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SOME CONSEQUENCES OF THE SHIFT THEOREM FOR MULTISPACECRAFT MEASUREMENTS.

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Abstract. We show how shift theorem yields a routine quantitative test to determine whether a structure seen by two or more spacecraft is a quasistatic convecting object, such as a boundary layer. The test indicates the frequency range over which the data is consistent with a structure which is coherent between spacecraft, planar, and time independent in its rest frame. A cluster of four non coplanar spacecraft is required to determine the velocity of the structure in the spacecraft frame. Whether or not the data is found to be consistent with quasistatic convecting object, this analysis of the data from four non coplanar spacecraft yields the convection velocity of the plasma given the dispersion relation or vice versa, as well as the wavevectors corresponding to given frequency components. The test for coherence, planarity and time independence is shown to be robust against detector and systematic inter-spacecraft timing and positional uncertainties. Random errors will affect a finite frequency range, in principle this can be determined to restrict the frequency range over which the test can be applied.

Introduction

One of the purposes of missions such as Cluster [*'The Cluster Mission'*, 1988], [*Rolfe,* 1990] is to employ multispacecraft measurements to distinguish between spatial and temporal changes in the field and plasma. There are two approaches to the ordering of sets of multispacecraft data. The first is to use the (four) spacecraft to measure specific quantities, such as $\nabla \wedge B$, or low frequency magnetic waves (the 'wave telescope'), these techniques have been discussed elsewhere (eg in the case of the magnetometer [*Dunlop et al.,* 1988], [*Dunlop et al.,* 1990a], [*Neubauer et al.,* 1990a], [*Dunlop et al.,* 1990b], [*Neubauer et al.,* 1990b]). The second approach, employed here, is to develop tests for specific classes of events. As tests of this type are not directed at the measurement of a particular quantity, they are not in principle instrument dependent.

Specifically we develop a technique to determine to what extent an event can be described as a static structure convecting over the spacecraft cluster. This technique relies on the analysis of data in ω, **k** space. If this is possible, the technique yields the frequency interval of the data consistent with a structure which is quasistatic and convecting over the cluster and the degree of spatial and temporal coherence of the structure. In principle this defines the filter which allows this segment of the data to be seperated for further analysis. If the plasma convection velocity is known the technique also determines the dispersion relation of waves in the rest frame of the plasma (or vice versa).

Generally it is essential to evaluate the restrictions on a given technique that result from the uncertainties in the inter-spacecraft positions and timing, and from the characteristics of the individual instrument response. The formalism in ω, **k** space used here naturally allows these characteristics of the 'measurement' to be distinguished from the characteristics of the event itself. In ω, **k** space we are able to define the frequency range over which the technique is robust against detector and inter-spacecraft timing and positional uncertainties.

The technique discussed here has been developed for the particular case of four (or more) non coplanar spacecraft, however some aspects are applicable to the analysis of data from pairs of spacecraft.

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The Ideal Spacecraft Cluster

We begin by considering a quasi-static planar structure which convects over the 'ideal' spacecraft cluster. This can be treated as a coherent planar wavepacket which has no dispersion, ie all frequency components move at the same velocity across the spacecraft, which sample the wavepacket, or structure, at different times. This relative motion between the structure and the spacecraft can be due either to wave motion in the plasma, or convection of the plasma across the spacecraft, or a combination of the two; this analysis, along with knowledge of the appropriate dispersion relation, will allow these to be distinguished. We will then consider the effect of departures from these assumptions of planarity, coherence, time independence and non-dispersion, and the effects of the properties of the detector, and positional and timing uncertainties in the spacecraft cluster.

For simplicity we consider pairs of spacecraft at different locations $\mathbf{r}_j$ and $\mathbf{r}_{j+1}$ which are at rest w.r.t. each other. We assume that the spacecraft take data over a period ΔT_M which is longer than the time taken for the structure to convect over all of the spacecraft. The jth spacecraft sees a signal $S(\mathbf{r}_j, t)$ which has Fourier transform $F(\mathbf{r}_j, \omega)$:

$$S(\mathbf{r}_j, t) \Longleftrightarrow F(\mathbf{r}_j, \omega) = \Psi(\mathbf{r}_j, \omega) e^{i\theta(\mathbf{r}_j, \omega)} \qquad (1)$$

where Ψ and θ are real. If the wavefield $S(\mathbf{r}, t)$ represents a planar, static structure convecting over, and coherent between, the jth and $j+1th$ spacecraft then the $j+1th$ spacecraft sees the same structure as the jth spacecraft, but at a time δt_j later, so that

$$S(\mathbf{r}_{j+1}, t) = S(\mathbf{r}_j, t - \delta t_j) \qquad (2)$$

From shift theorem

$$S(\mathbf{r}_j, t - \delta t_j) \Longleftrightarrow F(\mathbf{r}_j, \omega) e^{-i\omega\delta t_j} \tag{3}$$

Then at the $j+1th$ spacecraft (1) and (2) give

$$F(\mathbf{r}_{j+1}, \omega) = \Psi(\mathbf{r}_{j+1}, \omega) e^{i\theta(\mathbf{r}_{j+1}, \omega)} = \Psi(\mathbf{r}_j, \omega) e^{i\theta(\mathbf{r}_j, \omega)} e^{-i\omega\delta t_j} \tag{4}$$

The amplitude spectrum at the $j+1th$ spacecraft is just that at the jth spacecraft;

$$\begin{aligned} A_{j+1} &= F(\mathbf{r}_{j+1}, \omega)^{\dagger} F(\mathbf{r}_{j+1}, \omega) \\ &= \Psi(\mathbf{r}_{j+1}, \omega)^2 = \Psi(\mathbf{r}_j, \omega)^2 = A_j \end{aligned} \tag{5}$$

and is independent of the time delay δt_j. The phase spectrum at the $j+1th$ spacecraft

$$\theta(\mathbf{r}_{j+1}, \omega) = \theta(\mathbf{r}_j, \omega) - \omega\delta t_j \tag{6}$$

contains the time delay δt_j.

The time period ΔT_M over which a given spacecraft takes data then corresponds to a lower limit to the frequency range of information contained in the A_j and θ, which is just $\omega_{min} = \frac{2\pi}{\Delta T_M}$. The time interval between sucessive samples ΔT_s (which defines the sampling frequency ω_s) corresponds to an upper limit to the frequency range $\omega_{max} = \frac{1}{2}\omega_s = \frac{\pi}{\Delta T_s}$, where these frequencies are measured in the spacecraft rest frame. The amplitude and phase spectra at the $j+1th$ spacecraft contain different information describing the behaviour of the wavepacket or convecting structure.

Amplitude Spectrum

For a coherent, quasi-static planar structure then

$$R_j = \frac{A_{j+1}(\mathbf{r}_{j+1}, \omega)}{A_j(\mathbf{r}_j, \omega)} = 1 \tag{7}$$

so that a plot of R_j versus ω provides a routine test for coherence, planarity and time independence of the structure. This is most easily envisaged if we treat the signal as a moving wavepacket, the envelope of which is the convecting structure. For R_j to be independent of frequency then:

1. both the jth and the $j+1th$ spacecraft must sample the wavepacket (ie it is coherent between them).
2. all frequency components must be time independent so that they have the same magnitudes at times t and $t + \delta t_j$. This is consistent with the envelope of the wavepacket, the structure, being time independent.
3. the wave must be planar at all frequencies; otherwhise the amplitude will change with distance (eg a spherical wave representing an expanding bubble)

If $R_j(\omega)$ over some range of frequencies only, then the data can be filtered to remove those frequency components; the remaining signal can then be analysed as a static convecting structure. However, if there is an overlap in the frequency ranges containing the static convecting component of the signal, and other components, the information carried by the wave modes in the region of overlap in frequency cannot be retrieved by this process.

If the wavepacket is dispersive, however, R_j will still be independent of frequency, as the components of the wavepacket will all still arrive with unchanged magnitudes at the different spacecraft, but at different times. The dispersive wavepacket will not of course have a static envelope, and will not represent a static convecting structure. Information on these time delays is given in the phase spectrum.

Phase Spectrum

From (6) the delay between the arrival times of the static convecting structure δt_j is simply

$$\delta t_j = \frac{\theta(\mathbf{r}_j, \omega) - \theta(\mathbf{r}_{j+1}, \omega)}{\omega} \tag{8}$$

The significance of this time delay can again be understood if we treat the moving structure as the envelope of a propagating wavepacket. The wavepacket will have a number of frequency components, and if all components move at the same velocity then δt_j will be independent of ω and will simply yield the velocity with which the envelope of the wavepacket (the structure) moves w.r.t. the spacecraft. The components of the velocity along each spacecraft seperation vector are then given by

$$\mathbf{v}.(\mathbf{r}_{j+1} - \mathbf{r}_j) = v^2 \delta t_j \tag{9}$$

so that four spacecraft at non coplanar $\mathbf{r}_j$ are needed to determine $\mathbf{v}$. This velocity is just the phase velocity of the waves in the wavepacket in the spacecraft rest frame. Since the phase velocity $\mathbf{v} = \frac{\omega}{k}\hat{\mathbf{k}}$ where ω is also measured in the spacecraft rest frame, determining $A(\omega)$ and $\mathbf{v}$ also determines the $\mathbf{k}$ modes present in the structure. If the waves can then be identified, so that the true propagation velocity of the waves in the plasma rest frame $\mathbf{v}_p$ is known, then the convection velocity $\mathbf{v}_c$ (ie the bulk velocity of the plasma) can be distinguished from the propagation velocity as $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_c$. This convection velocity can be independently determined by the plasma instrument, giving a consistency check. Alternatively, if the convection velocity $\mathbf{v}_c$ (plasma bulk velocity) is known independently, the frequency in the plasma rest frame ω' can be obtained by doppler shifting the frequency measured in the spacecraft rest frame, ie $\omega' = \omega - \mathbf{v_c}.\mathbf{k}$ yielding the propagation velocity $\mathbf{v}_p = \frac{\omega'}{k}\hat{\mathbf{k}}$.

If $\delta t_j = \delta t_j(\omega)$ then $\mathbf{v} = \mathbf{v}(\omega)$ is the phase velocity of individual components of a wave-packet which has dispersive behaviour. Again this velocity is measured in the spacecraft, not the plasma, rest frame so that knowledge of the dispersion relation (ie $\mathbf{v}_p(\omega')$) should, with the measured $\mathbf{v}(\omega)$, in principle determine the plasma rest frame, or vice versa. A dispersive, but coherent, wavepacket will have the same amplitude spectra at the different spacecraft.

Again, a plot of δt_j versus ω would reveal those frequency ranges (if any) over which the wavepacket is non-dispersive; if the wavepacket does represent a convecting static structure a corresponding frequency range should appear in a R_j versus ω plot as discussed above.

Comparison With Models and Simulated Data

The values of ω_{max} and ω_{min} needed to include the information in ω space that characterizes a given structure will depend upon the spacecraft velocity w.r.t. that structure, since ω_{max} and ω_{min} are measured in the spacecraft rest frame. In addition, the multispacecraft analysis techniques discussed here require all spacecraft to make measurements of an event over an interval which is longer than the time taken for the structure to convect over all spacecraft, ie, for:

$$\Delta T_M > \frac{(\mathbf{r_{j+1}} - \mathbf{r_j}).\mathbf{v}}{v^2} \tag{10}$$

Hence to determine ΔT_M for a given spacecraft seperation, and the desirable sampling interval Δt_s, before a measurement is made requires a priori knowledge of the 'bandwidth' that characterizes an event (ie ω_{max} and ω_{min}) in some known frame, and the velocity of the spacecraft w.r.t. that frame. This can be provided to some extent by models, or self consistent simulations of given events.

If $\mathbf{v}$ is different between different spacecraft pairs then in general the structure is not coherent over the spacecraft seperation, although special cases of non planar, but coherent structures (expanding spherical wavefronts for example) will be consistent with different values of $\mathbf{v}$. These special cases, for which $R_j(\omega)$ and $\delta t_j(\omega)$ can still be used to identify properties of the structure can only be investigated by using models or 'data' generated by numerical simulations to calculate the behaviour of R_j and δt_j which can then be compared with spacecraft data. For example, for the simple model of an expanding spherical wavefront, we expect the amplitude of waves to fall with $\frac{1}{r_0}$ dependence, where r_0 is distance from the centre of the sphere. If the $j+1$th spacecraft is at a distance r_1 from the centre, and the jth spacecraft is at a distance r_0, then $R_j = \frac{r_0^2}{r_1^2}$ is not unity, but is independent of ω.

Instrument Response

We will now consider some effects of instrument response and spacecraft uncertainties, non coherence and non-quasistatic structures on these conclusions. The response of a given detector, and systematic timing uncertainties between spacecraft, can be represented by a 'filter' in frequency space:

$$G(\omega) = G_R(\omega)e^{iG_\phi(\omega)} \tag{11}$$

where the (real) $G_R(\omega)$ gives the amplitude response of the detector (ie the range of frequencies over which measurements can be made) and the (real) $G_\phi(\omega)$ gives systematic phase lags introduced both by the detector, and by inter-spacecraft timing errors. Timing and positional uncertainties (in the spacecraft $\mathbf{r}_j$) will be manifested in the determination of the convection velocity $\mathbf{v}$ from (9). Offsets measured by the instrument (for example, the spacecraft magnetic field which is seen at the magnetometer in addition to the local field) can also be shown not to invalidate this analysis provided that the offset is either small compared to the signal, or approximately constant over the timescale of the measurement (ie the frequency range corresponding to the offset is much lower than that required to resolve the signal structure).

The measured time series at the jth spacecraft will be a convolution between the fourier transform of $G(\omega) \Longleftrightarrow g(t)$ and the time series $S(\mathbf{r}_j, t)$, i.e.

$$F_M(\mathbf{r}_j, \omega) = \Psi(\mathbf{r}_j, \omega)G_{R,j}(\omega)e^{i(\theta(\mathbf{r}_j,\omega)+G_{\phi,j}(\omega))} \tag{12}$$

then at the $j+1th$ spacecraft

$$F_M(\mathbf{r}_{j+1}, \omega) = F(\mathbf{r}_{j+1}, \omega)G_{R,j+1}(\omega)e^{iG_{\phi,j+1}(\omega)} \tag{13}$$

and since $F(\mathbf{r}_{j+1}, \omega) = \Psi(\mathbf{r}_j, \omega)e^{i(\theta(\mathbf{r}_j,\omega)-\omega\delta t_j)}$ then

$$F_M(\mathbf{r}_{j+1}, \omega) = \Psi(\mathbf{r}_j, \omega)G_{R,j+1}(\omega)e^{i(\theta(\mathbf{r}_j,\omega)-\omega\delta t_j+G_{\phi,j+1}(\omega))} \tag{14}$$

then

$$R_{jM} = R_j(\frac{G_{R,j+1}}{G_{R,j}})^2 \tag{15}$$

and

$$\delta t_{jM} = \delta t_j + \frac{G_{\phi,j}(\omega) - G_{\phi,j+1}(\omega)}{\omega} \tag{16}$$

The functions $R_{jM}(\omega)$ and $\delta t_{jM}(\omega)$ will have a systematic dependence on ω which is dependent on the *differences* in instrument response between the spacecraft. This has significance in particular for the Cluster mission, where all spacecraft carry the same instrumentation, with similar $G(\omega)$. If the instrument response as a function of ω is not well known, calibration could in principle take place in-situ by event comparison. A systematic error in interspacecraft timing for a given spacecraft should only appear as a systematic, frequency independent factor in $G_{\phi,j}/\omega$ which then contributes to δt_j. Systematic timing uncertainties will then not affect conclusions as to whether the event is quasistatic, but will give a corresponding uncertainty in $\mathbf{v}$, as will positional uncertainties in the $\mathbf{r}_j$.

Random errors produce effects which will be more subtle. A random error in δt_{jM} will directly affect the determination of $\mathbf{v}$, as will random errors in spacecraft seperation. This will not yield an uncertainty in R_{jM} directly, but will, unlike systematic errors, lead to an uncertainty in measurements of the signal ω and hence for example the test for whether the event is quasistatic. This uncertainty in ω will become increasingly significant at higher frequencies, as we would expect, effective measurements can only be made of frequency components for which the corresponding period is longer than the random error in t. If the size of the random error in t is known, then the frequency range over which the test is valid can be determined.

The instrument response $G(\omega)$, and the random time errors discussed above define a 'window' (range of ω) within which measurements can be made, where ω is in the spacecraft rest frame. Given $\mathbf{v}$ this corresponds to an $\omega, \mathbf{k}$ window, since for any frequency component $\mathbf{v}(\omega) = \frac{\omega}{k}\hat{\mathbf{k}}$ in the spacecraft rest frame also.

Conclusions

We have examined how shift theorem yields a simple technique which can be used to order multispacecraft datasets. By first characterizing the properties of the frequency spectra of data from pairs of 'ideal' spacecraft in terms of two quantities R_j and δt_j, which effectively compare their amplitude and phase spectra respectively, it has been shown that:

1. The data, or specifically a subset over some ω range, is consistent with a single coherent quasistatic planar structure convecting over a spacecraft pair if both R_j and δt_j are independent of ω over that ω range.

2. If R_j is independent of ω over some ω range then over that range the structure or wavepacket envelope is coherent and planar, and may be time independent.

3. If δt_j is independent of ω over some ω range then over that range the wavepacket is non dispersive and may have a time independent envelope that defines a convecting structure.

4. As each spacecraft pair yields the component of velocity of the envelope (or structure) relative to the spacecraft, four non coplanar spacecraft are needed to determine the velocity of the structure in the spacecraft frame.

5. Since $\delta t_j(\omega)$ computed from the phase spectra from four non coplanar spacecraft determines the velocity of a given frequency component in the spacecraft frame, at a frequency which is also measured in the spacecraft frame, knowledge of the convection velocity of the plasma determines the dispersion relation, or alternatively, knowledge of the dispersion relation determines the convection velocity. In any frame, determination of the velocity of a given frequency component also determines the corresponding wavevector.

The quantities R_j and δt_j, and hence conclusions (1)-(3) are found to be robust against the effects of instrument response and systematic uncertainties in the spacecraft seperation and timing. The effect of random errors is more subtle, however these should only affect a clearly identifiable range of ω so that it should be possible to determine the range over which the test for a quasistatic convecting structure can be applied.

The shift theorem techniques require the time series measured at the spacecraft to be transformed into ω space; the comparison between simulation generated, and spacecraft data require the data to be examined in ω, $\mathbf{k}$ space. It is therefore suggested that routine analysis tools should include the ability to manipulate data in ω, $\mathbf{k}$, as well as $\mathbf{r}$, t space.

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