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Properties of single-particle dynamics in a parabolic magnetic reversal with general time dependence

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We investigate single charged particle dynamics in the earth's magnetotail by examining a reversal with simple spatial dependence (the field lines are parabolic) but with general time dependence, which includes the associated induction electric field. The parabolic spatial dependence has, in static models, been shown by previous authors to imply that various regular and stochastic regimes of behavior exist, with particles remaining in a given regime for all time t. Here we show that, in general, time dependence yields transitions in behavior between the various regimes of regular and stochastic behavior. We identify three independent parametric coordinates which are functions of the field spatial and temporal scales, and the particle gyroscales, and time, which will indicate the regime of particle behavior at any given t. For specific time dependent field models these parametric coordinates can be inverted to directly give the timescales for transitions between one regime of behavior and another in terms of the field and particle spatial and temporal scales. These parametric coordinates have no direct analogue in static models. For a general thinning and folding sheet (representing the magnetotail magnetic field close to the center plane just before substorm onset) this characterization implies that stochastic dynamics may only occur over a finite time period and may not occur under certain circumstances. This may have implications for our understanding of the role played by single particle dynamics in the destabilization or reconfiguration of the magnetotail current sheet associated with substorms.

Introduction

A considerable body of work exists [e.g., *Speiser*, 1965; *Sonnerup*, 1971; *Chen and Palmadesso*, 1986; *Buchner and Zelenyi*, 1989; *Chen*, 1992; *Wagner et al.*, 1979] on single charged particle dynamics in simple static models of the magnetic field reversal in the earth's magnetotail. These have led to studies of particle populations in the quasi-static magnetotail [e.g., *West et al.*, 1978; *Chen et al.*, 1990; *Chen and Burkhart*, 1990; *Burkhart and Chen*, 1991; *Ashour-Abdalla et al.*, 1991] and of conditions in the pre-substorm plasma sheet [e.g., *Pulkkinen et al.*, 1991] and its stability [*Buchner and Zelenyi*, 1987].

Generally, the characterization of particle dynamics for a given static model requires the numerical integration of large numbers of trajectories (sufficient to sample all possible regions of phase space) for different values of the model scaling [*Gray and Lee*, 1982, *Chen and Palmadesso*, 1986]. A given particle will map out the surface of a static torus in phase space and the nature of this torus (that is, whether it is regular or not) will determine the nature of the motion (that is, whether it is regular or stochastic).

Special cases in which the field model is scale free can be used to analytically characterize the particle dynamics. A well-studied example is the model where the field lines are parabolas which are static. In this static model, regular and stochastic regimes of particle behavior were identified, the particle remaining within a given regime for all time t. The dynamics were shown analytically to be specified largely (but not completely) by a constant parameter κ, a function of the characteristic field strength and thickness of reversal and the particle gyroradius [*Buchner*, 1986; *Buchner and Zelenyi*, 1986, 1989]. In the static parabolic model, this parameter plays the same role as the normalized energy $\hat{H}$ defined by *Chen and Palmadesso* [1986]. The parameter κ (or $\hat{H}$) will not completely specify the nature of the dynamics (i.e., whether it is regular or stochastic) [*Chen and Palmadesso*, 1986; *Chen*, 1992] there being ordering with respect to a second parameter (e.g., Figure 19 of *Buchner and Zelenyi* [1989]; see also *Chapman and Watkins* [1993]). The precise nature of some regimes of behavior in the static parabolic model have also yet to be completely elucidated [*Chen*, 1992]. However, it has been suggested that κ will characterize the limiting behavior of particles (*Buchner and Zelenyi* [1989]; see also *Chen* [1992] for a discussion of the validity of these limits) and in this sense is a controlling parameter. In the static model, the value of the constant κ then "characterizes" the regime of behavior of the particles for all t.

More recently, dynamics in a parabolic field model with simple time dependence (and corresponding induction electric field) was investigated [*Chapman and Watkins*, 1993; *Chapman*, 1993], and it was shown that although the same regimes of behavior existed as in the static parabolic model, the particle dynamics migrated from one regime to another. In the time dependent system, the torus in phase space which the particle maps out changes with time. Specifically, for a single trajectory, the torus may at early t be regular (so that the motion is adiabatic, conserving an approximate invariant, such as μ) in the sense that the system is close to an integrable limit, but at later t be irregular (so that the motion is stochastic). For the time dependent system in general we cannot therefore determine the regime of particle behavior (i.e. the nature of the torus) at some t by integrating the particle trajectory over large t.

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General time dependence is therefore addresed analytically within the framework of the scale free spatial dependence of the static model with parabolic field lines. In principle, this field model could be used to describe the thinning plasma sheet close to the magnetotail center plane just prior to substorm onset. The spatial dependence of the parabolic field lines again produces the same regimes of behavior found in the static model, the time dependence allows the system to migrate between one regime and another.

In this paper we will analytically derive three (time dependent) parametric coordinates in the system with general time dependence by examining the particle Hamiltonian equation of motion. These parametric coordinates have no direct analogue in any static model. The values of the parametric coordinates at any instant will characterize the regime of particle behavior of the system at that t. This "characterization" does not correspond to an exact specification of the dynamics of a given particle. Instead, we will show that the parametric coordinates will indicate at any t whether the system is regular (i.e., close to the limiting case of integrable behavior) or stochastic.

We establish the conditions on these parametric coordinates which indicate a transition from one class of behavior to another. These conditions then allow the time spent by the particles in any one regime of behavior to be determined for specific field models. For some models, including examples which describe the thinning presubstorm plasma sheet, particles will only spend a finite time executing stochastic behavior. If the reversal is sufficiently thin and changing sufficiently quickly, intervals of stochastic behavior may not occur. The transitions in behavior are illustrated with numerically integrated trajectories in a simple "thinning" magnetic reversal (with time-varying linking field $B_z(t)$).

The organization of the paper is as follows. We first discuss the geometrical constraints which allow the Hamiltonian to be written in a sufficiently simplified form to allow analytical parameterization. We then review the normalization of the static model required for parameterization, and establish the extent to which behavior can be "characterized" analytically. This normalization is then used to parameterize the general time dependent system. We derive the three parametric coordinates, and indicate the possible regimes of particle behavior as regions in parametric coordinate space. The consequences of possible trajectories in this parametric coordinate space for motion in a thinning reversal are then discussed, and numerically integrated trajectories in the simple thinning model are presented.

Constraints Required to Obtain Parametric Coordinates

Two significant constraints are required to analytically obtain a simplified Hamiltonian equation of motion and hence parametric coordinates which characterize the dynamics.

First, we shall analytically examine a system of coupled oscillators by essentially obtaining appropriate scaling of the particle pseudopotential Ψ so that we require a system with Hamiltonian of the form

$$H = \sum \frac{p_j^2}{2m_j} + \Psi(q_j, t) \qquad (1)$$

where p_j and q_j are the generalized momenta and position coordinates of the particle. Given the form of the Hamiltonian in an electromagnetic field

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi \qquad (2)$$

we have the constraint that in an appropriate coordinate system $\mathbf{p}\cdot\mathbf{A}(q_j) = F(q_j)$. In the Earth's magnetotail this is readily achievable by choosing invariance in the direction of the cross-tail current (i.e. the GSE (Geocentric Solar Equatorial) $\hat{y}$ direction) so that $\mathbf{A} = A_y\hat{y}$, with independent coodinates $q_1 = x$, $q_2 = z$. The canonical momenta are then $p_1 = m\dot{x}$, $p_2 = m\dot{z}$ with $p_3 = m\dot{y} + eA_y$ as a constant of the motion (which is zero in an appropriate frame). The magnetic field is then confined to the x, z plane and is given by contours of A_y. In this Hamiltonian description the system therefore has two degrees of freedom and four time dependent coordinates $\dot{x}$, $\dot{z}$, x and z, which describe the trajectory in four-dimensional phase space.

Second, we seek to obtain parametric coordinates which vary with time and the (constant) field and particle scales only. If the parametric coordinates are a function of one variable (t) only it is, in principle, possible to then invert them to obtain the times at which transitions in particle behavior will occur for a given field model. As we will see next, the choice of a field model that is scale free in space effectively allows the spatial dependence to be separated from the field and particle scales. For static field models, this leads (via appropriate normalization) to the determination of parameters such as κ [*Buchner and Zelenyi*, 1989] which characterize the motion for all t. For time dependent models this yields time dependent parametric coordinates which characterize the motion at any given t.

Regimes of Behavior in the Static Parabolic Model

The particle behavior in the static parabolic model has been analyzed in detail by *Buchner and Zelenyi* [1989] and *Chen and Palmadesso* [1986]. Here we briefly reiterate the properties of the system to establish the regimes of particle behavior that arise as a consequence of the geometry of this model. In this static field, particles will remain within a given regime for all t; with the addition of time dependence these regimes are preserved, but particles can migrate from one regime to another with time. We will then identify the extent to which parameters such as κ "characterize" particle behavior.

With X, Y and Z corresponding to GSE coordinates, the static model is

$$\mathbf{B} = \left(\frac{B_{x0}Z}{h_0}, 0, B_z\right) \qquad (3)$$

$$\mathbf{E} = 0 \qquad (4)$$

The Larmor scales associated with the constant linking field B_z define the spatial and temporal scales with which we can "measure" all other spatial and temporal scales. Normalizing the magnetic field to the linking field B_z, temporal scales to the inverse of the gyrofrequency in this field $\Omega_z = eB_z/m$ and spatial scales to the particle gyroradius $\rho_z = v/\Omega_z$, the static field model becomes:

$$\mathbf{B} = (\alpha_0 z, 0, 1) \qquad (5)$$

$$\mathbf{E} = 0 \qquad (6)$$

The parameter

$$\alpha_0 = \frac{B_{x0}}{B_z}\frac{\rho_z}{h_0} = \frac{1}{\kappa^2} = \sqrt{(2\hat{H})} \tag{7}$$

partially determines the particle dynamics [*Buchner and Zelenyi*, 1989; *Chen and Palmadesso*, 1986; *Chapman and Watkins*, 1993]. In addition, it can be demonstrated analytically that at least one other parameter is required to completely specify the regime of behavior of a given particle in the parabolic model [*Buchner and Zelenyi*, 1989; *Chapman and Watkins*, 1993].

Since the static parabolic model has no intrinsic scale height, this is effectively specified by the range of z values of a given trajectory [see *Chapman and Cowley*, 1985]. This can be illustrated as follows: if we renormalize the above equations to give new variables $z^* = \alpha_0 z$, $t^* = t$ (that is, normalizing (3) to length ρ_z/α_0 instead of length ρ_z), the equations of motion describing the system become functions of z^* (and x^*, y^*) and t^* only (note that htis also renormalizes velocities, i.e. $\mathbf{v}^*(t^*) = \alpha_0 \mathbf{v}(t)$). The single trajectory given by $z^*(t^*)$ is either that of a particle with small α_0 and large $z(t)$ or large α_0 and small $z(t)$. These correspond to a trajectory with mirror points far from the center plane in a "weak" (e.g., large-scale height h) reversal or a trajectory with mirror points close to the center plane in a "strong" (e.g., small-scale height h) reversal, respectively. Hence the choice of α_0 does not uniquely distinguish dynamics in the static model but does, as we shall see next, characterize the dynamics by scaling the coupling between the x and z motion.

In the static parabolic system the equations of motion are

$$\ddot{x} = \alpha_0 \frac{z^2}{2} - x \tag{8}$$

$$\ddot{z} = -\alpha_0(\alpha_0 \frac{z^2}{2} - x)z \tag{9}$$

The Hamiltonian may be written

$$H_0 = \frac{\dot{x}^2}{2} + \frac{\dot{z}^2}{2} + \Psi_0(\alpha_0, x, z) \tag{10}$$

where H_0 is a constant of the motion and

$$\Psi_0 = \frac{1}{2}(\alpha_0 \frac{z^2}{2} - x)^2 = \frac{1}{2}A_0^2 \tag{11}$$

These equations describe the behavior of two nonlinear coupled oscillators in x and z. The strength of the coupling between the two oscillators will essentially dictate the nature of the particle dynamics. To parameterize the coupling strength, we can attempt to write the Hamiltonian in the form for two decoupled oscillators. If the x and z oscillators were decoupled, then the Hamiltonian

$$H = h_x(\alpha_0, \dot{x}, x) + h_z(\alpha_0, \dot{z}, z) \tag{12}$$

where

$$h_x = \frac{\dot{x}^2}{2} + \psi_x(\alpha_0, x) \tag{13}$$

$$h_z = \frac{\dot{z}^2}{2} + \psi_z(\alpha_0, z) \tag{14}$$

The Hamiltonian for the coupled system (10) can be written as $H_0 = h_{x0} + h_{z0}$ with

$$h_{x0} = \frac{\dot{x}^2}{2} + \frac{1}{2}x^2 \tag{15}$$

$$h_{z0} = \frac{\dot{z}^2}{2} + \frac{1}{2}z^2(\alpha_0\frac{z}{2})^2\left(1 - 2x(\alpha_0\frac{z^2}{2})^{-1}\right) \tag{16}$$

or

$$h_{z0} = \frac{\dot{z}^2}{2} + \frac{1}{2}x^2\left(1 - 2\frac{1}{x}\alpha_0\frac{z^2}{2}\right) \tag{17}$$

$$h_{z0} = \frac{\dot{z}^2}{2} + \frac{1}{2}z^2(\alpha_0\frac{z}{2})^2 \tag{18}$$

In the limit of a "strong" reversal

$$|\ \alpha_0\frac{z^2}{x}\ | \gg 1 \tag{19}$$

and in the limit of a "weak" reversal

$$|\ \alpha_0\frac{z^2}{x}\ | \ll 1 \tag{20}$$

the oscillators are weakly coupled, and (15)-(18) imply segments of motion with distinct x and z frequencies $\omega_{x0} \approx 1$ and $\omega_{z0} \approx \alpha_0 z/2$. One limit in which the motion completely decouples corresponds to a zero reversing field $B_x = 0$ (or an infinite sheet thickness compared to the gyroradius $h/\rho_z \rightarrow \infty$), so that $\alpha_0 = 0$ and from (17) and (18) (for finite z) the z oscillator vanishes leaving gyromotion about the constant linking field B_z at the unnormalized gyrofrequency $\Omega_z = eB_z/m$, which has been normalized to $\omega_{x0} = 1$. The other limit, of vanishing linking field $B_z = 0$ (or equivalently a vanishing sheet thickness to gyroradius $h/\rho_z \rightarrow 0$) so that $\alpha_0 \rightarrow \infty$ has $\omega_{z0} \rightarrow \infty$ and $\omega_{x0} = 1$ and does not decouple the oscillators as written in (15)-(16). This limit does not represent a neutral sheet as in the formalism used here the scaling (7) cannot distinguish the physically meaningful case of a vanishing linking field from the non-physical case of a vanishing sheet thickness (see also *Chen* [1992]). However, the time dependent parametric coordinate that will be obtained from a generalization of the scaling (7) will be meaningful for realistic time dependent field models appropriate for the pre-substorm geotail in which the magnetic reversal thins with increasing t (i.e., those that do not include an infinitely thin current sheet at finite t).

Two regimes of interest here can then be identified [see also *Buchner and Zelenyi*, 1989; *Chen and Palmadesso*, 1986; *Chen*, 1992]. In a reversal which is sufficiently "weak" with respect to the particle motion (i.e., if α_0 is sufficiently small that $|\ \alpha_0 z^2/x\ | \ll 1$) the oscillators are weakly coupled. The motion will be regular (in the sense that it approaches the integrable decoupled limit $\alpha_0 = 0$) conserving an approximate adiabatic invariant $\mu = v_\perp/B$. The motion is characterized by fast x oscillations about the field, with slow z bounce motion between mirror points (i.e. $\omega_{x0} \gg \omega_{z0}$) .

For larger values of α_0, such that $|\ \alpha_0 z^2/x\ | \sim 1$ the motion will become strongly coupled and will be stochastic. This regime of behavior, characterized by $\alpha_0 > 1$ or $\alpha_0 \sim 1$, corresponds to motion which is composed of segments of μ-conserving motion when the particle is far from the center plane, and fast oscillations in z which are approximately $I_z(= \oint v_z dz)$ conserving as the particle crosses the center plane.

The simple scale free parabolic model can represent some aspects of more phenomenologically accurate models of the tail current sheet in a region close to the center plane. For μ-conserving and stochastic trajectories, stochastic behavior resulting, for example, in jumps in the particle action μ, occurs as the particle crosses the center plane $z = 0$, so that in order to examine the conditions for stochasticity in single-particle dynamics we accurately model the spatial region close to the center plane. The model should therefore be relevant to trapped particles in more complex systems provided that the segment of the particle trajectory close to the center plane of the reversal which results in its stochasticity (e.g., where the pitch angle of μ conserving particles is being scattered) is contained within a region where the field lines are approximately parabolic. Far from the center plane, the particle motion will be regular in both the parabolic model and more phenomenologically accurate models. An additional feature of more phenomenologically accurate models will be the inclusion of loss cones, so that particles exiting the parabolic region with sufficiently small pitch angle will be lost from the system (e.g., transient orbits [*Chen and Palmadesso*, 1986]).

THE PARABOLIC MODEL WITH GENERAL TIME DEPENDENCE

We will now demonstrate that the introduction of time dependence allows particle behavior to migrate between the different regimes summarized above and will obtain convenient parametric coordinates which characterize this migration. We will consider a magnetic field of the form

$$\mathbf{B} = \left(f_x(T)B_x\frac{Z}{h}, 0, f_z(T)B_z\right) \tag{21}$$

with vector potential

$$\mathbf{A} = \left(0, f_zB_zX - f_xB_x\frac{Z^2}{2h} + A_t(T), 0\right) \tag{22}$$

with corresponding induction electric field

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial T} = \hat{\mathbf{y}}\left(-\dot{f}_zB_zX + \dot{f}_xB_x\frac{Z^2}{2h} + E_c(T)\right) \tag{23}$$

where, in general, the magnetic field model does not constrain $E_c(T) = -\dot{A}_t(T)$, which depends upon the chosen frame of reference. If, for example, the $\hat{z}$ component of the field is constant (f_z constant) this "convection" electric field E_c can be removed by a de Hoffman Teller frame transformation [*Chapman and Watkins*, 1993]. Generally, and for examples of specific interest such as a thinning model where f_x decreases with time, this will not be the case.

We can again use the characteristic normalization of the static model, that is, to normalize the magnetic field to the characteristic z field B_z, and distances and times to the Larmor scales in this field. This yields

$$\mathbf{B} = \left(f_x(\frac{t}{\Omega_z})\alpha_0 z, 0, f_z(\frac{t}{\Omega_z})\right) \tag{24}$$

$$A_y = f_z(\frac{t}{\Omega_z})x - f_x(\frac{t}{\Omega_z})\alpha_0\frac{z^2}{2} + A_t(\frac{t}{\Omega_z}) \tag{25}$$

and

$$\mathbf{E} = \hat{\mathbf{y}}\left(-\dot{f}_zx + \dot{f}_x\alpha_0\frac{z^2}{2} + E_c(\frac{t}{\Omega_z})\right) \tag{26}$$

where α_0 is as defined for the static model.

Note that since the spatial dependence of the model is still scale free, we can again rewrite the equations that specify the system as independent of α_0 if we renormalize such that the new position coordinates $z^* = \alpha_0 z$; the choice of $z^*(t^* = 0)$ then specifies the α_0 of the system. Since the time dependence is in general not scale free this does not eliminate the additional dependence on the intrinsic timescales contained in $f_x(t)$ and $f_z(t)$.

The equations of motion are

$$\ddot{x} = \dot{y}f_z = -f_z^2x + f_z\left(f_x\alpha_0\frac{z^2}{2} - A_t\right) \tag{27}$$

$$\dot{y} = -f_zx + f_x\alpha_0\frac{z^2}{2} - A_t = -A_y \tag{28}$$

$$\ddot{z} = -\dot{y}\alpha_0f_xz = -\alpha_0f_xz\left(-f_zx + f_x\alpha_0\frac{z^2}{2} - A_t\right) \tag{29}$$

The time dependent Hamiltonian equation of motion is

$$\dot{H} = \frac{d}{dt}\left(\frac{1}{2}(\dot{x}^2 + \dot{z}^2) + \Psi\right) \tag{30}$$

where

$$\Psi = \frac{1}{2}A_y^2 = \frac{1}{2}f_z^2\left(x - \frac{f_x}{f_z}\alpha_0\frac{z^2}{2} + \frac{A_t}{f_z}\right)^2 \tag{31}$$

so that time dependence results in a pseudopotential which, as well as having the same spatial dependence as in the static model, is a function of time dependent parameters

$$\lambda_1(t) = \frac{f_x(\frac{t}{\Omega_z})}{f_z(\frac{t}{\Omega_z})}\alpha_0 \tag{32}$$

$$\lambda_2(t) = f_z(\frac{t}{\Omega_z}) \tag{33}$$

and the frame dependent $A_t(t)$. We can transform to the (primed) frame in which $A_t'(t) = 0$ with $x' = x + A_t/\lambda_2$, $z' = z$, and $t' = t$, provided we do not attempt to take the limit of vanishing linking field (i.e., provided $\lambda_2 \not\to 0$). The pseudopotential then becomes

$$\Psi' = \frac{1}{2}A_y'^2 = \frac{1}{2}\lambda_2^2\left(x' - \lambda_1\frac{z^2}{2}\right)^2 \tag{34}$$

Once again these equations describe the behavior of two nonlinear coupled oscillators in x and z. If in the accelerating frame, we attempt to write them in decoupled form in analogy to (16)-(19), then

$$h_x' = \frac{\dot{x}'^2}{2} + \frac{1}{2}\lambda_2^2x'^2 \tag{35}$$

$$h_z' = \frac{\dot{z}'^2}{2} + \frac{1}{2}\left(\lambda_1\lambda_2\frac{z'}{2}\right)^2z'^2\left(1 - 2x'\left(\lambda_1\frac{z'^2}{2}\right)^{-1}\right) \tag{36}$$

or

$$h'_x = \frac{\dot{x'}^2}{2} + \frac{1}{2}\lambda_2^2 x'^2 \left(1 - 2\frac{1}{x'}\lambda_1 \frac{z'^2}{2}\right) \quad (37)$$

$$h'_z = \frac{\dot{z'}^2}{2} + \frac{1}{2}\left(\lambda_1 \lambda_2 \frac{z'}{2}\right)^2 z'^2 \quad (38)$$

so that now in the weakly coupled limit the frequency of the x oscillator $\omega_x \approx \lambda_2 = f_z$ is time dependent, the frequency of the z oscillator $\omega_z \approx \lambda_1 \lambda_2 z'/2$ is also explicitly time dependent, and the ratio of the two scales with λ_1.

In analogy to the static model, and as was demonstrated for the simple time dependent model [*Chapman and Watkins*, 1993], the value of the parametric coordinate $\lambda_1(t)$ scales the strength of the coupling between the two oscillators. If $\lambda_1 = 0$, then the motion decouples. If λ_1 is sufficiently small then the coupling will be weak, and the characteristic motion can be regular μ-conserving. However if λ_1 becomes sufficiently large (again "characterized" by $\lambda_1 \simeq 1$ or $\lambda_1 > 1$) at some later t, there will be transition to stochastic motion.

The behavior identified in the static model for different α_0 (constant) for all t by $\alpha_0 \ll 1$, $\alpha_0 \simeq 1$ and $\alpha_0 \gg 1$ will be just the behavior exhibited by the time dependent system at different times, that is for $\lambda_1(t)$ for times when $\lambda_1 \ll 1$, $\lambda_1 \simeq 1$, and $\lambda_1 \gg 1$. Hence for example, for $\lambda_1(t)$ increasing with time, particles at initially small $\lambda_1(t)$ will be μ conserving and, as $\lambda_1(t)$ increases will exhibit stochastic behavior. We can estimate the time at which the transition between particles being μ conserving to being stochastic takes place as when $\lambda_1 \simeq 1$, provided the particles cross the center plane on that timescale.

In addition, since

$$\Psi(\lambda_1, \lambda_2, x, z, A_t) = \frac{1}{2}\lambda_2^2 \left(x - \lambda_1 \frac{z^2}{2} + \frac{A_t}{\lambda_2}\right)^2 \quad (39)$$

then

$$\begin{aligned} \dot{H} &= \dot{\lambda_1}\frac{\partial H}{\partial \lambda_1} + \dot{\lambda_2}\frac{\partial H}{\partial \lambda_2} + \dot{A}_t \frac{\partial H}{\partial A_t} \\ &= \dot{\lambda_1}\frac{\partial \Psi}{\partial \lambda_1} + \dot{\lambda_2}\frac{\partial \Psi}{\partial \lambda_2} + \dot{A}_t \frac{\partial \Psi}{\partial A_t} \\ &= \dot{\lambda_1}\frac{\partial \Psi}{\partial \lambda_1} + \dot{\lambda_2}\frac{\partial \Psi}{\partial \lambda_2} + \dot{y} E_c \end{aligned} \quad (40)$$

since $E_c = -\dot{A}_t$ and using (28).

The change in the particle Hamiltonian with time is, as we would expect, given by the change in the pseudopotential, with an additional frame dependent contribution to $\mathbf{v} \cdot \mathbf{E}$ from the y-directed "convection" electric field E_c. The latter will contribute to the (frame dependent) energy of the particle at any given t but should not affect the characterization of the overall behavior, that is, whether the particle motion is regular or stochastic.

Equation (40) suggests that $\dot{\lambda_1}$ and $\dot{\lambda_2}$ scale the rate of change of the pseudopotential and therefore also must be parametric coordinates which characterize the particle motion. These two parametric coordinates refer to two classes of adiabaticity, which appear since there are two timescales on which the pseudopotential changes. These timescales are just the characteristic particle Larmor period given by $\lambda_2 = \omega_x = f_z$ and the transition timescale given by λ_1. If the pseudopotential changes slowly with respect to the Larmor period then the particle motion is "adiabatic" in the commonly referred to sense, that is,

$$\frac{|\dot{\omega_x}|}{\omega_x} \ll \omega_x \quad (41)$$

then the parameter

$$d\lambda_2 = \frac{|\dot{\lambda_2}|}{\lambda_2^2} = \frac{|\dot{f_z}|}{f_z^2} \quad (42)$$

must indicate "particle" adiabaticity with respect to Larmor oscillations, so that if $d\lambda_2 \ll 1$ segments of μ-conserving motion may exist, and if $d\lambda_2 \simeq 1$ or $d\lambda_2 > 1$, there can be no μ-conserving motion.

The nature of the system (i.e. that it is two nonlinear coupled oscillators with time dependence) has introduced a second timescale: the transition timescale given by $\lambda_1 \simeq 1$. This timescale gives a slow change in the pseudopotential with respect to the Larmor period, that is, the system is adiabatic if

$$|\dot{\lambda_1}| \ll \omega_x \quad (43)$$

so that the parameter

$$d\lambda_1 = \frac{|\dot{\lambda_1}|}{\lambda_2} = \frac{\alpha_0}{f_z^3} \, | \dot{f_x} f_z - \dot{f_z} f_x | \quad (44)$$

indicates system adiabaticity. If $d\lambda_1 \ll 1$ (the "slow passage limit"), then the transition is slow; that is, it occurs over many gyroperiods, and segments of μ-conserving motion may exist, whereas if $d\lambda_1 \simeq 1$ or $d\lambda_1 > 1$ (the "fast passage limit") there can be no μ-conserving motion.

We can illustrate this by estimating the time taken for λ_1 to change from 0 to 1. Since

$$\frac{d\lambda_1}{dt}\Delta t \approx \Delta\lambda_1 \quad (45)$$

the change $\Delta\lambda_1 \approx 1$ will be slow (i.e., occur over many gyroperiods) if the corresponding $\Delta t \gg 1/\omega_x$ so that

$$\frac{d\lambda_1}{dt}\frac{1}{\omega_x} \ll 1 \quad (46)$$

as above.

In order to characterize trajectories we only need to identify the various regimes of behavior in λ_1, $d\lambda_1$, $d\lambda_2$ space. The location of a particle at any t is known in this space as λ_1, $d\lambda_1$, and $d\lambda_2$ are just functions of t and are specified by the magnetic field model in which the particles move. Possible classes of trajectory are shown on Figure 1, which is a plot of the possible regimes of behavior in this parametric coordinate space. The notional trajectories shown begin in a region of regular "adiabatic" motion, at a time when $\lambda_1 \ll 1$, $d\lambda_1 \ll 1$ and $d\lambda_2 \ll 1$ so that the motion is μ-conserving. This initial condition could correspond to motion in a reversal at the early stages of thinning, so that the spatial scale of the reversal is large compared to the gyroradius, and the field is changing slowly with respect to both the gyroperiod and the transition timescale.

Two possible paths are shown for the particles as t increases and the reversal thins. If the transition $\lambda_1 \simeq 1$ occurs while $d\lambda_1 \ll 1$ and $d\lambda_2 \ll 1$ (a slow passage through the transition), the particles can continue to execute segments of μ- conserving motion once $\lambda_1 > 1$. The transition is therefore into stochastic behavior, which includes segments of both μ-conserving motion and fast (I_z-conserving) z oscillations across the center plane. The motion can only continue to execute segments of μ-conserving motion while $d\lambda_1 \ll 1$ and $d\lambda_2 \ll 1$, if at some later t this is no longer the case

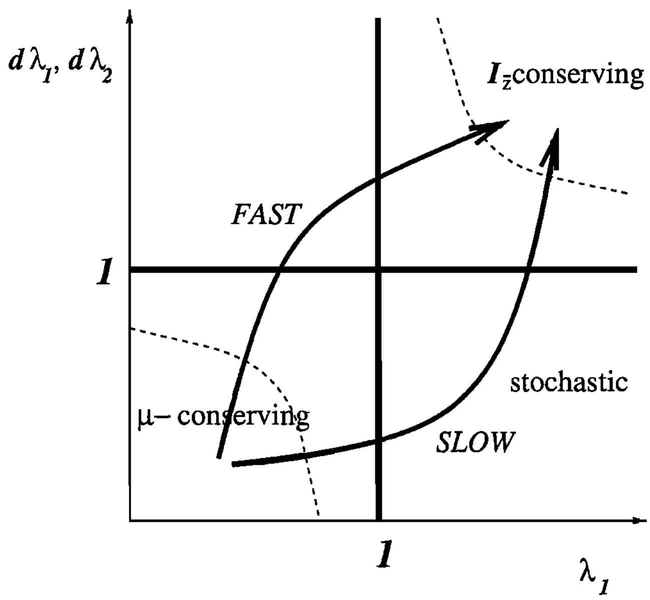

Figure 1. Notional trajectories for slow and fast passage transitions for a thinning reversal are marked on a plot in parametric coordinate $d\lambda_1$, $d\lambda_2$ versus λ_1 space. The different regimes of particle motion are marked on the plot (dashed lines indicate their notional boundaries).

(as shown), the motion will be entirely composed of fast (I_z-conserving) z oscillations. If on the other hand the transition $\lambda_1 \simeq 1$ occurs after $d\lambda_1 \simeq 1$ or $d\lambda_2 \simeq 1$ (a fast passage through the transition) then segments of μ-conserving motion cannot exist after the transition $\lambda_1 \simeq 1$. In this case, stochastic motion composed of segments of both μ-conserving motion and fast (I_z-conserving) motion cannot occur, and the transition will be from a single period of regular μ-conserving motion to a single period of regular I_z-conserving motion.

It should be emphasised that the condition for a slow or fast transition is not simply whether the system is changing slow or fast with respect to the particle gyroperiod (indicated by $d\lambda_2$) but also whether the change is slow or fast with respect to the transition timescale (indicated by $d\lambda_1$).

Numerically Integrated Trajectories

Previously, trajectories were numerically integrated in the simple time dependent model of *Chapman and Watkins* [1993] using a model field where $f_x = T/\tau = t/(\Omega_z\tau)$ (where τ is the timescale on which the field changes) and $f_z = 1$ so that $\lambda_1 = \alpha_0 t/(\Omega_z\tau) = \alpha t$, $\lambda_2 = 1$, then $d\lambda_1 = \alpha$ was a constant and $d\lambda_2 = 0$. Trajectories were then found to exhibit a transition in behavior when $t \simeq 1/\alpha$, that is, when $\lambda_1 \simeq 1$, and to be ordered in behavior by the constant $d\lambda_1 = \alpha$. On the plot in parametric coordinate space, trajectories would be horizontal lines, the particle moving to the right (increasing λ_1) with increasing t. The parametric coordinate $d\lambda_2$ could not be identified in this model as it was zero for all trajectories.

In order to further illustrate the above results, a simple field model has been selected which has the property that $\lambda_1 = \lambda_1(t)$, $d\lambda_1 = d\lambda_1(t)$, and $d\lambda_2$ is constant; trajectories in this model then exhibit transitions with respect to both λ_1 and $d\lambda_1$ and are ordered with respect to $d\lambda_2$.

Unnormalized, the magnetic field is

$$\mathbf{B} = \left(\frac{B_x Z}{h}, 0, \frac{B_z}{(T_O + \frac{T}{\tau})}\right) \tag{47}$$

which represents a "thinning" field; that is, the linking field decreases with t. The corresponding electric field

$$\mathbf{E} = \hat{\mathbf{y}}\left(\frac{B_z X}{\tau(T_O + \frac{T}{\tau})^2}\right) \tag{48}$$

has $E_c(T) = 0$, this corresponds to a frame of reference in which the field line passing through $X = 0$, $Z = 0$ is at rest. We may move into an $E_c(T) \neq 0$ frame by acceleration in the $\pm x$ direction; in the moving frame the particle initial velocity and subsequent energisation will differ (from equation (40)), but the $z(t)$ will be unchanged.

Again, normalizing the magnetic field to a characteristic linking field B_z, time to the inverse of the gyrofrequency in field B_z (that is, to $1/\Omega_z$, and length to a characteristic gyroradius $\rho_z = v/\Omega_z$ yields

$$\mathbf{B} = \left(\alpha_0 z, 0, \frac{1}{(\frac{t_0}{\Omega_z} + \frac{t}{\Omega_z\tau})}\right) \tag{49}$$

The parametric coordinates for this system are

$$\lambda_1 = \alpha_0\left(\frac{t_0}{\Omega_z} + \frac{t}{\Omega_z\tau}\right) \tag{50}$$

$$\lambda_2 = \frac{1}{(\frac{t_0}{\Omega_z} + \frac{t}{\Omega_z\tau})} \tag{51}$$

$$d\lambda_1 = \frac{\alpha_0}{\Omega_z\tau}\left(\frac{t_0}{\Omega_z} + \frac{t}{\Omega_z\tau}\right) = \frac{1}{\Omega_z\tau}\lambda_1 \tag{52}$$

$$d\lambda_2 = \frac{1}{\Omega_z\tau} \tag{53}$$

Examples of a fast and a slow passage in this model are sketched in parametric coordinate space in Figures 2a and 2b, which show lines given by (52) and (53), respectively.

In order for a particle to undergo a slow passage, we require that as t increases $\lambda_1 \ll 1$ to $\lambda_1 > 1$, while both $d\lambda_1 \ll 1$ and $d\lambda_2 \ll 1$. From Figure 2a we see that this particle (represented by the line marked slow) will, if $\lambda_1(t=0) \ll 1$, initially, execute μ-conserving behavior. Once $\lambda_1 > 1$ the motion will become stochastic, exhibiting segments of both μ-conserving behavior when far from the center plane, and fast z oscillations across the reversal when close to the center plane. Once $d\lambda_1 > 1$ the motion will be entirely composed of fast z oscillations across the reversal. This scenario requires $d\lambda_2 \ll 1$, as shown by the horizontal line marked slow on Figure 2b. From (52) and (53) this trajectory requires $\Omega_z\tau \gg 1$ so that both $d\lambda_2 \ll 1$ and the gradient of (52), which is $1/(\Omega_z\tau) \ll 1$. In addition, t_0/Ω_z must be sufficiently small that all three parametric coordinates are much less than one at the start of the trajectory, so that the motion is initially μ-conserving.

A numerical solution of such a trajectory is shown in Figures 3a and 3b. The equations of motion were integrated using an adaptive order and adaptive stepsize scheme [*Shampine and Gordon*, 1975]. The normalization is the same as used in the above discussion, (i.e. to length ρ_z and time $1/\Omega_z$). The field model then has $\alpha_0 = 0.01$ (a weak reversal) , $\Omega_z\tau = 25$ (a slowly changing reversal),

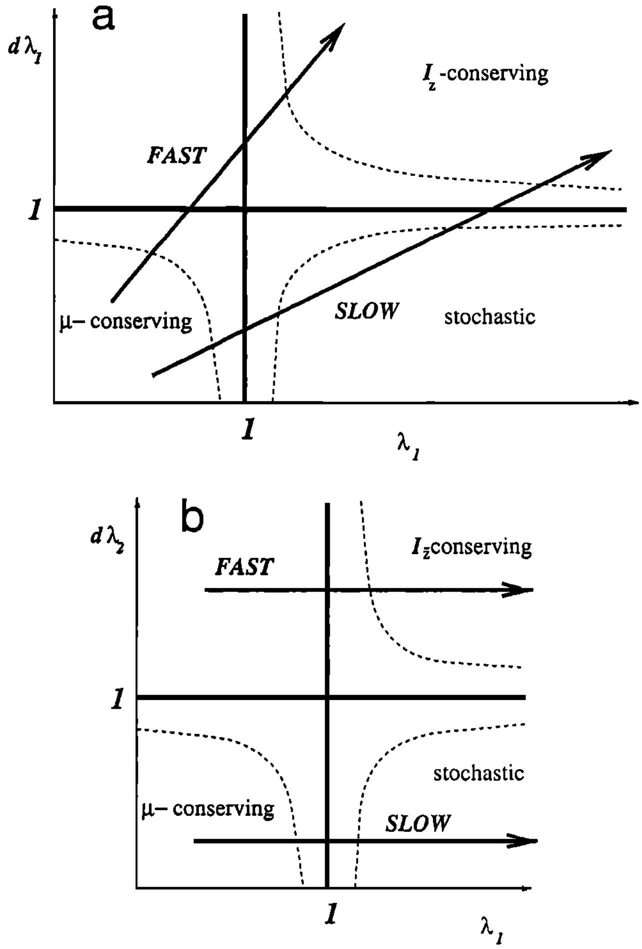

Figure 2. Notional trajectories for slow and fast passage transitions in the simple thinning model given in the text marked on plots in parametric coordinate $d\lambda_1$ (a) and $d\lambda_2$ (b) versus λ_1 space. The different regimes of particle motion are marked on the plot (dotted lines indicate their notional boundaries).

and $t_0/(\Omega_z) = 0.01$. The trajectory has initial position $\mathbf{r} = (5 \times 10^{-2}, 0, 1)$, with the z position chosen such that the renormalized system would have $z^*(t^* = 0) = \alpha_0 z(t = 0) = \alpha_0$ and the x position chosen such that initially the particle lies on the rest field line defined by the choice of $E_c(T) = 0$. The initial $\mathbf{v} = (0, 0.5, -0.01)$ at $t = 0$, with the velocity components chosen to ensure that the particle crosses the center plane $z = 0$ on a shorter timescale than the transition timescales of the system.

Once $t/(\Omega_z \tau) \gg t_0/\Omega_z$, that is, once $t \gg 0.25$ (approximately a quarter of a gyroperiod) the parametric coordinates are just

$$\lambda_1 \simeq \frac{\alpha_0}{\Omega_z \tau} t \qquad (54)$$

$$d\lambda_1 \simeq \frac{\alpha_0}{(\Omega_z \tau)^2} t \qquad (55)$$

$$d\lambda_2 = \frac{1}{\Omega_z \tau} \qquad (56)$$

which yield the transition times, that is, $\lambda_1 \simeq 1$ at $t = \Omega_z \tau / \alpha_0 = 2500$, and $d\lambda_1 \simeq 1$ at $t = (\Omega_z \tau)^2/\alpha_0 = 62500$ ($d\lambda_2 = 1/25 \ll 1$ for all t). We can compare these times with the numerically integrated trajectory shown in Figures 3a and 3b, which are plots of $x(t)$ and $z(t)$. The period in time spanning from $t = 0$, so that $\lambda_1 < 1$, to $t = 10000$, so that $\lambda_1 > 1$, is shown in Figure 3a. The x motion appears almost linear due to acceleration in the induction electric field, and the z motion is oscillatory. From the z motion we see that before $t = 2500$ the motion is composed of fast oscillations combined with a slow bounce motion, the particle is not strongly pitch angle scattered on crossing the center plane (the mirror points at $t \approx 750$ and $t \approx 2000$ are at approximately the same z). However, once $t > 2500$, an interval of stochastic motion begins; on each crossing of the center plane the particle executes fast z oscillations before being ejected to execute a segment of μ-conserving motion away from the center plane. In Figure 3b the same trajectory

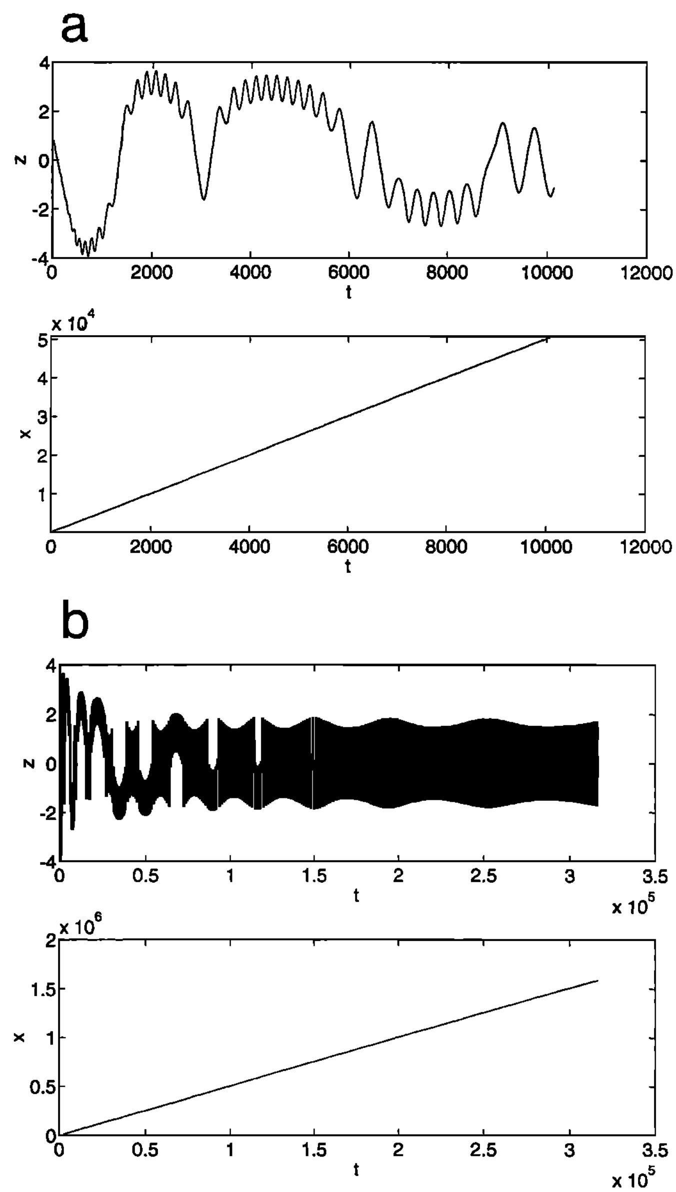

Figure 3. A numerically integrated trajectory undergoes slow passage in the simple thinning model. The panels show $z(t)$ (top) and $x(t)$ (bottom). The plot Figure 3(a) starts at the initial condition ($t = 0$) and shows a time period spanning $\lambda_1(t) < 1$ to $\lambda_1(t) > 1$. The plot Figure 3(b) starts at the initial condition ($t = 0$) and shows a longer time period which also spans $d\lambda_1(t) < 1$ to $d\lambda_1(t) > 1$. In this model $d\lambda_2$ is constant and $d\lambda_2 < 1$ for this trajectory.

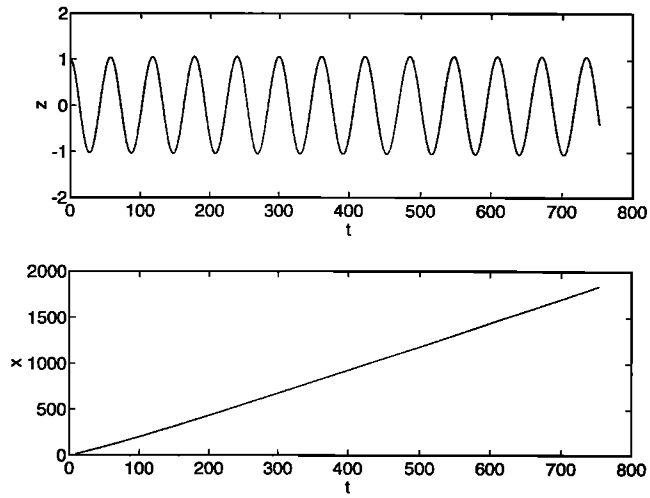

Figure 4. A numerically integrated trajectory undergoes fast passage in the simple thinning model. The panels show $z(t)$ (top) and $x(t)$ (bottom). The time period starts at the initial condition ($t = 0$). The initial conditions and model parameters are identical to the plotted trajectory in Figure 3, except that now $d\lambda_2 > 1$.

is plotted from $t = 0$ to $t = 3 \times 10^5$, that is, until $d\lambda_1 > 1$. The particle continues to execute stochastic behavior; at each subsequent crossing of the center plane the segment of fast z oscillations is longer in time until $t \approx 1.2 \times 10^5$ ($d\lambda_1 \simeq 2$) when the motion becomes entirely composed of fast z oscillations. Hence as implied by the above analysis, the conditions $\lambda_1 \rightarrow 1$ and $d\lambda_1 \rightarrow 1$ (with $d\lambda_2 \ll 1$) give the approximate times after which a transition in dynamics will occur, the exact times at which the dynamics can be identified to change being dependent upon initial conditions (being sensitive to the times at which the particle crosses the center plane, for example).

The fast trajectory sketched in parametric coordinate space in Figure 2 corresponds to the gradient of (52), $1/(\Omega_z\tau) > 1$. However this implies $d\lambda_2 > 1$, so that segments of μ-conserving motion cannot exist. A numerical solution of such a trajectory is shown in Figure 4, in the same format as Figure 3. The field model in this case has identical parameters to above, except that now $\Omega_z\tau = 0.02$ ($d\lambda_2 = 50$); that is, the reversal now changes quickly. The initial position and velocity of the particle is also identical to the previous case. From Figure 4 the motion can be seen to be entirely composed of fast z oscillatory motion and appears to be regular.

CONCLUSIONS

Under certain constraints we can obtain a system which yields parametric coordinates which are functions of the field and particle spatial and temporal scales, and time, that characterize particle dynamics of a time dependent reversal with parabolic field lines.

Provided that the particles cross the center plane on the transition timescales identified in the system we have shown the following:

1. In general, the different regimes of particle behavior may be ordered with respect to three time dependent parametric coordinates which are readily obtainable for a field model of any given time dependence. Essentially, these are given by the (time dependent) scaling of the strength of the coupling between the x and z oscillatory motion and the particle and system adiabaticity.

2. For the case of a thinning reversal, two types of transition are possible: a slow transition from μ-conserving to stochastic motion, then subsequently to regular motion entirely composed of z oscillations across the center plane, and a fast transition into z oscillatory motion with no interval of stochastic motion.

3. For specific models for the field the parametric coordinates can be inverted to give the approximate times at which transitions between the different types of motion occur.

Multispacecraft in situ measurements, such as those of the proposed Cluster mission [*Rolfe*, 1990 and references therein] are needed to unambiguously determine both the characteristic length scale and timescale of the field in terms of the particle Larmor scales. This will allow the parametric coordinates to be determined and hence give estimates of the times at which transitions between one class of motion and another should occur. In particular, this should allow the duration of the period of stochastic motion to be determined.

It has been suggested that the change in the particle distribution as the dynamics evolves in the presubstorm thinning plasma sheet may produce substorm-related instabilities [*Buchner and Zelenyi*, 1987] or may thermalize the distribution sufficiently to lead ultimately to a reconfiguration of the magnetotail [*Cowley*, 1991; *Mitchell et al.*, 1990, and references therein]. The implication of these results, that stochastic motion may persist for a finite time or under certain circumstances may not occur, may therefore have implications for our understanding of the role played by single-particle dynamics in the substorm cycle.

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