \tau$ the field $x$ component has characteristic strength $B_x$. The inductive electric field $B_x z^2/2h\tau$ is chosen to satisfy $\nabla \wedge \mathbf{E} = -\partial \mathbf{B}/\partial t$ with the additional constraint that there should be no $x$ dependence in the model so that a constant cross-tail, in ($\hat{\mathbf{y}}$), convection electric field $\mathbf{E_C} = v_T B_z \hat{\mathbf{y}}$ can be introduced by moving in the $\hat{\mathbf{x}}$ direction with constant velocity $v_T \hat{\mathbf{x}}$ [*Chapman and Cowley*, 1985], to represent plasma convection in the Earth's magnetotail. This requirement for the inclusion of a convection electric field by frame transformation effectively constrains the linking field $B_z$ to be constant for all $x$ and $t$, so that the model is appropriate provided that on the time scales to be discussed below the linking field does not change substantially (i.e., provided the particle is not convected along a substantial length of the tail). In addition to being constrained by $\nabla \wedge \mathbf{E} = -\partial \mathbf{B}/\partial t$, the induction electric field is also chosen to satisfy $\nabla . \mathbf{E} = 0$ consistent with quasi-neutrality.

By normalizing time to $1/\Omega_z$ (the inverse of the gyrofrequency in the field at $z = 0$, $\Omega_z = eB_z/m$) and length scales to $B_z \tau \Omega_z h / B_x$ it can be verified that this field model is scale free, that is, the Lorentz force law becomes (for positively charged particles)

$$\frac{d\mathbf{v}^*}{dt^*} = \mathbf{v}^* \wedge (z^* t^* \hat{\mathbf{x}} + \hat{\mathbf{z}}) \quad (5)$$

In this scale free normalization the vector potential may be written

$$\mathbf{A}^* = (0, x^* - \frac{z^{*2} t^*}{2}, 0) \quad (6)$$

with $\phi = 0$ in the frame where $\mathbf{E}_c = 0$. Field lines (i.e., contours of constant $A_y$) are parabolic, folding toward the $z = 0$ plane with increasing $t$.

This simple scale free model represents the more phenomenologically accurate model of the tail current sheet

$$\mathbf{B} = (C \tanh(\frac{z}{h}), 0, B_z) \quad (7)$$

with intrinsic scale height $h$, over the restricted region $z < h$. Here we also introduce time dependence with

$$C = B_x \frac{t}{\tau} \quad (8)$$

Previous steady state studies [e.g., *Chen and Palmadesso*, 1986 and references therein] have a constant $C = B_x$. Here it will be shown that stochastic behavior, that is, jumps in the particle action $\mu$, occurs as the particle crosses the center plane $z = 0$, so that in order to examine the conditions for the onset of chaos we need only to accurately model the spatial region close to the center plane $z < h$, for which the above $\mathbf{B}$ field reduces to equation (3) and is scale free as required. This approximation does imply that the simple field model does not have a loss cone, that is, all particles will ultimately mirror and return to the center plane [*Chen and Palmadesso*, 1986]; this, however, will not affect conclusions concerning the pitch angle scattering (onset of chaos) in initially trapped, $\mu$-conserving trajectories. In addition the $x$ component of the magnetic field varies linearly in $t$, giving parabolic field lines for arbitrarily small $t > 0$ and a current density which increases linearly in $t$. This linear variation in $t$ in general represents a small time interval of any fluctuation in the current density (e.g., $\sin t/\tau$ or $1 - e^{-\frac{t}{\tau}}$) for $t \ll \tau$, or in particular a system which has approximately linear time variation in $\mathbf{J}$ for arbitrary $t$. As it will be shown that in the time-dependent field, a transition occurs in the particle dynamics on a finite time scale, it is appropriate to consider a model which is valid for some finite time interval. Our conclusions will then be valid if the variation in the current density at the center plane is not strongly nonlinear in $t$ over the time taken for the transition in behavior, i.e., for $\alpha t \approx 1$.

## 3. ANALYTICAL DESCRIPTION OF THE PARTICLE DYNAMICS

It has been demonstrated that a trajectory is uniquely specified by three independent parameters. Here we will show that most, but not all, features of the particle behavior are determined by two parameters: $\alpha t$ which indicates the transition time of the particle behavior, and $\alpha$, which yields the "adiabaticity" of the system with respect to this transition time.

### *3.1. The Controlling Parameters $\alpha$ and $\alpha t$*

The controlling parameters are revealed by renormalizing the particle and Hamiltonian equations of motion in the scale free model. Normalizing time to the inverse of the gyrofrequency at $z = 0$, $1/\Omega_z$, and length to $\rho_z$, the gyroradius at $z = 0$, the equations of motion are

$$\ddot{x} = \alpha t \frac{z^2}{2} - x = F_x - x \quad (9)$$

$$\ddot{z} = -\alpha t(\alpha t \frac{z^2}{2} - x) z = -\omega_z^2 z \quad (10)$$

where we have used the first integral of the $y$ equation of motion,

$$\dot{y} = \alpha t \frac{z^2}{2} - x \quad (11)$$

The Hamiltonian equation of motion can be written as

$$\frac{dH}{dt} = \frac{d}{dt}(\frac{\dot{x}^2}{2} + \frac{\dot{z}^2}{2} + \Psi) = \alpha \frac{z^2}{2}(\alpha \frac{z^2 t}{2} - x) = \mathbf{v}.\mathbf{E} \quad (12)$$

$$\Psi = \frac{1}{2}(\alpha t \frac{z^2}{2} - x)^2 = \frac{x^2}{2} + \omega_z^2 \frac{z^2}{2} - \frac{1}{2} F_x^2 \quad (13)$$

where the control parameter $\alpha$ is given by

$$\alpha = \frac{B_x}{B_z} \frac{\rho_z}{h} \frac{\Omega_f}{\Omega_z} \quad (14)$$

and $\Omega_f = 1/\tau$, the fold-up "frequency" of the field.

The role of $\alpha$ and $\alpha t$ as controlling parameters is apparent from the above Hamiltonian equation of motion.

First, at any instant in time, we have written the Hamiltonian as

$$H = H(\dot{x}, \dot{z}, x, z, \lambda) = \frac{\dot{x}^2}{2} + \frac{\dot{z}^2}{2} + \Psi(x, z, \lambda) \quad (15)$$

where

$$\lambda = \alpha t \tag{16}$$

and $\Psi$ has been written as the pseudopotential for two coupled oscillators in $x$ and $z$. The parameter $\lambda = \alpha t$ scales the Hamiltonian by scaling the pseudopotential $\Psi$ within which the particles move at any time $t$, so that the ratio of oscillation frequencies will also depend on $\alpha t$.

Second, $dH/dt \neq 0$ as the v.E term is nonzero, so that we can write this Hamiltonian equation of motion as

$$\frac{dH}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} \tag{17}$$

where

$$\frac{d\lambda}{dt} = \alpha \tag{18}$$

The parameter $\dot{\lambda} = \alpha$ scales the rate at which the Hamiltonian changes, and since

$$\frac{\partial H}{\partial \lambda} = \frac{\partial \Psi}{\partial \lambda} \tag{19}$$

here, $\alpha$ also scales the rate at which the pseudopotential changes. In this sense, $\alpha$ indicates the "adiabaticity" of the system. The limit of an infinitesimally slowly changing system is

$$\frac{dH}{dt} \rightarrow 0 \tag{20}$$

implying

$$\alpha \rightarrow 0 \tag{21}$$

which corresponds to neglecting the induction electric field.

We will now discuss the particle behavior in this system in detail.

The above equations (9)-(13) describe the motion of two nonlinear coupled oscillators, in $x$ and $z$. The $x$ equation of motion contains an oscillator with frequency 1, the (normalized) Larmor frequency at the center plane $z = 0$. Since the magnetic field minimizes at the center plane, this is also the minimum value of the Larmor frequency $\Omega_L$ at any point on the trajectory. In addition, the driving term $F_x(z, \alpha t)$ represents a force in the $x$ direction. The $z$ equation of motion contains an oscillator with frequency $\omega_z(\alpha t)$, which it is important to note is explicitly time dependent. In section 6 we will show that in the steady state model this frequency only has implicit time dependence through the particle position. This explicit time dependence in $\omega_z$ will now be shown to imply that no trajectories in the time-dependent model can have regular (adiabatic) behavior for all $t$, and will yield the transition time $\alpha t \approx 1$.

Particle trajectories will be regular (adiabatic) for some finite time interval over which the two oscillation frequencies remain dissimilar. There are in general two possible modes of regular behavior for which this time interval is longer than the period of both the fast and slow oscillation. If $\Omega_L \gg \omega_z$ over both the fast and slow oscillatory motion, and hence $1 \gg \omega_z$, the motion is adiabatic, conserving an invariant $\mu = v_\perp^2/B$ associated with the fast oscillation about the field at the Larmor frequency $\Omega_L$, the frequency $\omega_z$ corresponding to the frequency of the slow bounce motion between mirror points. If $\Omega_L \ll \omega_z$, the motion, if completely regular, is "ring-type" (in the definition of *Buchner and Zelenyi* [1989]), the particle executing fast $z$ oscillations across the $z = 0$ plane which are approximately $I_z = \oint v_z dz$ conserving, accompanied by a slow $x$ gyration about the field threading the center plane. The force term $F_x\hat{\mathbf{x}}$ should, if the motion is $\mu$ conserving, result in a drift in the $\hat{\mathbf{x}} \wedge \mathbf{B}$ direction, i.e., in $\hat{\mathbf{y}}$ consistent with guiding center drift, and if the motion is $I_z$ conserving will produce a drift in the $\hat{\mathbf{x}}$ direction.

A third class of motion will arise if at some point on the trajectory $\omega_z \approx \Omega_L$. This weakly chaotic (stochastic [*Chen and Palmadesso*, 1986] or cucumberlike [*Buchner and Zelenyi*, 1989]) motion consists of approximately regular segments over which $\omega_z \not\approx \Omega_L$. When the particle is far from the center plane, the motion is approximately $\mu$ conserving. When the particle crosses the center plane, it executes fast oscillations in $z$ which are approximately $I_z$ ($= \oint v_z dz$) conserving. The particle completes a half turn about the center plane field $B_z$ during this section of the trajectory.

This type of interaction, where the particle motion switches from $\mu$ conserving to $I_z$ conserving as it interacts with the maximum field radius of curvature at the center plane, is reminiscent of both Speiser [*Speiser*, 1965] and cucumberlike [*Buchner and Zelenyi*, 1986] trajectories. These trajectories are distinguished by [*Chen and Palmadesso*, 1986] whether they are transient (Speiser) particles which can escape the system, or stochastic (cucumberlike) particles which mirror and return to interact with the center plane. In the scale free parabolic field all particles of this type will ultimately mirror and return to the center plane, so that this discussion concerns cucumberlike, or stochastic, chaotic particles.

We will now establish that the time scale for the transition between different classes of motion is $t \rightarrow 1/\alpha$. Since we have shown that $\alpha$ yields the adiabaticity of the system, we expect different behavior for different $\alpha$. There are two classes of behavior, for the two regimes $\alpha < 1$ and $\alpha \geq 1$.

### 3.2. $\alpha < 1$

In this case the time interval $t < 1/\alpha$ (where $t$ is normalized to $\Omega_z^{-1}$) corresponds to many Larmor periods. At early $t$, that is, for sufficiently small $\alpha t$, $\omega_z \ll 1$ and hence $\omega_z \ll \Omega_L$ over the entire motion (since $\Omega_L \geq 1$). During this interval the trajectories are regular, characterized by fast oscillation about the field at the local Larmor frequency $\Omega_L$ and slow bounce motion (at frequency $\omega_z$) between mirror points, that is, they are $\mu$ conserving. As $\omega_z \rightarrow 1$ (with increasing $\alpha t$), it will approach $\Omega_L$ in magnitude, the two frequencies first becoming equal at the center plane where $\Omega_L$ minimizes (i.e., where $\Omega_L = 1$). As the particle crosses the $z = 0$ plane (where the field curvature maximises), $\mu$ will jump. The particle will undergo strong pitch angle scattering before evolving into weakly chaotic cucumberlike behavior. For sufficiently large $\alpha t$ we may have $\omega_z \gg 1$ (when $\omega_z$ is real) over almost the entire trajectory so that the motion becomes regular, that is, $\omega_z \gg \Omega_L$ over the entire motion. This corresponds to the regular $I_z$ conserving ("ring type") orbits discussed above. On the other hand, if $\omega_z$ remains of the order of $\Omega_L$ close to the center plane, the motion will remain cucumberlike. These two classes of behavior are distinguishable by a third parameter, which is related to the particle energy associated with the $z$ component of the motion as will be demonstrated in section 3.4 (see also *Buchner and Zelenyi* [1989]).

By inspection of the Hamiltonian equation of motion we have however established the conditions for "chaotization",

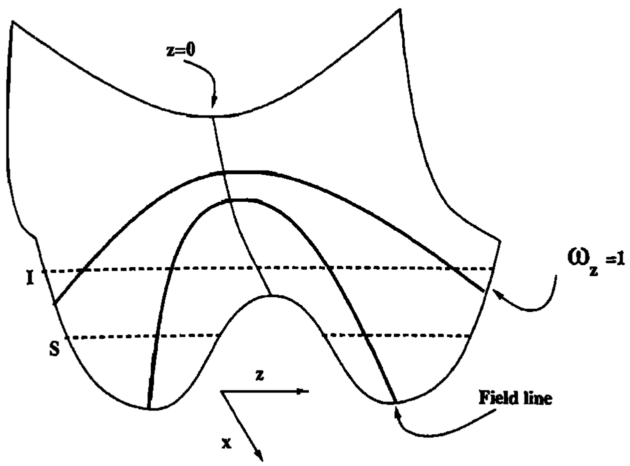


Fig. 1. Sketch of the pseudopotential $\Psi(x,z)$. The inner solid line marks the minimum $\Psi = 0$ corresponding to the location of the field line or contour of constant vector potential $A_y = 0$. The outer solid line is the locus of points where $\omega_z = 1$ where the $x$ and $z$ oscillation frequencies become approximately equal. Both of these lines collapse toward the center plane $z = 0$ with increasing $t$. The dotted lines indicate the energies associated with the $z$ motion of particles with two classes of behavior, S is either $\mu$ conserving or cucumberlike and I is $I_z$ conserving.

that is, the conditions for the transition from $\mu$ conserving to cucumberlike motion.

This is illustrated in Figure 1, which shows a sketch of the function $\Psi(x,z)$ at some $t$. Marked on the sketches are the lines $\Psi = 0$, and $\omega_z = 1$. The inner line $\Psi = 0$ corresponds to the field line, that is, the contour of constant vector potential $A_y = 0$ (see equation (6)) about which a particle described by the above Hamiltonian equation of motion gyrates during the $\mu$-conserving segments of its trajectory. This line, that is, the field lines of the model, moves toward the center plane $z = 0$ with increasing $t$.

When a trajectory approaches the outer $\omega_z = 1$ line, jumps in the action ($\mu$ or $I_z$) are expected to occur. Provided that $\Omega_L \approx 1$, the $\omega_z = 1$ line divides the $x - z$ plane into two regions of particle behavior: the region extending into $+x$ corresponding to ($\omega_z < 1$) $\mu$-conserving, $\omega_z \ll \Omega_L$ motion, and the region extending into $-x$ corresponding to $\omega_z \gg \Omega_L$ or $I_z$-conserving motion. Motion on the $+x$ side of the field line $\Psi = 0$ yields an imaginary $\omega_z$ (see equation (10)) corresponding to exponential growth in $z$. This simply results in a net $z$ oscillatory motion that is not sinusoidal. The $\omega_z = 1$ line is closest to the $\Psi = 0$ line at the center plane $z = 0$, and it intersects this plane at $x = -1/\alpha t$ in the above normalization. At early times, when $\alpha t \ll 1$, this intersection is at large negative $x$, and most of the $x - z$ plane, and the trajectories within it, are in a region where $\omega_z/\Omega_L \ll 1$. These trajectories are $\mu$ conserving, and averaging over the fast Larmor oscillation follow the field line $\Psi = 0$. As we increase $\alpha t$, the $\omega_z = 1$ boundary between $\omega_z/\Omega_L \ll 1$ ($\mu$-conserving) and $\omega_z/\Omega_L > 1$ ($I_z$ conserving) behavior moves closer to the trajectory until at $\alpha t = 1$ the intersection is at $x = -1$ corresponding in unnormalized units to $x = -\rho_z$, the particle gyroradius in the center plane. Hence at $\alpha t \to 1$ this chaotic boundary just intersects the trajectory once at the center plane, and jumps in the fast action $\mu$ will occur as the particle crosses the center plane. As $\alpha t > 1$ the $\omega_z = 1$ boundary moves closer to the origin, intersecting the center plane at $x > -\rho_0$ and the trajectory at least twice in the region close to the center plane. The motion will now be cucumberlike (or stochastic): far from the center plane the trajectory is still located in the $\omega_z/\Omega_L \ll 1$ or $\mu$ conserving region, but close to the center plane a segment will be in the $\omega_z/\Omega_L > 1$ or $I_z$-conserving region. As the trajectory crosses the $\omega_z = 1$ line, a jump in the action occurs. The motion will remain cucumberlike if segments of the trajectory remain wholly within the $\mu$-conserving, $\omega_z < 1$ region.

In summary then, we expect that as $t \to 1/\alpha$ trajectories will exhibit jumps in $\mu$ as they cross the center plane, and that the motion will evolve from $\mu$ conserving to cucumberlike. Since jumps in the action $\mu$ occur just as the particle crosses the center plane, this discussion only applies to trajectories that cross the center plane on time scales shorter than, or of the order of, the chaotization timescale $t = 1/\alpha$. This is appropriate for particles interacting with the Earth's magnetotail as in the frame containing a zero convection electric field most particles will have a significant field-aligned velocity directed into the field reversal.

### *3.3.* $\alpha \geq 1$

In this case the interval $t < 1/\alpha$ corresponds to less than a Larmor period, and the particle cannot execute $\mu$ conserving motion for which $\Omega_L \gg \omega_z$. Once $t \to 1/\alpha$, provided the particle crosses the center plane, the trajectory will intersect the $\omega_z = 1$ line as above, and the motion must become either cucumberlike or $I_z$ conserving ("ring-type"), depending upon the subsequent motion, that is, whether segments of the trajectory remain within the $\omega_z < 1$ region. Since $\omega_z$ increases with $\alpha$, we might expect that for sufficiently large $\alpha$ the motion will become regular $I_z$ conserving. In section 4 we will numerically demonstrate that for $\alpha > 1.55$, particles execute regular, $I_z$ conserving (or "ring type") trajectories, the motion instead being cucumberlike for smaller values of $\alpha$.

### *3.4. The Third Controlling Parameter*

Both classes of particle, those that remain cucumberlike for $\alpha t > 1$ discussed above, and those that exhibit regular $I_z$-conserving behavior, move in the pseudopotential $\Psi(x,z)$. These two classes of behavior can be distinguished by their energies associated with the $z$ component of the motion at a given $x$ (see also Figure 7 of *Buchner and Zelenyi* [1989]). From the Hamiltonian equation of motion the total energy at any time $t$ is

$$H(\alpha t) = \frac{\dot{x}^2}{2} + \frac{\dot{z}^2}{2} + \Psi(x,z,\alpha t) \tag{22}$$

which in terms of the energy associated with the $z$ motion can be written as

$$H(\alpha t) = \frac{\dot{x}^2}{2} + h_z(x,z,\dot{z},\alpha t) \tag{23}$$

so that

$$h_z(x,z,\dot{z},\alpha t) = \frac{\dot{z}^2}{2} + \Psi(x,z,\alpha t) \tag{24}$$

At some fixed $x$, notional values of $h_z$ are marked S for a cucumberlike and I for an $I_z$-conserving trajectory on Figure 1. For a given particle in the time dependent field, the value of $h_z(\alpha t)$ will not be constant as implied by the straight lines on the figure, and particles may migrate between the two classes of behavior. The limiting case for particle motion to be regular $I_z$ conserving is just that [cf. *Buchner and Zelenyi*, 1989, equations (21a)-(21c)]

$$h_z(\alpha t) = \Psi(z_{max}) \geq \Psi(z = 0) = \frac{x^2(z = 0)}{2} \qquad (25)$$

This implies that the third parameter that characterizes the particle motion will depend upon $h_z(\alpha t)$.

## 4. Numerically Integrated Trajectories

The two classes of behavior, corresponding to $\alpha < 1$ and $\alpha \geq 1$, have been examined numerically. Trajectories with different values of $\alpha$ have been integrated using an adaptive time step, adaptive order Adams method [*Shampine and Gordon*, 1975]. We use the scale free normalization to length $L = \rho_z/\alpha$ and time $T = 1/\Omega_z$; the value of $\alpha$ for each trajectory is then specified by the starting position $z_0^* = z/L = \alpha z/\rho_z$. All trajectories have initial conditions $t = 0$ (**B** constant), and initial velocities which are arbitrary (a constant $v_x$ or $v_y$ may be added by frame transformation) except that the motion is initially toward the center plane and that the particle crosses the center plane at least once every $t = 1/\alpha$. Trajectories are independent of the initial $x$ and $y$ position, and are for positively charged particles (the equivalent trajectory for a negatively charged particle is obtained by reflection in the $x - z$ plane).

### 4.1. $\alpha < 1$ Behavior

The overall behavior for a trajectory started at $z_0^* = 0.008$ is shown in Figures 2a and 2b. Figure 2a spans the period $t = 0$ to $t > \frac{1}{\alpha}$ and shows the transition from $\mu$ conserving to chaotic behavior. As suggested by the Hamiltonian equation of motion, the particle $\mu$ is reasonably well conserved for $t < 1/\alpha = 125$, the oscillations in $\mu$ about the average are a result of $\mu$ being calculated at the particle, rather than the guiding center location. The first crossing of the center plane, at $t \simeq 50$, is adiabatic, the second, at $t \simeq 80$, is approximately so. After $t = 125$, however, the $z$ motion is clearly not a slow oscillation between mirror points, and $\mu$ is no longer well conserved. Three-dimensional plots of the trajectory over this period are shown in Figures 3 and 4. The approximately adiabatic part of the trajectory, for $t = 0 - 100$, is shown in Figure 3; this is composed of fast oscillations about the parabolic field, with a slow bounce motion between mirror points. Figure 4 shows the trajectory for $t = 100 - 190$, spanning the chaotization time $\alpha t \rightarrow 1$. During the first center plane crossing at $t = 150$ the particle is strongly pitch angle scattered, from Figure 2a the plot of $z$ versus $t$ shows that just after $t = 150$, $\Omega_L \approx \omega_z$, and on

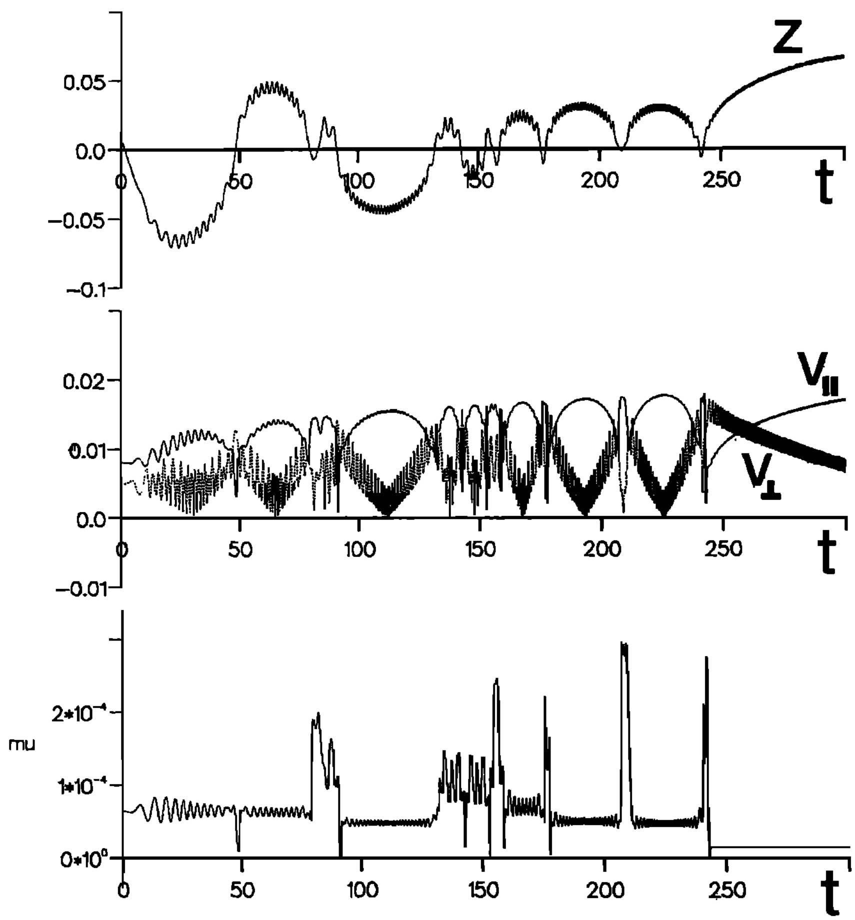


Fig. 2a Plot of $\mu$, the perpendicular and parallel velocity components, and the particle $z$ coordinate versus time for a trajectory with $\alpha = 0.008$, that is, with $z^* = 0.008$ at $t = 0$. The full initial conditions were $\mathbf{r}^* = (0, 0, 0.008)$, $\mathbf{v}^* = (0, 0.008, -0.005)$. The plot spans the interval $t = 0$ to $t > 1/\alpha$.

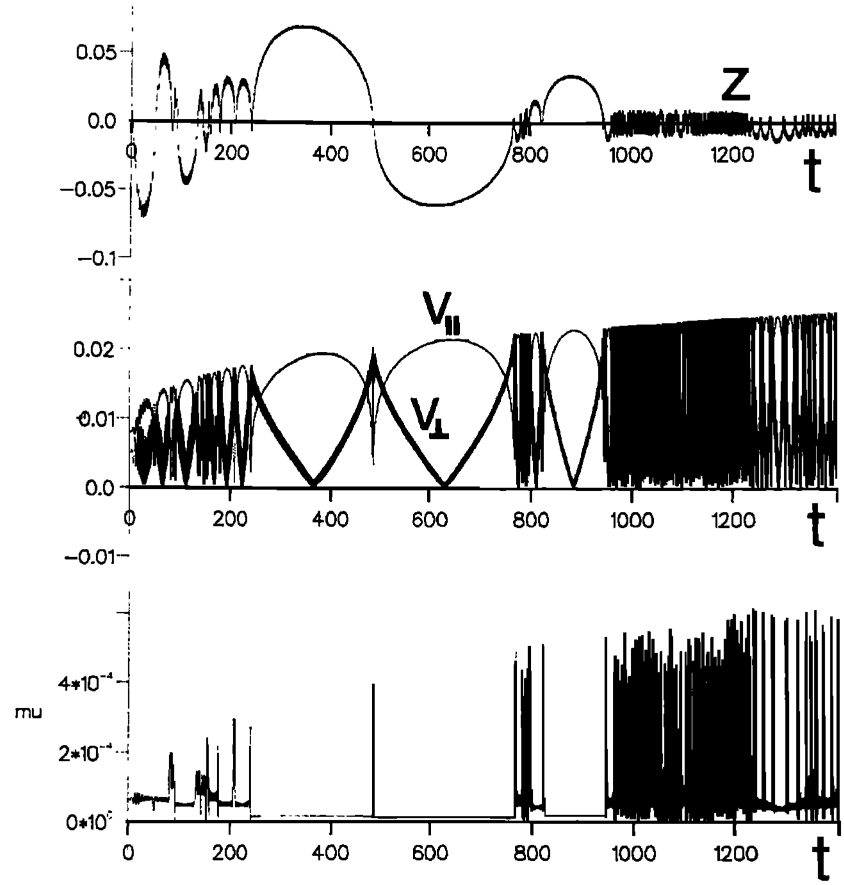


Fig. 2b Plot of $\mu$, the perpendicular and parallel velocity components, and the particle $z$ coordinate versus time for a trajectory with $\alpha = 0.008$, spanning the interval from $t = 0$ to $\alpha t \gg 1$.

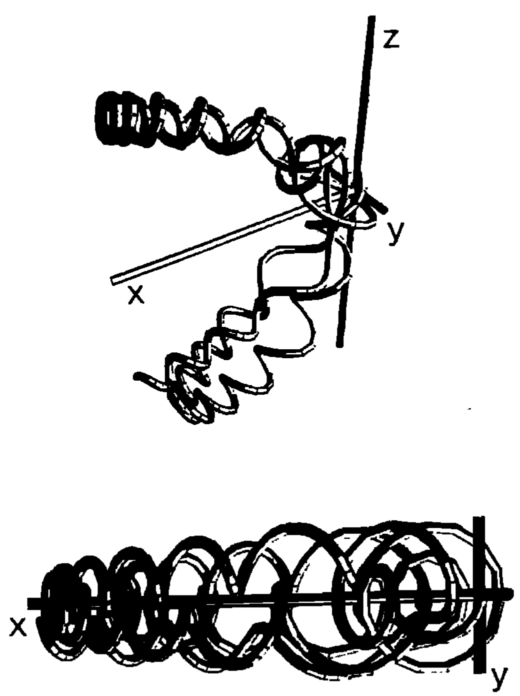


Fig. 3. Three dimensional plot of the trajectory with $\alpha = 0.008$, for the interval $t = 0 - 100$, that is, for $t < 1/\alpha$. The motion is approximately $\mu$ conserving.

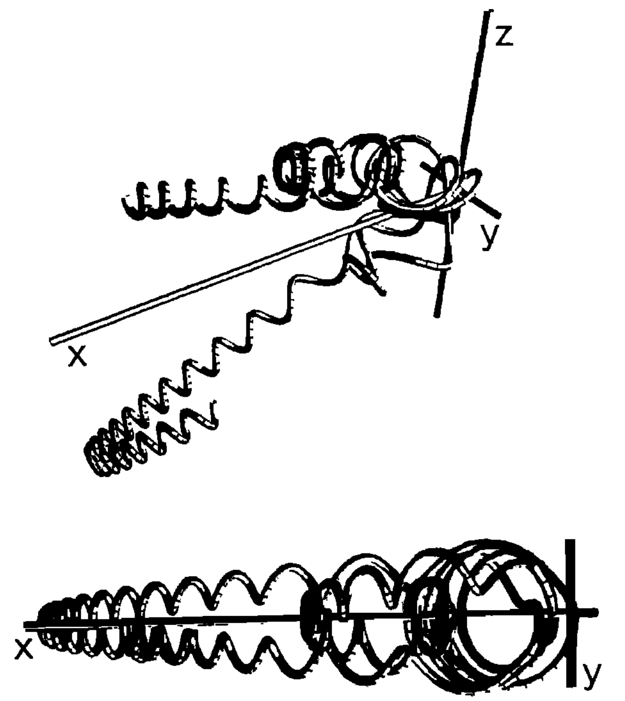


Fig. 4. Three dimensional plot of the trajectory with $\alpha = 0.008$, for the interval $t = 100 - 190$, that is, spanning $t \to 1/\alpha$. The particle is strongly pitch angle scattered.

subsequent crossings the motion is more cucumberlike, in that the particle reaches the center plane, executes a single half turn, and then exits in the same direction from which it entered.

Figures 5-8 show the same trajectory for times $t \gg 1/\alpha$, selected from the time period spanned by Figure 2b. Up to $t \approx 900$ the motion is cucumberlike, an example (for $t = 760 - 825$) is shown in Figure 5. From $t \approx 900$ to 1200 the motion is interspersed with segments of regular, $I_z$-conserving behavior. Figure 6 shows the evolution from cucumberlike to a period of $I_z$-conserving behavior, during this interval ($t = 945 - 980$) the particle first interacts twice with the center plane with cucumberlike motion before executing regular, $I_z$ conserving motion. The motion continues to be $I_z$ conserving for several $x$ oscillations as shown in Figure 7, but ultimately reverts to cucumberlike behavior as shown in Figure 8.

A number of trajectories have been examined over the range of $\alpha < 1$, and in all cases, chaotization, that is, the commencement of pitch angle scattering and the transition of behavior from $\mu$ conserving to cucumberlike has been found to occur at $\alpha t \approx 1$. Some trajectories subsequently exhibit intervals of $I_z$-conserving behavior as shown here, but in all cases the motion does not continue to be regular, reverting to cucumberlike motion at later $t$.

### *4.2. $\alpha \geq 1$ behavior*

For these trajectories the time $t < 1/\alpha$ is shorter than a Larmor period, so that the particles do not execute $\mu$-conserving motion. The overall behavior of an $\alpha = 1$ trajectory is shown in Figure 9. From the time of the first crossing of the center plane at $t = 6$ this particle executes cucumberlike motion. The trajectory, for $t = 0 - 40$ shown in Figure 10 is similar to the cucumberlike segments of the $\alpha \ll 1$ trajectory discussed above. This trajectory again is found to remain cucumberlike for large $t \gg 1/\alpha$, with no evolution into $I_z$-conserving behavior. Integration of numerous trajectories reveals the same behavior for $\alpha < 1.55$. For much greater values of $\alpha$ the behavior evolves directly into regular, $I_z$-conserving motion once $t > 1/\alpha$, with no interval in which the motion is cucumberlike. An example of this type of trajectory is shown in Figures 11 and 12, for an $\alpha = 5$ particle. The behavior of $\mu$ shown in Figure 11 clearly differs from that of a cucumberlike trajectory. As discussed in

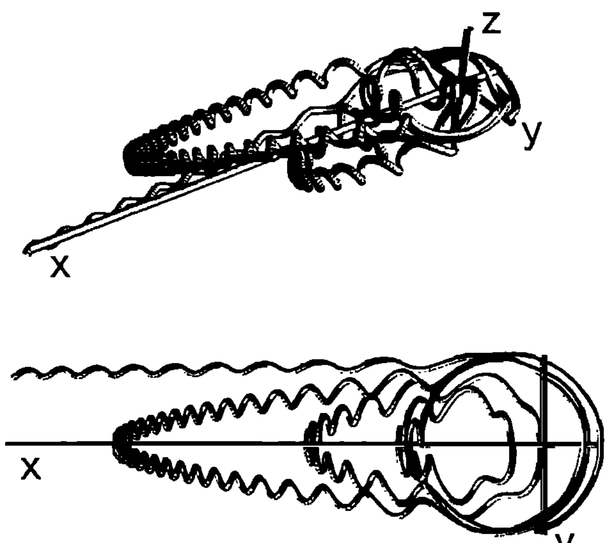


Fig. 5. Three dimensional plot of the trajectory with $\alpha = 0.008$, for the interval $t = 760 - 825$, that is, for $t > 1/\alpha$. The motion is now cucumberlike.

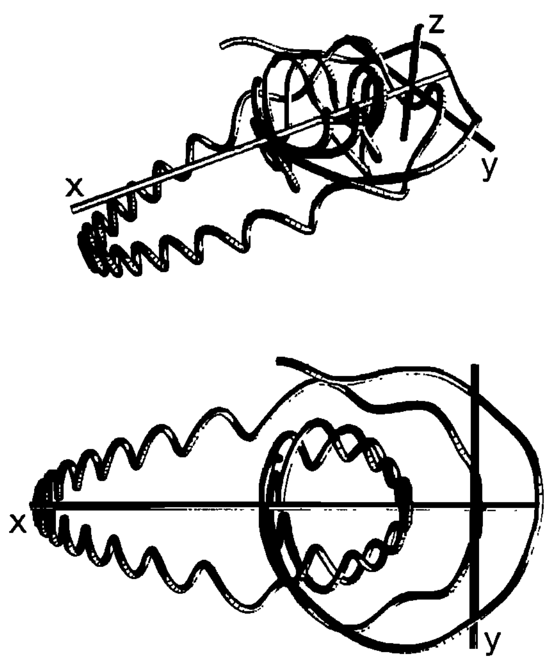


Fig. 6. Three dimensional plot of the trajectory with $\alpha = 0.008$, for the interval $t = 945 - 980$, that is, for $t > 1/\alpha$. The motion is evolving from cucumberlike to $I_z$ conserving.

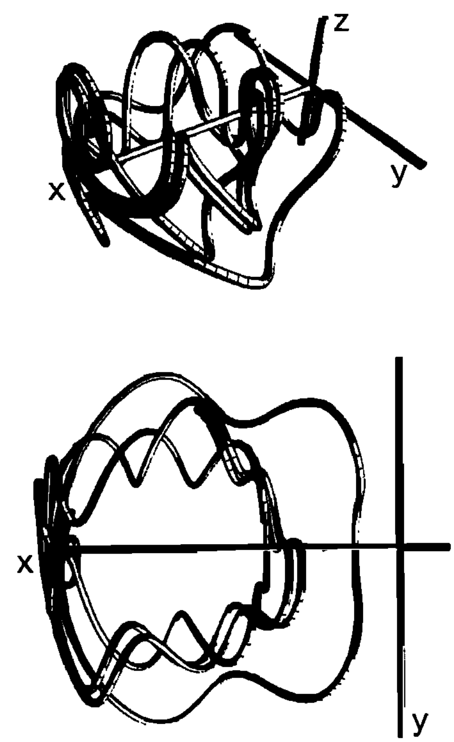


Fig. 7. Three dimensional plot of the trajectory with $\alpha = 0.008$, for the interval $t = 975 - 1000$, that is, for $t > 1/\alpha$. The motion is now completely regular, i.e., $I_z$ conserving.

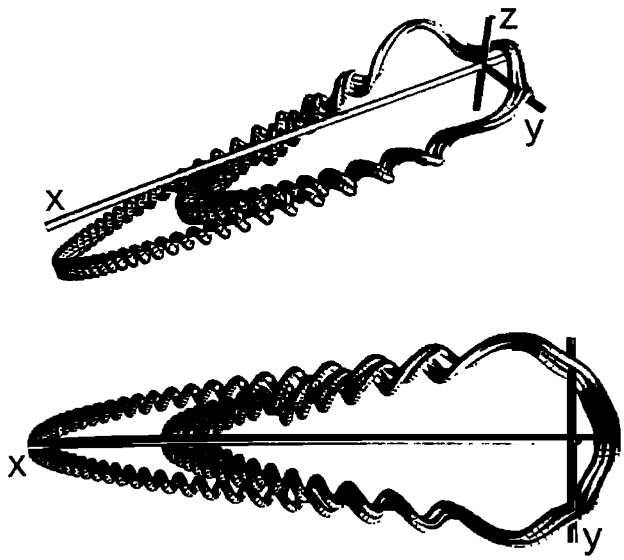


Fig. 8. Three dimensional plot of the trajectory with $\alpha = 0.008$, for the interval $t = 1250 - 1320$, that is, for $t > 1/\alpha$. The motion is now cucumberlike.

section 3, the particle slow $x$ motion is sinusoidal with a net drift in $+x$, whereas the fast $z$ motion is not sinusoidal, being composed of exponential growth in $z$ for that part of the trajectory closest to the center plane where $\omega_z$ is imaginary (enclosed by the $\Psi = 0$ line). The trajectory itself (Figure 12) shown for $t = 0-15$ is plotted for two periods of the slow motion, which can be seen in the $x-y$ plot to be a rotation about the linking field $B_z\hat{z}$ in the $x-y$ plane, accompanied by a drift in the $+x$ direction. The fast motion is an oscillation in the $z$ direction. These numerical results suggest that the $\alpha \geq 1$ behavior is separated into two distinct classes by the value of $\alpha$ and that particles do not migrate between the two. Since numerically it is difficult to ensure that all of phase space has been adequately explored, this conclusion needs to be confirmed analytically.

## 5. Summary of Analytical and Numerical Results

The analytical and numerical results are summarized in Figure 13, which is a plot of the particle behavior ordered with the two controlling parameters $\alpha t$ and $\alpha$. The parameter $\alpha t$ is plotted along the ordinate, so that the trajectory of a particle of a given $\alpha$ is a horizontal line on the plot, the particle coordinate moving to the right with increasing $t$. Lines denoting the numerically integrated examples are marked on the plot. The diagonal $t = 1$ line indicates how far a particle of a given $\alpha$ must move across the plot to have executed of the order of a single Larmor orbit in the center plane field (i.e., a time $t = 1/\Omega_z$ in un-normalized units). The plot shows the following:

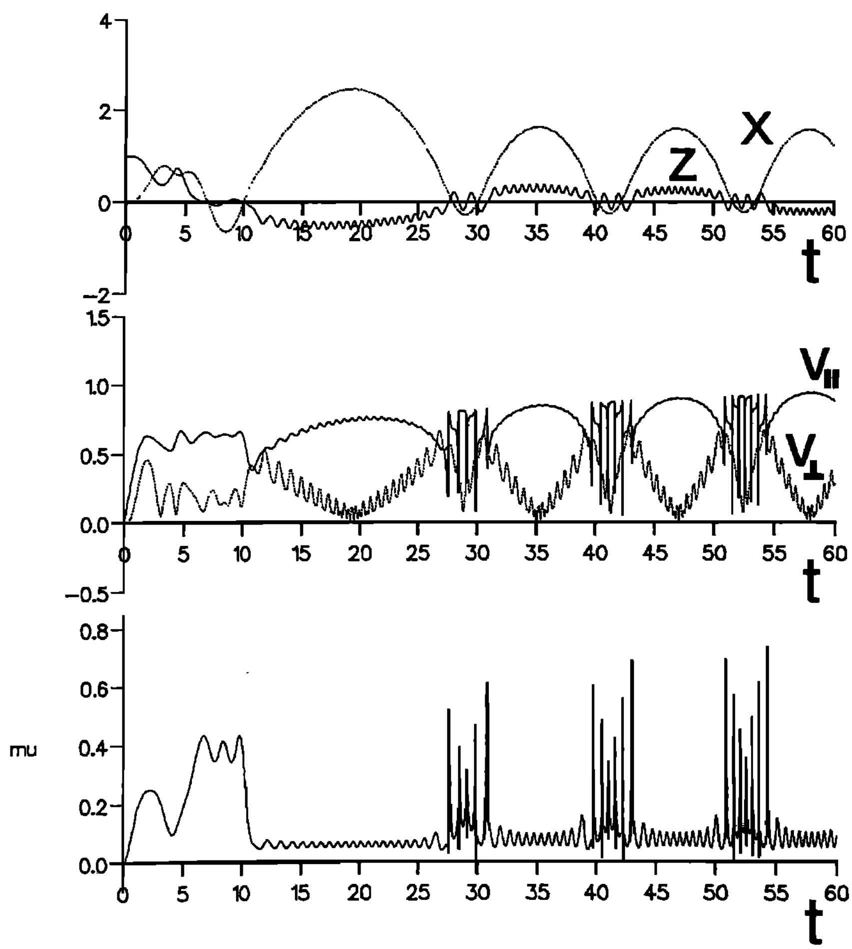


Fig. 9. Plot of $\mu$, the perpendicular and parallel velocity components, and the particle $x$ and $z$ coordinate versus time for a trajectory with $\alpha = 1$, that is, with $z^* = 1$ at $t = 0$. The full initial conditions were $\mathbf{r}^* = (0,0,1)$, $\mathbf{v}^* = (0,0,0)$. The plot spans the interval $t = 0$ to $t > 1/\alpha$.

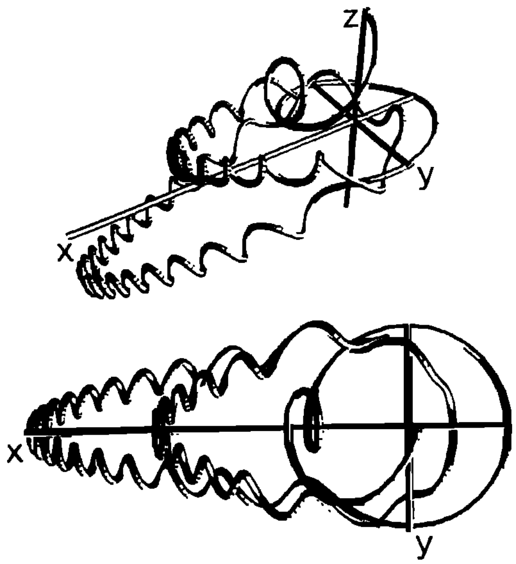


Fig. 10. Three dimensional plot of the trajectory with $\alpha = 1$, for the interval $t = 0 - 40$, that is, spanning the interval $t = 0$ to $t > \frac{1}{\alpha}$. For $t > 1/\alpha$ the motion is cucumberlike.

1. For an $\alpha \ll 1$ system, the particle executes many Larmor orbits before reaching the time $t \to 1/\alpha$. For $t < 1/\alpha$ the motion is regular, and the particle conserves $\mu$; once $t > 1/\alpha$ the motion is in general cucumberlike.

2. For $\alpha \approx 1$, the particle executes of the order of a single Larmor orbit before $t \to 1/\alpha$, once $t > 1/\alpha$ the motion is cucumberlike. This type of behavior is numerically found for $\alpha < 1.55$ approximately.

3. For $\alpha > 1.55$ the time $t \to 1/\alpha$ is much shorter than a Larmor period. Once $t > 1/\alpha$ the particle executes regular, $I_z$-conserving ("ring-type") trajectories.

## 6. Relationship with the Steady State Model Parameter $\kappa$

We will now establish the relationship between the time-dependent system discussed above, and the steady state parabolic field model [e.g., *Buchner and Zelenyi*, 1989]. Application of the steady state model to time dependent fields, by treating the time-dependent system as a series of steady state magnetic field models, each characterized by a slightly different value of $\kappa$, has been used to imply that regular, $\mu$-conserving motion becomes chaotic (cucumberlike) when $\kappa \approx 1$ [e.g., *Pulkkinen*, 1991]. We have seen that this corresponds to a particular case of motion in the time-dependent model, that is, the limit of a slowly changing system $\alpha \ll 1$. We will now establish the exact relationship between $\kappa$ and $\alpha t$, and show that the system adiabaticity $\alpha$ does not have an analogue in the static model.

In the steady state model:

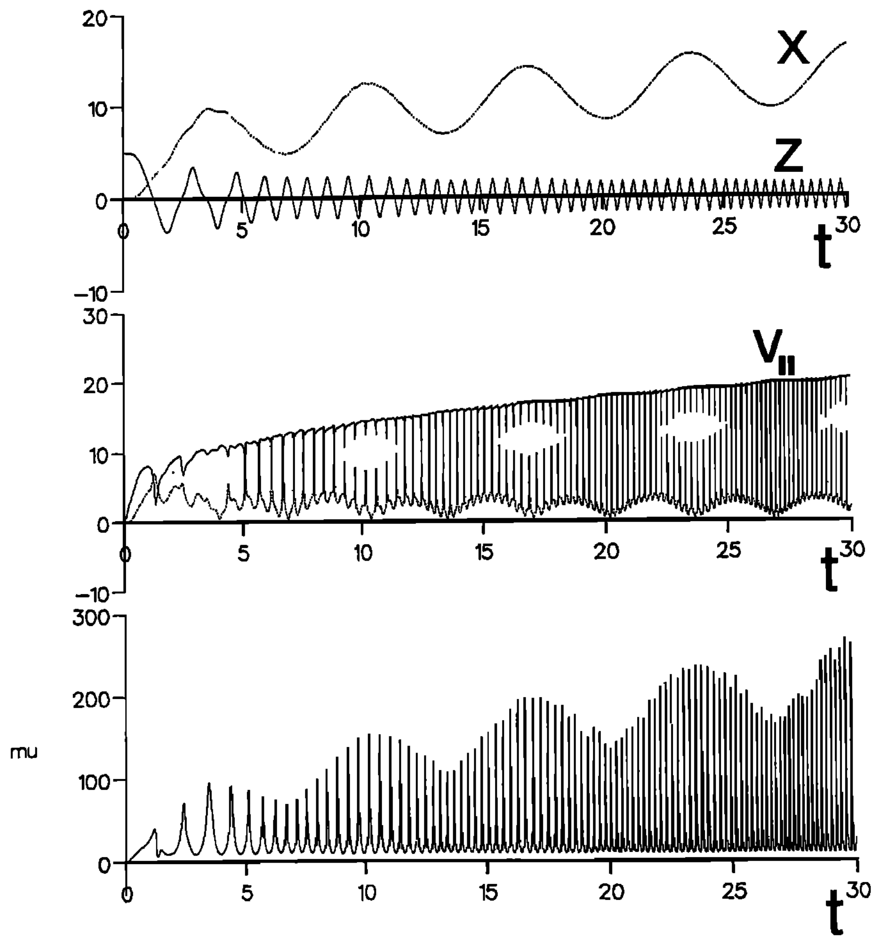


Fig. 11. Plot of $\mu$, the perpendicular and parallel velocity components, and the particle $x$ and $z$ coordinate versus time for a trajectory with $\alpha = 5$, that is, with $z^* = 5$ at $t = 0$. The full initial conditions were $\mathbf{r}^* = (0,0,5)$, $\mathbf{v}^* = (0,0,0)$. The plot spans the interval $t = 0$ to $t > 1/\alpha$ For $t > 1/\alpha$ the motion is $I_z$ conserving.

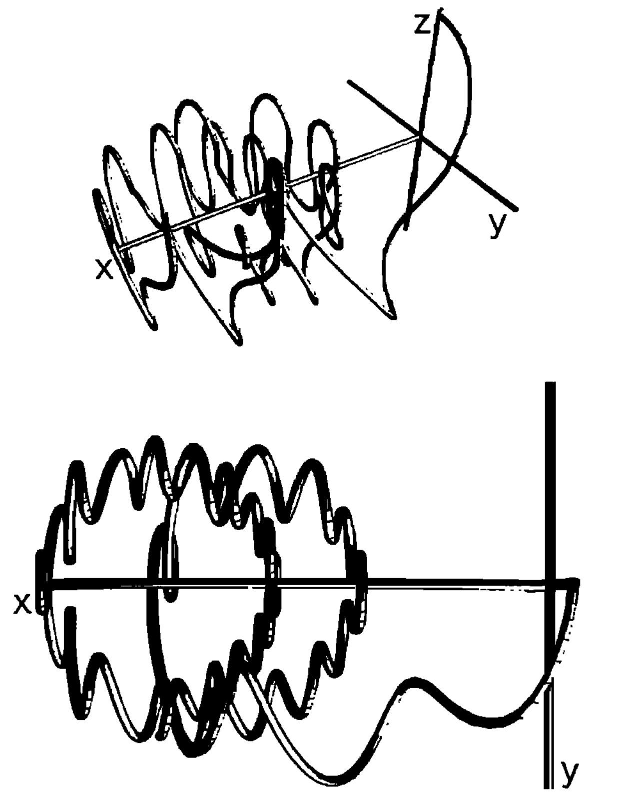


Fig. 12. Three-dimensional plot of the trajectory with $\alpha = 5$, for the interval $t = 0 - 15$, that is, including $t > 1/\alpha$. For $t > 1/\alpha$ the motion is $I_z$ conserving.

$$\mathbf{B} = (\frac{B_x z}{h}, 0, B_z) \tag{26}$$

$$\mathbf{E} = 0 \tag{27}$$

the Hamiltonian may be written

$$H_0 = \frac{\dot{x}^2}{2} + \frac{\dot{z}^2}{2} + \Psi_0 \tag{28}$$

where

$$\Psi_0 = \frac{x^2}{2} + \omega_{z0}^2 \frac{z^2}{2} - \frac{1}{2} F_{x0}^2 \tag{29}$$

with

$$\omega_{z0}^2 = \alpha_0(\alpha_0 \frac{z^2}{2} - x) \tag{30}$$

$$F_{x0} = \alpha_0 \frac{z^2}{2} \tag{31}$$

$$\alpha_0 = \frac{B_x}{B_z}\frac{\rho_z}{h} = \frac{1}{\kappa^2} \tag{32}$$

where $\kappa$ is defined in equation (5) of [*Buchner and Zelenyi*, 1989]. As in the time dependent case, the pseudopotential $\Psi_0$ describes two coupled oscillators in $x$ and $z$. The constant Hamiltonian has been written as

$$H_0 = H_0(\dot{x}, \dot{z}, x, z, \lambda) = \frac{\dot{x}^2}{2} + \frac{\dot{z}^2}{2} + \Psi_0(x, z, \lambda) \tag{33}$$

where

$$\lambda = \alpha_0 \tag{34}$$

is now a constant. The parameter $\lambda = \alpha_0$ scales the Hamiltonian by scaling the pseudopotential $\Psi_0$ within which the particles move at any time $t$, so that the ratio of the oscillation frequencies will also depend on $\alpha_0$ and will no longer have explicit time dependence. The parameter $\alpha_0$ (or $\kappa$) in the steady state model hence plays an analogous role to the parameter $\alpha t$ in the time dependent model (see equation (16)). As a consequence, and as shown by previous authors [*Buchner and Zelenyi*, 1989; *Chen and Palmadesso*, 1986] particles in a given regime of behavior (dictated by $\alpha_0$, or $\kappa$) now remain in that regime for all $t$. If $\alpha_0 \not\approx 1$, then the motion can be regular (integrable) for all $t$, conserving an adiabatic invariant associated with the fast oscillatory motion. The trajectory will be confined to a specific region in phase space [*Chen and Palmadesso*, 1986]. If $\alpha_0 \ll 1$, then for all $t$, $\omega_{z0} \ll 1$, and the trajectory will be characterized by fast oscillations about the field at the Larmor frequency, and slow oscillations between mirror points at frequency $\omega_{z0}$ and will conserve $\mu$. If $\alpha_0 \gg 1$, and if, for all $t$, $\omega_{z0} \gg 1$, the trajectory will be ring type (as defined by *Buchner and Zelenyi* [1989]) and will conserve an invariant $I_z$. If, on the other hand, $\omega_{z0} \approx 1$ at some point on the trajectory, the particle will be cucumberlike for all $t$. The condition for chaotic behavior is satisfied for all $t$ where the particle intersects the center plane ($z = 0$) at $x = -1$ (i.e., a distance $x = -\rho_z$ from the field line in unnormalized units) if $\alpha_0 \to 1$ ($\kappa \to 1$).

In this steady state model (and in any model in which the induction electric field is neglected) we also have

$$\frac{dH_0}{dt} = 0 \tag{35}$$

so that

$$\frac{d\lambda}{dt} = \frac{d\alpha_0}{dt} = 0 \tag{36}$$

so that there is no counterpart of the parameter $\alpha = \dot{\lambda}$, which in the explicitly time-dependent model indicated the "adiabaticity" of the system. In this sense, the steady state system, or any magnetic field model with no induction elec-

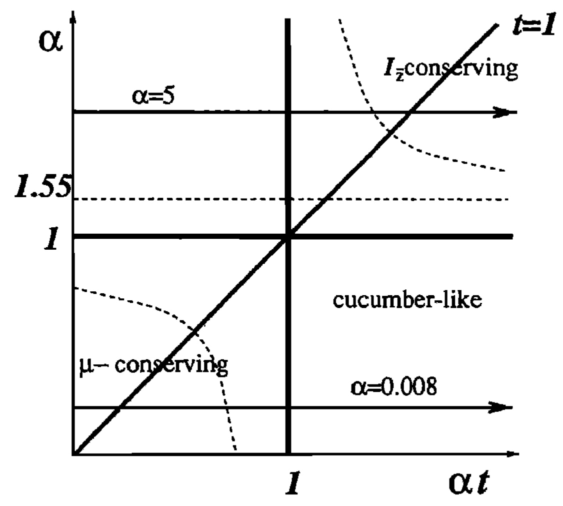


Fig. 13. Particle behavior in the time-dependent model ordered by the two controlling parameters $\alpha$ and $\alpha t$. The diagonal line $t = 1$ denotes the time taken for a particle of a given $\alpha$ to execute a Larmor orbit. The paths of the numerically integrated examples are marked on the plot.

tric field (with Hamiltonian of the form $H(q, \dot{q}, \lambda)$), corresponds to the explicitly time dependent system in the limit in which the system changes infinitesimally slowly, that is,

$$\alpha \rightarrow 0 \tag{37}$$

Since

$$\alpha = \frac{B_x}{B_z}\frac{\rho_z}{h}\frac{\Omega_f}{\Omega_z} = \alpha_0 \frac{\Omega_f}{\Omega_z} \tag{38}$$

this corresponds to a finite $\alpha_0$ only if the fold up timescale of the field $\tau \rightarrow \infty$ ($\Omega_f \rightarrow 0$), so that the time dependent field effectively becomes steady state. Alternatively, if $\alpha_0 \rightarrow 0$ so that $\alpha \rightarrow 0$, the models are identical, both simply describing a particle that gyrates in the constant field $\mathbf{B} = (0, 0, 1)$.

To summarize, the parameter $\alpha_0$ (or $\kappa$), which determines (for all $t$) whether a trajectory is regular or chaotic in the steady state system, plays an analogous role to the parameter $\alpha t$ in the time dependent system. Hence the condition $\alpha_0 \approx 1$ ($\kappa \approx 1$) dictates that a trajectory is chaotic in the steady state system for all $t$, and the condition $\alpha t \rightarrow 1$ gives the time at which a ($\alpha \ll 1$) trajectory becomes chaotic in the time dependent system. These two systems are however generally not identical, that is, the equations of motion in the above steady state system cannot be transformed into those describing the time dependent system with the substitution $\alpha_0 \rightarrow \alpha t$. Hence the regimes of behavior traversed by a single trajectory in the time dependent model as $\alpha t \ll 1$, $\alpha t \approx 1$, $\alpha t \gg 1$ cannot be identified in a single trajectory in the steady state model, instead being given by the behavior of a collection of distinct trajectories specified by $\alpha_0 \ll 1$, $\alpha_0 \approx 1$ and $\alpha_0 > 1$ in the steady state model [*Buchner and Zelenyi*, 1989].

Furthermore, trajectories in the time dependent model exhibit different behavior that depends on the $\alpha$ of the system. This ordering of behavior with the "adiabaticity" of the system does not have a counterpart in any field model which is static or does not include the inductive electric field associated with the time dependence.

## 7. CONCLUSIONS

Single charged particle dynamics in a simple scale free, time dependent model for a magnetic field reversal have been examined analytically and numerically. Two constraints have been imposed: the magnetic field and the associated induction electric field are independent of $x$ (to allow the inclusion of a cross-tail convection electric field by frame transformation), and the model is scale free (to allow parameterization of the particle behavior). The principal results are as follows:

1. Unlike the steady state system, behavior at any given $t$ in the time-dependent system is ordered by two parameters; $\alpha t$, which at any $t$ scales the pseudopotential in which the particles move, and $\alpha$, which scales the rate of change of the pseudopotential. The time at which a transition occurs in the dynamics is $t \approx 1/\alpha$, and the nature of the transition depends upon $\alpha$.

2. The explicit time dependence produces two possible classes of motion:

If $\alpha \ll 1$ representing a thick reversal which is folding slowly, the motion is a transition in behavior; while $t < 1/\alpha$ it is regular, conserving an invariant $\mu$; once $t > 1/\alpha$, the particle is pitch angle scattered and the motion generally becomes chaotic and "cucumberlike".

If $\alpha \gg 1$, representing a reversal which is thin and folds quickly, the motion becomes regular ("ring-type") within a Larmor period, exhibiting no extended interval of chaotic motion.

3. The parameter $\alpha t$ plays an analogous role to a constant parameter $\alpha_0$ (or in a different normalization, $\kappa$ [*Buchner and Zelenyi*, 1989]) which scales the steady state system. Hence whilst $\alpha t \approx 1$ indicates the time at which initially $\mu$ conserving particles become chaotic in the time dependent system (with $\alpha \ll 1$), particles in the steady state system with $\alpha_0 \approx 1$ (or $\kappa \approx 1$) are chaotic for all $t$.

4. The parameter $\alpha$ in the time-dependent system, and as a consequence the ordering of behavior with $\alpha$, do not have a counterpart in any field model which is static or does not include the inductive electric field associated with time dependence.

Multispacecraft in-situ measurements, such as those of the proposed Cluster mission [*Rolfe*, 1990, and references therein] are needed to unambiguously determine both the characteristic length scale and time scale of the field in terms of the particle Larmor scales, to obtain the $\alpha$ of the time-dependent model.

Taking both space and time derivatives of the normalized time-dependent field model

$$\frac{\partial^2}{\partial z \partial t}(\alpha z t, 0, 1) = (\alpha, 0, 0) \tag{39}$$

yields the $\alpha$ of the system, and will distinguish between a system which is time-dependent, and one which is static, since

$$\frac{\partial}{\partial z}(\alpha_0 z, 0, 1) = (\alpha_0, 0, 0) \tag{40}$$

We can only crudely estimate $\alpha$ from single, or two spacecraft in-situ data, which cannot unambiguously distinguish between spatial and temporal changes in the field. We will consider two cases: a "weak" reversal which is thick (half thickness $h = 5000$ km) and folds slowly ($\tau = 20$ mins) [e.g., *Hones*, 1977], and a "strong" reversal which is thin ($h = 200$ km) and folds quickly ($\tau = 5$ min) [*Mitchell et al.* 1990]. In both cases $B_x = 20$ nT, and the linking field $B_z$ is 5 nT. Electrons then have $\alpha \ll 1$ in both cases, for example, 50 keV electrons have $\alpha_e \approx 5.3 \times 10^{-7}$ in the weak reversal and $\alpha_e \approx 6.4 \times 10^{-5}$ in the strong reversal. Protons, however, can be in the $\alpha > 1.55$ regime. In the weak reversal, 1 keV protons have $\alpha_p \approx 6.0 \times 10^{-3}$, and in the strong reversal they have $\alpha_p \approx 0.7$. Higher energy protons, at 50 keV, have $\alpha_p \approx 0.04$ in the weak reversal and $\alpha_p \approx 5$ in the strong reversal. The ordering of particle behavior with $\alpha$ is therefore significant for the evolving proton distribution in the time dependent, presubstorm magnetotail.

The transition time $t \approx 1/\alpha$ should provide a prediction for the evolution in time of both the local electron and proton particle distribution functions. If the change in the particle distribution as the dynamics evolves produces substorm-related instabilities, or thermalizes the distribution sufficiently to lead ultimately to a reconfiguration of the magnetotail [*Cowley*, 1991]. *Mitchell et al*, 1990, and references therein], then a characteristic time scale for substorm "evolution", will be $t \approx 1/\alpha$.

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