TOWARDS SYNTHESIS OF SOLAR WIND AND GEOMAGNETIC SCALING EXPONENTS: A FRACTIONAL LÉVY MOTION MODEL

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Abstract. Mandelbrot introduced the concept of fractals to describe the non-Euclidean shape of many aspects of the natural world. In the time series context, he proposed the use of fractional Brownian motion (fBm) to model non-negligible temporal persistence, the 'Joseph Effect'; and Lévy flights to quantify large discontinuities, the 'Noah Effect'. In space physics, both effects are manifested in the intermittency and long-range correlation which are by now well-established features of geomagnetic indices and their solar wind drivers. In order to capture and quantify the Noah and Joseph effects in one compact model, we propose the application of the 'bridging' fractional Lévy motion (fLm) to space physics. We perform an initial evaluation of some previous scaling results in this paradigm, and show how fLm can model the previously observed exponents. We suggest some new directions for the future.

Keywords: auroral indices, scaling, solar wind turbulence, fractional Lévy

1. Introduction

Ever since it became clear that Earth's magnetosphere is influenced by the sun, significant effort has been devoted in establishing the relationship between fluctuations in the energy delivered by the solar wind to the magnetosphere and variations in the magnetospheric response. A particularly important diagnostic for the response has been the family of geomagnetic indices, especially the auroral electrojet index *AE* (Davis and Sugiura, 1966). A common proxy for the solar wind input is the ε function (Perreault and Akasofu, 1978) which estimates the fraction of the solar-wind Poynting flux through the dayside magnetosphere.

One approach is to investigate causal relationships, and considerable sophistication has now been developed in this (e.g. Ukhorskiy *et al.*, 2004; March *et al.*, 2005 and references therein). However, even without examining causality, significant information can be obtained by examining the scaling behaviour of fluctuations. A first analysis of this, in the Fourier domain, was done by Tsurutani *et al.* (1990) using the power spectrum. Subsequent analyses have introduced other methods for detecting scale invariance (e.g. Takalo *et al.*, 1993; Freeman *et al.*, 2000a,b; Uritsky

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et al., 2001). Most recently, Hnat *et al.*, (2002a,b, 2003a,b, 2005) and Chapman *et al.* (2005) have studied the scaling collapse of the increments of time series.

A fundamental problem has been raised by the evidence for multifractality in some solar wind quantities (e.g. Hnat *et al.*, 2002a, and references therein) and the *AE* index (Consolini *et al.*, 1996). Multifractality is physically well motivated – at least for solar wind quantities – in that it arises naturally from the intermittency of multiplicative turbulent cascade models (Frisch, 1995). Multifractality would imply that the Hurst's 'roughness' exponent H is not constant but varies from scale to scale. This evidence for multifractality in the indices thus means that any comparison of pairs of scaling exponents derived from solar wind and geomagnetic indices may be problematic (Chang and Consolini, 2001; Watkins, 2002). Preliminary comparisons of solar wind and geomagnetic field measurements made using multiscaling measures (Vörös *et al.*, 1998) showed similarity at low orders after low pass filtering of the magnetospheric quantities. However, Hnat *et al.* (2002a,b, 2003a,b, 2005), in examining a range of solar wind quantities, have recently found some apparent simplifications. They see the intriguing result that although some quantities (notably v and B) *do not* show a simple scaling collapse, consistent with their well-known multifractality, others (such as B^2) *do*, i.e. they are, in this sense, effectively monofractal. Recently Hnat *et al.* (2003b) have extended the 1-year *AE*/*U*/*L* dataset studied by Hnat *et al.* (2002b) to 10 years used by Freeman *et al.* (2000a). They find that when such long auroral index datasets are examined, *AE* and ε do indeed have discernably different PDFs.

Such analyses are not easy to compare. Some used overlapping index and solar wind time series (Uritsky *et al.*, 2001), other did not (Freeman *et al.*, 2000a). Techniques which impose finite limits on the integral used to evaluate structure functions have also been explored (Chapman *et al.*, 2005, and references therein). The choice of solar wind measures and geomagnetic time series has also varied. It seems to us thus imperative to try to start to reconcile the various studies and understand why some show much greater similarity between the solar wind signal and indices than others. We also believe that the synthesis of observations will help towards a goal we have proposed elsewhere: The definition of models which are either (I) simple, statistical, 'strawman' models which may nonetheless capture some relevant fluctuation phenomenology, e.g. the fractional lognormal model sketched by Watkins (2002) or (II) more clearly statistical physics-based, e.g. the generalised Fokker–Planck model of Hnat *et al.* (2005) and Chapman *et al.* (2005).

By analogy with mathematical economics we may think of the Type I models as modelling the 'stylised facts' of the coupled solar wind magnetospheric system (Watkins, 2002). In this paper, we shall introduce one such model: fractional Lévy motion (Mandelbrot, 1995; Chechkin and Gonchar, 2000b), in order to see how well it can describe the solar wind ε function and the *AE* family of indices (*AE* itself, *AU* and *AL*). Preliminary comparison is made with some of the measurements listed above, and it is shown that the model provides a good quantitative explanation for the difference between two scaling exponents first noted in this context by

Hnat *et al*. (2002a) as well as a possible qualitative explanation for the multifractal behaviour seen by Hnat *et al*. (2003b). Where relevant, the effect of the truncation (finite variance) implicit in both a natural data series and a computer model are noted. Future directions are then sketched.

2. Datasets Used

The *AE* and ε data are a 1-year subset of those studied by Hnat *et al*. (2002b, 2003a). They correspond to the years 1978 and 1995, respectively. As with (Hnat *et al*., 2002b) solar wind data is taken only for periods when WIND is definitely in the solar wind, see (Freeman *et al*., 2000a) for details. We follow Hnat *et al*. (2002b) by firstly differencing the time series $X(t)$ of the indices AE, AU, AL and ε at intervals τ of 1, 2, 3 ... times the fundamental sampling period (1 min for the indices and 46 s for ε) to generate difference time series $\delta X(t, \tau) = X(t + \tau) - X(t)$. For further details of the dataset and preprocessing techniques (see Hnat *et al*., 2002b, and references therein).

3. Motivation for and Testing of a Fractional Lévy Motion Model

3.1. Fractional Lévy Motion as a Bridge Between Lévy Flights and Fractional Brownian Motion

As noted by Mandelbrot (1995):

> The ‘normal’ model of natural fluctuations is the Wiener Brownian motion process (WBm). By this standard, however, many natural fluctuations exhibit clear-cut ‘anomalies’ which may be due to large discontinuities (‘Noah Effect’) and/or non-negligible global statistical dependence (‘Joseph Effect’). [Mandelbrot’s book ‘The Fractal Geometry of Nature’] ... shows that one can model various instances of the Noah effect by the classical process of [standard Lévy motion] (sLm), and various instances of the Joseph effect by the process of [fractional Brownian motion] (fBm).

Takalo *et al*. (1993) were the first to use fBm as a model of the auroral indices, but it subsequently could not describe the highly non-Gaussian leptokurtic distributions seen in differenced solar wind and geomagnetic index quantities. This can for example be seen in Figure 7 of Chapman *et al*. (2005) where the pdf of differences δX of *AE* is contrasted with the Gaussian pdf of an fBm with equal Hurst exponent H. Similarly, we are aware of only a small number (Consolini *et al*., 1997; Kabin and Papitashvili, 1998; Hnat *et al*., 2002a; Bruno *et al*., 2004) of discussions on the use of truncated sLm as a model for in-situ solar wind, magnetotail or ground-based magnetometer time series. One reason why sLm has not seen wider use here is because it cannot reproduce the correlated increments seen for both these types of data and also because it models superdiffusive ($H > 0.5$) rather than the observed subdiffusive ($H < 0.5$) behaviour. The term 'truncated Lévy flight' usually indicates standard Lévy motion with a finite variance introduced deliberately by means of a finite range cutoff (c.f. the discussion in Section 8.4 of Mantegna and Stanley(2000)); however, any finite (and thus finite-variance) series of sLm must also be naturally truncated, but in an uncontrolled fashion (Nakao, 2000).

Mandelbrot (1995) went on to note that:

> sLm and fBm, however, are far from exhausting the anomalies found in nature ... many phenomena exhibit *both* the Noah and Joseph effects and fail to be represented by either sLm or fBm .... One obvious bridge, fractional Lévy motion, is interesting mathematically, but has found no concrete use'.

Since those words were written, fLm has found applications, notably in geophysics (Painter and Patterson, 1994) and telecommunications-network modelling (Laskin *et al.*, 2002). We here apply it to essentially the same need; to compactly describe and unify the 'stylised facts' of the well-demonstrated Noah and Joseph effects in space plasma physics time series (Watkins, 2002).

3.2. Mathematical Definition of Fractional Lévy Motion

Fractional Lévy motion can be defined using a Riemann–Liouville fractional integral generalising the better-known expression for fractional Brownian motion (Voss, 1985). We here adapt the notation of Equation (5) of Laskin *et al.* (2002), which defines a process $W_{\mu,\beta}$:

$$W_{\mu,\beta}(t) = \frac{1}{\Gamma(\beta/2)} \int_0^t (t-\tau)^{(\beta/2-1)}\, dW_\mu(\tau) \tag{1}$$

Equation (1) can be unpacked as a summation of Lévy stable increments $dW_\mu(\tau)$ each weighted by a response function $(t-\tau)^{(\beta/2-1)}$. The μ parameter describes the power law tail of the pdf of dW which falls off as $P(x) \sim x^{-(1+\mu)}\mu = 2$ is the special, Gaussian, case corresponding to fBm. β is the parameter which controls long-range dependence. It is well-known to be related to the power spectral density $S(f) \sim f^{-\beta}$ for fractal processes with finite variance (Voss, 1985), but can also be rigorously defined through fractional differentiation in other cases (Chechkin and Gonchar, 2000b).

With $\mu = 2$ and taking in addition $\beta = 2$ the response function becomes unity giving an uncorrelated random Gaussian walk (WBm). Keeping $\beta = 2$, but allowing μ to vary in the range 0–2 describes sLm. fLm is thus in general a process with μ, β allowed to vary in the range $[0 < \mu \leq 2, 1 \leq \beta \leq 3]$ and so forms a bridge between the $\beta = 2$ sLm and $\mu = 2$ fBm 'axes'. fLm thus by construction exhibits both the sources of anomalous diffusion identified by Mandelbrot above.

These limits have corresponding simplified fractional kinetic equations (FKE) for the pdf $P(W)$, see Section 5.2 of (Zaslavsky, 2002). Putting $W = W_{\mu,\beta'}(x,t)$ with $\beta' = \beta/2$, WBm is given by the diffusion equation $\partial_t^1 P(W_{2,1}) = \partial_x^2(\mathcal{A}P(W_{2,1}))$; fBm by $\partial_t^{\beta'} P(W_{2,\beta'}) = \partial_x^2(\mathcal{A}P(W_{2,\beta'}))$; and sLm by $\partial_t^1 P(W_{\mu,2}) = \partial_x^{\mu}(\mathcal{A}P(W_{\mu,2}))$. fLm should thus correspond to Equation (132) of Zaslavsky (2002):

$$\frac{\partial^{\beta'}}{\partial t^{\beta'}} P(W_{\mu,\beta'}) = \frac{\partial^{\mu}}{\partial |x|^{\mu}}(\mathcal{A}P(W_{\mu,\beta'})) \tag{2}$$

All cases have a fixed diffusion constant $\mathcal{A}$. Future work is required to establish if this simplified form of Equation (127) of Zaslavsky (2002), the full FKE, can map on to the Fokker–Planck equation of (Hnat *et al.*, 2005) or whether the full equation, including fractional drift and diffusion terms, is needed. After initial submission of this paper we also became aware of the relevance of the work of Milovanov and Zelenyi (2001) to the interpretation of fLm as an FKE; see in particular their Equation (3).

3.3. Self-Similarity, the Hurst Exponent and Peak Scaling

We now follow Laskin *et al.* (2002) to show that $W_{\mu,\beta}$ is indeed an H-self-similar process. To see this we first put $\tau = cs$ in (1). We then use the fact that the increments $dW_\mu(cs)$ are defined to be $1/\mu$ self-similar, i.e. are equal in distribution ($\stackrel{d}{=}$) to $c^{1/\mu} dW_\mu(s)$. Then

$$W_{\mu,\beta}(ct) \stackrel{d}{=} c^H W_{\mu,\beta}(t) \tag{3}$$

with a self-similarity parameter H given by

$$H = \frac{\beta}{2} + \frac{1}{\mu} - 1 = \left[\frac{\beta}{2} - \frac{1}{2}\right] + \left[\frac{1}{\mu}\right] - \frac{1}{2} \tag{4}$$

more usually known as the Hurst exponent. Note that we would not necessarily expect this equation to hold for more general fractal processes. In the fBm case $\mu = 2$ and for that case only we recover the well-known expression that $\beta = 2H+1$. In the sLm case $\beta = 2$ and we find $H = 1/\mu$. Recently Mandelbrot (2002) has proposed writing

$$H = J + L - \frac{1}{2} \tag{5}$$

where he defines a Joseph (long-range dependence) exponent $J (= \beta/2 - 1/2)$ and a Noah (heavy tail) exponent $L (= 1/\mu)$.

The first property that needs to be shown in a time series for fLm to be a candidate model is thus H-self similarity. This can be tested by a number of methods. The first is scaling collapse, which was shown for the datasets in our paper by Hnat *et al.* (2002b, 2003a).

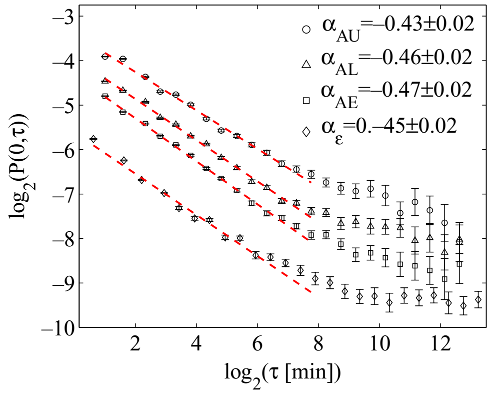

Figure 1. Estimation of Hurst exponent H via scaling of peaks $P(0)$ of pdfs of differenced time series $X(t+\tau) - X(t)$ as a function of differencing interval τ. Plots are for (i) auroral indices 1978: $X = AU(\circ)$, $X = AL(\Delta)$ and AE (box) and (ii) solar wind $\varepsilon(\Diamond)$ for 1995. Plots have been offset vertically for clarity.

An fLm model also implies that the pdf of returns i.e $P(\delta X = 0, \tau)$ will scale with τ with exponent also equal to H. This was shown in Figure 2 of (Hnat *et al.*, 2002b). For convenience in Figure 1, we show a comparison of the scaling regions of the 1-year signals taken from the natural time series AE, AU, AL and ε. All are seen to scale up to approximately 2^6 min ($\approx$1 h). Caution is however necessary because in a natural dataset the moments $|\delta X|^q = |X(t+\tau) - X(t)|^q$ would be expected to be dominated in the small τ limit by the scaling of the measurement noise on the differences δX rather than that of the physical variables themselves (Hnat *et al.*, 2005).

Interestingly, although the exponent needed to rescale the pdfs $P(\delta X, \tau)$ of difference δX taken from fLm is the 'full' extended $H = H(\mu, \beta)$ defined in Equation (4), the difference pdfs have the same shape they would have for an sLm with the same μ value. This is analogous to the way in which fBm retains the same Gaussian distribution as the steps from which it is composed, despite their statistical dependence, and is why fLm is also known as 'linear fractional stable motion'.

3.4. Structure Functions S_q and Their Scaling Exponents $\zeta(q)$: H as $\zeta(1)$, While the pdf of Returns Gives $\zeta(-1) \equiv -H$

One may extend the idea of self similarity expressed by H to the generalised qth order structure functions (Frisch, 1995):

$$S_q = \langle |x(t+\tau) - x(t)|^q \rangle \tag{6}$$

where q need not be an integer. If a given S_q is empirically found to be a power law we can then define an exponent $\zeta(q)$ from $S_q \sim \tau^{\zeta(q)}$.

For a stable self-similar process where all moments are finite $\mu = 2$, i.e. WBm ($H = 0.5$) or fBm ($0 \leq H \leq 1$), the exponents of the structure functions $\zeta(q)$ follow $\zeta(q) = qH$, as we have checked by simulating an fBm using the same fLm algorithm as used for the figures, in the $\mu = 2$ limit. By definition, we then have $\zeta(1) = H$. Additionally, in these Gaussian ($\mu = 2$) cases $\zeta(2) = 2H$, which from Equation (4) then implies $2H = \beta - 1$.

The exponent derived from the pdf of returns can be shown to be equivalent to $\zeta(-1)$ (Miriam Forman, private communication, 2002) so for self-similar processes (see also our Figure 5) the plot of $\zeta(q)$ versus q is antisymmetric about $q = 0$ at least insofar as $\zeta(-1) = -H = -\zeta(1)$.

3.5. Second-Order Moment and J: Pseudo-Gaussian Behaviour of Truncated Lévy Time Series

Because of the relation $\zeta(q) = qH$ for WBm and fBm, a complementary estimate of the self-similarity parameter H can, for these cases, be obtained from the well-known growth of the standard deviation $\sigma(\tau)$ of the difference time series $\delta X(\tau)$ with differencing interval τ. Indeed the growth of a measured σ as $\tau^{1/2}$ in the case of WBm defines diffusive behaviour. σ is the square root of variance and thus scales like S_2, i.e. as $\tau^{(\beta-1)/2}$, i.e. it follows Mandelbrot's (2002) Joseph exponent J (which from (4) will be identical to H in the Gaussian WBm or fBm cases).

In the case of Lévy motion, however, whether ordinary or fractional, the qth order moments S_q (where $q > \mu$) taken from a set of N data points are theoretically infinite as $N \to \infty$ in contrast to the convergence seen for Gaussians. It is thus not *a priori* obvious how the variance of a truncated, finite-N, time series would be expected to scale. This is significant because any simulation that we perform of fractional Lévy motion is effectively one of truncated Lévy motion; while a natural time series will also have a finite variance in practice. The possible relevance of this question to data is clearly illustrated by our Figure 2, (see also Table 1 of

TABLE I

Measured values of H (from Figure 1) and J (from Figure 2) for natural time series, and μ value predicted from Equation (4) on the assumption of naturally truncated fLm

Variable	Measured H	Measured J	Inferred β	Inferred L	Predicted μ
AE	0.47	0.45	1.90	0.52	1.92
AU	0.43	0.43	1.86	0.5	2
AL	0.46	0.43	1.86	0.53	1.88
ε	0.45	0.29	1.58	0.66	1.51

Note. All measured values are ± 0.02 except J for ε which is ± 0.03.

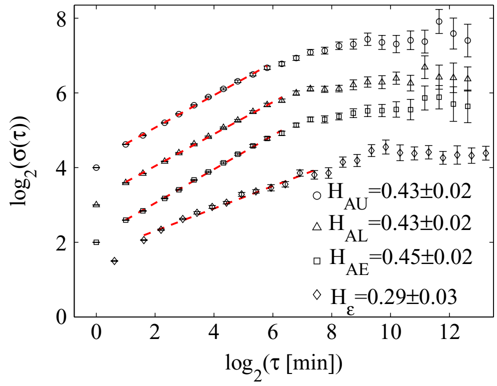

Figure 2. Estimation of exponent J for scaling of the standard deviation σ of the differenced series vs. τ for the same quantities as Figure 1. Notation as in Figure 1.

(Hnat *et al*., 2002b)) in which σ for the solar wind variable ε is seen to scale with an exponent of 0.29 as opposed to the values around 0.43–0.45 seen for the three geomagnetic-index quantities. Rather than scaling with H, σ still appears to be showing pseudo-Gaussian behaviour, i.e. following J, in that $\beta = 1.56$ for this time series (estimated by wavelet methods) giving $J = (1.56 - 1)/2 = 0.28$.

The apparent disadvantage of the loss of a second, independent, estimate of H seems to be compensated for by the possibility that we can use the growth of σ to measure β, i.e. we can effectively use it as a measurement of J. On the assumption that a naturally truncated fLm describes our data we can build a table (Table I) of the measured β and H values and then predict μ, using Equation (4).

On inspecting Table I, the first point is that the values of H and J are so close in the case of *AU* that if we assume they are exact the predicted μ becomes 2, eliminating fLm as a model for *AU*. The H is sub-diffusive, so fBm would remain a possible candidate model; however, the observed (Hnat *et al*., 2003a) difference pdfs $P(\delta X)$ for *AU* are non-Gaussian, eliminating fBm. The error bars quoted in Table I suggest these conclusions may be too harsh. fLm would, however, seem more suitable as a model for *AE, AL* and ε.

As a test we may also consider the values of H and J for solar wind B^2 obtained by Hnat *et al*. (2002a). Their Figure 3 gives $H = 0.42$ in our parlance, while they report a scaling exponent for σ of 0.28 (i.e. J). Inserting this into Equation (4) predicts $\mu = 1.56$, which is equivalent to the $1/\alpha$ of their Equation (3) (see also their Figure 4) which they find to be $1/0.66 = 1.5$, encouragingly good agreement.

3.6. Fractional Lévy Simulation: Comparison with First- and Second-Order Measures

We can then now simulate fLm using parameters drawn from natural data to see if the inferences we have drawn above are indeed consistent, and to qualify fLm as at

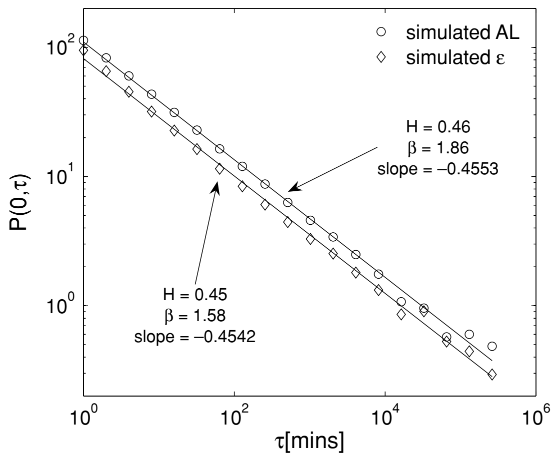

Figure 3. Estimation of H via scaling of peaks $P(0)$ of pdfs of differenced model fractional Lévy motion time series $X(t+\tau) - X(t)$ as a function of differencing interval τ. Plots are for (i) a synthetic $AL(\circ)$ time series and (ii) a series of synthetic solar wind $X = \varepsilon(\Diamond)$. Plots have been offset vertically for clarity.

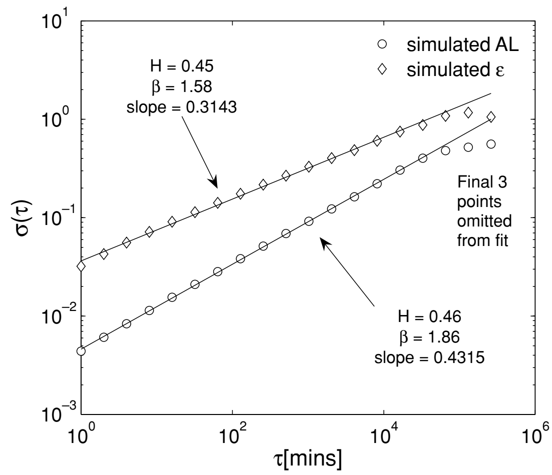

Figure 4. Estimation of J by scaling of the standard deviation σ of the differenced simulated (and thus truncated) series vs. τ for the same quantities as Figure 2. Notation as in Figure 2.

least a possible proxy for these time series. We use the published algorithm of (Wu *et al.*, 2004). This has the advantage of being linked more closely to the definition of fLm from Equation (1) than the (faster) approach of replacing (Chechkin and Gonchar, 2000b) a Gaussian random number generator by a Lévy generator in otherwise standard Fourier filter methods (Voss, 1985). A comparison of these two approaches will be reported in a future paper.

We show simulation results for synthetic *AL* and ε time series. These were specified by the ordered pairs (β, μ) of (1.86, 1.88) and (1.58, 1.51), respectively. The $P(\delta X = 0, \tau)$ scaling for both series (Figure 3) is seen to follow H as we expect, so both model series have very similar measured H values, as we also saw in their natural counterparts (Figure 1). Conversely, for finite samples of fLm, however, modelling *AL* and ε we see from Figure 4 that rather than following $\tau^{1/\mu}(= \tau^L)$, the σ measured on the difference time series δX still grows as $\tau^{(\beta-1)/2}(= \tau^J)$, i.e. it does, as postulated in subsection 3.5, measure J rather than L.

This effect requires some discussion. It seems to be a further manifestation of the 'pseudo-Gaussian' behaviour of truncated standard Lévy motion (Chechkin and Gonchar, 2000a), and known (Nakao, 2000) to be responsible for the result $\zeta(2) = 1$ in that case (see also Figure 5). Our simulations have clearly demonstrated that it generalises to the long-range dependent fLm case, i.e. that in general for fLm $\zeta(2)/2 = J = (\beta - 1)/2$. This conclusion is most clearly supported by Figure 5 where the $\zeta(2)$ value can be read off as following this relation over the range $\beta = 1.5$ to 2.5. The agreement is poorer at smaller β values tested. We currently think this reflects known difficulties with accurately simulating strongly anti-correlated fLm (Chechkin and Gonchar, 2000b). The effect has previously been remarked on in the truncated standard Lévy paradigm; for example the S&P 500 financial time series, depicted by (Mantegna and Stanley, 2000) where $\beta = 2$ (their Figure 11.4.a) so σ grows as $\tau^{1/2}$ (their Figure 11.3a), in contrast to an H value from peak scaling of 0.71 (their Figure 9.3).

In the multifractal modelling community, the power spectrum has long been seen as only just one of several ways of measuring $\zeta(2)$. For this reason a difference in the value of $\zeta(1) \neq \zeta(2)/2$ has sometimes been claimed as direct evidence of the inapplicability of *any* additive model and thus the immediate need for a multiplicative model (Schertzer and Lovejoy, 1987).

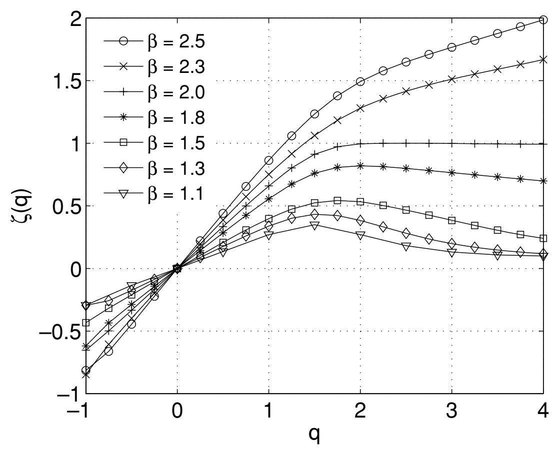

Figure 5. Zeta plots for simulated (and thus truncated) fLm with μ fixed at 1.5 and β ranging from 1.1 to 2.5. The relation $\zeta(2) = \beta - 1$ is seen to be well satisfied for $\beta \geq 1.5$.

Conversely our result would seem to suggest that any truncated stable additive model other than the fBm/WBm limiting cases is likely to show $\zeta(1) \neq \zeta(2)/2$, and $\zeta(2)/2 = J = (\beta - 1)/2$ without the need for a multiplicative model. This may be understood as being because truncated Lévy motion, whether standard or fractional, behaves as a bifractal (Nakao, 2000). There may be natural time series where additive fLm is actually the most natural model, or at least an economical and easily specified one.

3.7. ζ Plots and the Multifractality of Truncated Lévy Motions

At this point it may be objected that we have not tested any predictions of the fLm model against the behaviour of natural time series other than those properties used to specify it. Our first additional check is thus to examine the multi-affine behaviour seen in the data and the model using the 'ζ plots' defined in Section 3.4. Such a plot, showing scaling exponent $\zeta(q)$ versus moment q is shown for the data in Figure 6. Interestingly, *AU* most resembles a 'classic' multifractal, in that the points $\zeta(q)$ lie on a curve rather than a straight or broken line (Frisch, 1995). However *AE*, or at least *AL*, have ζ which arguably flattens out near 1 for higher moments. ε intriguingly even seems to *fall* as *m* increases. This behaviour is qualitatively similar to that seen for our simulated *AL* and ε time series, whose $\zeta(q)$ plots are superposed on the figure. In particular a change in the range of τ over which the simulated *AL* structure functions are taken to be power laws is enough

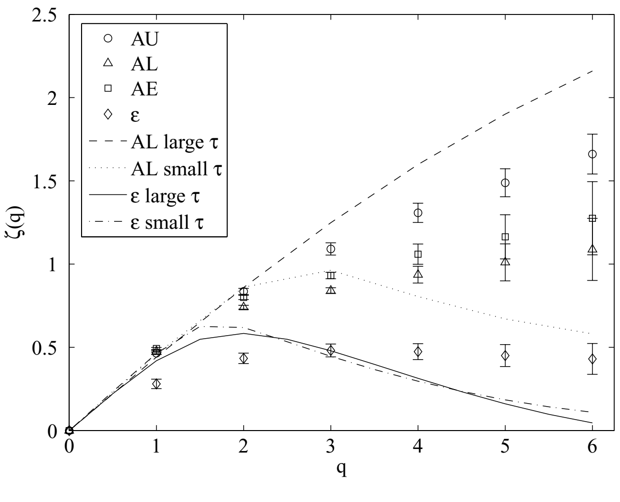

Figure 6. $\zeta(q)$ vs. q plots for three auroral indices (AU, AL and AE) during 1978, and solar wind ε from WIND for 1985. Overlaid are $\zeta(q)$ plots for our simulated *AL*, and simulated ε, where in both cases 'large τ' and 'small τ' indicate the range of estimates depending on the part of the structure function taken to be scaling.

to encompass the observed range of ζ plots for ε. More detailed comparison is at present prevented due to the difficulty of obtaining accurate values of S_q for high moments–an issue also afflicting analysis of real data.

4. Conclusions

A significant body of data and models now exists for the problem of solar wind and magnetic-index scaling. We have here suggested a complementary approach, motivated in particular by the need to (i) reconcile differing estimates of scaling exponents (in hindsight the Joseph and Hurst exponents J and H); (ii) model subdiffusive behaviour ($H < 0.5$); and (iii) model long-ranged correlation ($\beta \neq 2$). We proposed the use of a simple and economical model: fractional Lévy motion, to describe the scaling of the above quantities. Initial consistency checks with respect to the distribution of returns and the scaling of standard deviation support the use of fLm, and the multi-affine 'zeta plots' are more qualitatively similar. Importantly we find that the degree of similarity between model solar wind ε and the model *AL* index does indeed depend on the moment order at which comparison is made, but that this does not, however, require a multiplicative process to explain it. The difference can, rather, be understood as coming from the bifractality of a truncated fractional Lévy motion. This explains why some measures such as H from the distribution of returns or pdf rescaling are much closer to each other than, for example, the σ-based exponent (which we found to measure J, not H).

The present paper has been mainly concerned with the modelling of measured quantities rather than the extent to which they are artificial. For geomagnetic indices and other constructed quantities like ε, however, the extent to which scaling behaviour could be an artefact of the construction method is an important issue. We are aware of some progress in studying this problem (e.g. Edwards, 2001; Weigel and Baker, 2003), more will be needed. Further work is also underway to test the predictions of the fLm model against other scaling studies such as the cited burst lifetime and spreading exponent investigations.

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