

# Bifurcation to Chaos in Charged Particle Orbits in a Magnetic Reversal With Shear Field

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**Abstract**—Regular and stochastic behavior in single particle orbits in static magnetic reversals have wide application in laboratory and physical plasmas. In a simple magnetic reversal, the system has three degrees of freedom but only two global (exact) constants of the motion; the system is nonintegrable and the particle motion can, under certain conditions, exhibit chaotic behavior. Here, we consider the dynamics when a constant shear field is added. In this case, the form of the potential changes from quadratic to velocity dependent. We use numerically integrated trajectories to show that the effect of the shear field is to break the symmetry of the system so that the topology of the invariant tori of regular orbits is changed. In this case, invariant tori take the form of nested Moebius strips in the presence of the shear field. The route to chaos is via bifurcation (period doubling) of the Moebius strip tori.

**Index Terms**—Chaos, current sheets, single particle dynamics, virtual reality.

CONSIDERABLE literature exists concerning single-particle dynamics in static magnetic reversals where a shear field is absent (e.g., [1] and references therein). Magnetic field models which have no shear component have the advantage that a Hamiltonian with a quadratic potential exists to describe the particle dynamics, hence their attraction as an approach to astrophysical problems of interest. However, observations suggest that a shear field is present in the Earth's geotail and bow shock and is also important in magnetic reversals in solar flares. Here, we illustrate how the underlying dynamics in a magnetic reversal with shear component differs from that with zero shear. We consider single particle motion in a simple magnetic reversal with  $z$  dependence and shear, of the form  $B = B_0(f(z), b_2, b_1)$ . This simple model has application in astrophysical plasmas as a convective electric field can be included with a de Hoffman Teller frame transformation (here, we will work in the  $E = 0$  frame). The Hamiltonian is a constant and takes the form [2]

$$H = \frac{1}{2}(p_x + \alpha z)^2 + \frac{1}{2}(p_z + \beta x)^2 + \frac{1}{2}(F(z) - b_1 x)^2 \quad (1)$$

with canonical  $y$  momentum  $p_y$  as a constant of the motion. A further constant exists which is related to  $p_x$ ,  $y$  and  $z$  through  $I = p_x - b_1 y + \beta z$ . The Poisson bracket  $[p_y, I] = -2b_1$ , thus the two constants are in involution only when  $b_1 = 0$ , in which case  $I$  is the third global invariant and the equations of motion

can be solved exactly. However, islands of regular orbits can exist even in the nonintegrable system if a local constant of the motion exists to decouple the  $x$  and  $z$  motion; this was identified by [3] using a multiple timescale perturbation technique.

The full set of equations can only be solved numerically. The original six-dimensional (6-D) system, through the existence of its constants of the motion, requires three dimensions to characterize its dynamics geometrically. We, therefore, rely on 1) the existence of Kolomogorov, Arnold, and Moser (KAM) surfaces and 2) visualization in three-dimensions (3-D) to examine the dynamics. In the numerical treatment of the system a simple model for the field reversal was used:  $f(z) = \tan h(z)$ , with numerical integration performed using an adaptive time step, adaptive order Adams method [4]. The trajectories shown were initiated with  $y = z = 0$ . Phase space could then be explored by varying  $x$  to obtain different trajectories in phase space for a given field configuration. The initial velocity was chosen so all orbits have  $p_y = 0$ .

The principal effect of nonzero shear field  $b_2$  is that for certain values of  $b_2$  and  $H$  regular orbits are absent. Here, we characterize the transition between these regions of global chaos and regions of regular orbits by the topology of the corresponding 3-D trajectories. In the limiting case of infinite  $b_2$ , the system is integrable; at large  $b_2$ , phase space is dominated by simple KAM surfaces. As  $b_2$  is reduced, the KAM surfaces and central orbit distort. This is shown in Fig. 1(a), which is a zoom of the surface of section (SOS) in the region of the central orbit at the center of the nested KAM tori for  $b_2 \sim 0.188$ . In Fig. 1(b), we see that as  $b_2$  is decreased by one part in  $10^3$  the innermost KAM tori bifurcate. Only the first bifurcation can be resolved adequately on SOS plots. In 3-D, we can isolate the central periodic orbit and numerically identify three successive bifurcations [2].

Period 2 orbits exist that are the centers of the nested KAM surfaces shown. When moving slightly away from these centers, the orbits will be confined to KAM surfaces (tori in 3-D). The trajectory shown in Fig. 1(c) corresponds the KAM surface in Fig. 1(a) intersecting  $v_x = 0$  at  $x = 0.556$ . The trajectory defines a surface in the shape of a Moebius strip. On closer inspection of the strip it is found that the surface is a twisted and squashed torus. In Fig. 1(d), the bifurcated Moebius strip corresponding to the KAM surfaces shown in Fig. 1(b) intersecting  $v_x = 0$  at  $x = 0.589$  is shown.

Interactive use of 3-D trajectories together with SOS plots played an instrumental role in understanding the properties of the studied system. Key topological features of the system occur in extremely small regions of phase space. It was only through the interactive study of individual 3-D orbits that we were able to first identify these regions and then adequately populate the

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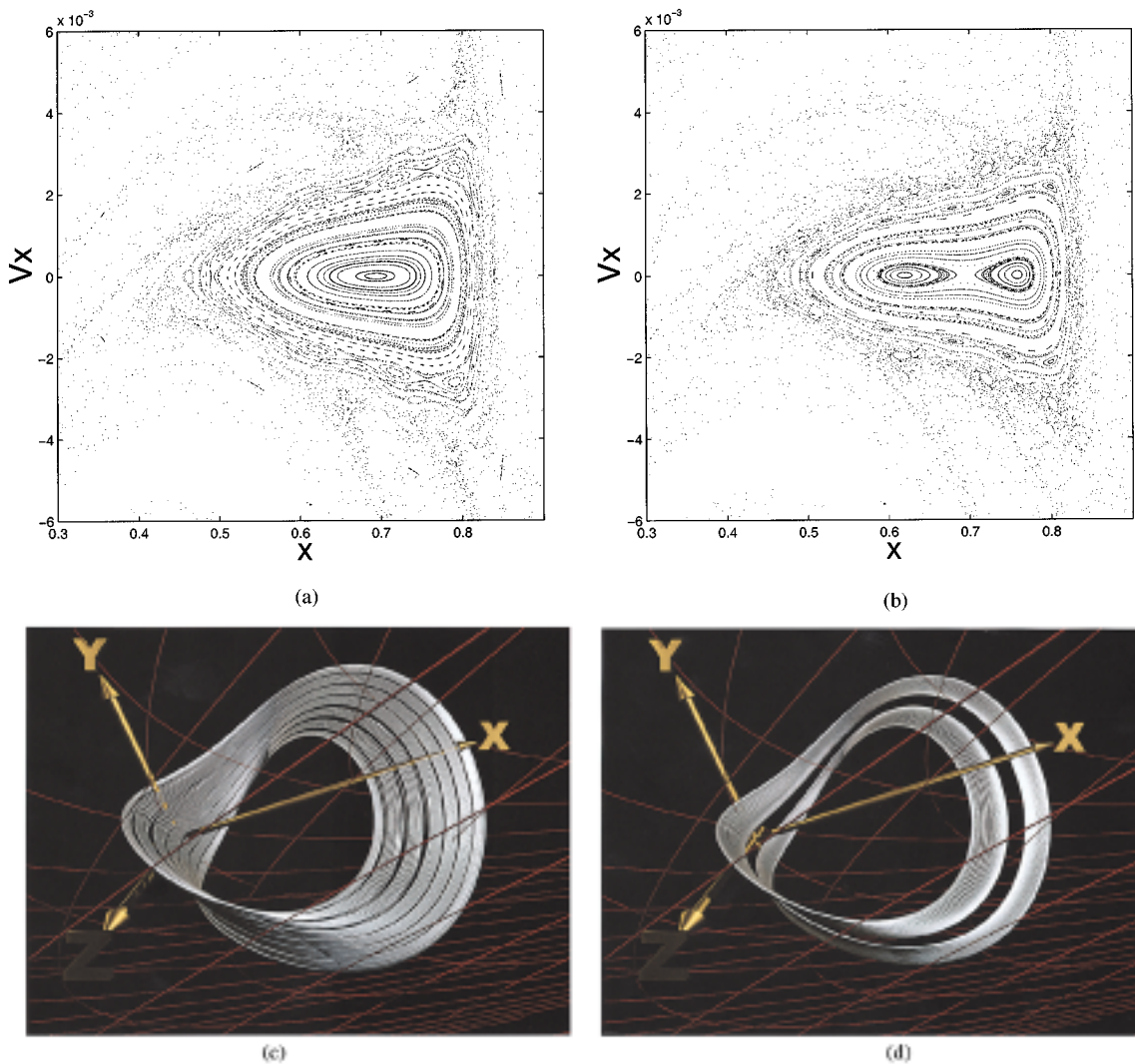


Fig. 1. Bifurcation of a single charged particle orbit. In (a), a Poincaré surface of section (SOS) plot for  $b_2 = 0.189$  is shown. The SOS is constructed from many trajectories by marking with a dot the coordinates where the trajectories cross the plane  $z = 0$ . Regular trajectories form closed loops (KAM surfaces) on such a plot. In (b), the corresponding SOS plot for  $b_2 = 0.188$  is shown and shows a bifurcation of KAM surfaces which marks the beginning of onset of chaos. In (c), the regular orbit in  $x$ ,  $y$ , and  $z$  space corresponding to one of the KAM surfaces in (a) appears as a Moebius strip. In the SOS, this region in phase space is closely packed with simple torii and island chains, these correspond to 3-D surfaces that are single, and multifold sets of closed Moebius strips respectively. The strip shows the symmetry breaking handedness of the system. In (d), the trajectory corresponding to the bifurcated KAM surfaces in (b) is shown. Field lines in the reversal are shown in red.

local region in phase space to produce the corresponding SOS plots. A package for computational steering of the 3-D trajectories based on a particle integration code [4] was developed. The package was instrumented with control and graphics routines to enable the computational steering of the field reversal model from a visual environment. The magnetic field lines in Fig. 1(c) and (d) were used to relate the positions of the orbit to the reversal coordinates, and thus facilitated a much needed intuitive navigation scheme during the interactive exploration.

The system was implemented on desktop systems as well as semi-immersive virtual reality environments such as the ImmersaDesk. The graphics modules were developed in OpenGL and the CAVE library was used for the semi-immersive virtual reality implementation. User interfaces were developed using Tcl/Tk and in the virtual reality implementation navigation and interaction using tracked devices are supported. During exploration of parameter space, it is possible to save “snapshots” of

the model in PovRay format. The images of particle trajectories shown in Fig. 1(c) and (d) are examples of such images rendered using PovRay. These images give a higher level of detail, and thus complements the interactive exploration by revealing additional features not resolved in real-time graphics.

## REFERENCES

- [1] J. Chen, “Nonlinear dynamics of charged particles in the magnetotail,” *J. Geophys. Res.*, vol. 97, pp. 15 011–15 050, 1992.
- [2] A. Ynnerman, S. C. Chapman, M. Tsalas, and G. Rowlands, “Identification of symmetry breaking and a bifurcation sequence to chaos in single particle dynamics in magnetic reversals,” *Physica D*, vol. 139, pp. 217–230, 2000.
- [3] S. C. Chapman and G. Rowlands, “Are particles detrapped by a constant  $B_y$  in static magnetic reversals?,” *J. Geophys. Res.*, vol. 103, pp. 4597–4603, 1998.
- [4] L. F. Shampine and M. K. Gordon, *Computer Solution of Ordinary Differential Equations: The Initial Value Problem*. New York: Freeman, 1990.