Problem Sheet

Problem 1 *Estimate in km the distance on the Sun corresponding to one arc-second when the Earth is in perihelion and is in 147.5 million km from the Sun and when it is in aphelion, in 152.6 million km from the Sun.*

Solution Consider a triangle with a vertex formed by the Earth (A) and the side ($|BC|$), opposite to the vertex and perpendicular to the line connecting the Earth and the Sun. The size of the side is to be determined. The angle at the vertex representing the Earth is α. The height $|AD|$ of the triangle is R_{AE} - the distance between the Earth and the Sun. The triangle ABC is isosceles. The height meets the side $|BC|$ at the point D and, consequently, divides the triangle ABC to two identical triangles ABD and ACD.

Consider the triangle ABD. The angle BAD is $\alpha/2$. Consequently, the side BD is expressed through $|AD| = R_{AE}$ and α as

$$|BD| = R_{AE} \tan(\alpha/2),$$

and $|BC| = 2|BD|$. Substituting the figures,

$$|BC| = 2|BD| = 2R_{AE} \tan(\alpha/2).$$

The angle α is one arcsec or, in radiants, $\alpha = \pi/180/60/60$.

For $R_{AE} = 147.5\times10^6$ km, $BC = 715.1$ km, and for $R_{AE} = 152.6\times10^6$ km, $BC = 739.8$ km. The average value of the distance corresponding to one arcsec is $(739.8 + 715.1)/2 = 727.5$ km.

Problem 2 *Observations show that the solar constant increases by 0.1%. Linking the solar constant with the Sun's absolute luminosity, and assuming the Sun emits as a black body, and that the solar radius does not change and the distance between the Earth and the Sun remains the same, determine the relative increase in the temperature of the solar surface.*

Solution Assuming that the solar radiation is spatially isotropic, we estimate the Sun's absolute luminosity as $L_\odot = f_\odot S$, where $f_\odot$ is the solar constant and S is the area of a sphere of radius 1 AU.

In the first measurement we have $f_\odot = f_{\odot 1}$ and $L_{\odot 1} = f_{\odot 1} S$,

and in the second $f_\odot = 1.001 f_{\odot 1}$ and $L_{\odot 2} = 1.001 f_{\odot 1} S$.

On the other hand, according to the Stefan–Boltzmann law, the Sun's absolute luminosity is determined by the temperature of the solar surface, $T_{\odot 1}$ and $T_{\odot 2}$, in the first and the second measurements, respectively.

Thus, we obtain

$$\begin{aligned} f_{\odot 1} S &= 4\pi\sigma R_\odot^2 T_{\odot 1}^4 \\ 1.001 f_{\odot 1} S &= 4\pi\sigma R_\odot^2 T_{\odot 2}^4. \end{aligned}$$

and

$$\frac{T_{\odot 2}}{T_{\odot 1}} = (1.001)^{1/4}.$$

Problem 3 *Show that the adiabatic equation can be expressed as*

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{V}. \tag{1.1}$$

How would the expression change in the incompressible case?

Solution Starting with the energy equation

$$\frac{\rho^\gamma}{\gamma - 1} \frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = -\mathcal{L}, \tag{1.2}$$

we observe that in the adiabatic case $\mathcal{L} = 0$ it can be rewritten as

$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0. \tag{1.3}$$

Then, we rewrite the left hand side of this equation as

$$\frac{1}{\rho^\gamma} \frac{dp}{dt} - \gamma p \rho^{-\gamma-1} \frac{d\rho}{dt} = 0, \tag{1.4}$$

or

$$\frac{dp}{dt} - \frac{\gamma p}{\rho} \frac{d\rho}{dt} = 0. \tag{1.5}$$

as ρ is not zero.

Consequently, we obtain

$$\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p - \frac{\gamma p}{\rho} \left[\frac{\partial \rho}{\partial t} + (\mathbf{V} \cdot \nabla) \rho \right] = 0. \tag{1.6}$$

Then, we use the continuity equation to get

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V}), \tag{1.7}$$

and obtain

$$\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p - \frac{\gamma p}{\rho} \left[-\nabla \cdot (\rho \mathbf{V}) + (\mathbf{V} \cdot \nabla) \rho \right] = 0, \tag{1.8}$$

or

$$\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p + \frac{\gamma p}{\rho} \left[\rho \nabla \cdot \mathbf{V} + \nabla \rho \cdot \mathbf{V} - \mathbf{V} \cdot \nabla \rho \right] = 0, \tag{1.9}$$

and, cancelling two last terms in the brackets, we arrive at

$$\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p + \gamma p \nabla \cdot \mathbf{V} = 0. \tag{1.10}$$

In the incompressible case $\nabla \cdot \mathbf{V} = 0$ and the adiabatic equation becomes

$$\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p = 0. \tag{1.11}$$

Problem 4 *Show that in the case of constant pressure, the energy equation*

$$\frac{\rho^\gamma}{\gamma - 1} \frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = -\mathcal{L}, \tag{1.12}$$

can be rewritten as

$$\rho c_p \frac{\mathrm{d} T}{\mathrm{d t}} = -\mathcal{L}, \tag{1.13}$$

where c_p is specific heat at constant pressure,

$$c_p = \frac{\gamma}{\gamma - 1} \frac{k_B}{m}.$$

Solution Differentiating the left hand side of Eq. (1.12), and taking into account at $p = \text{const}$, we have

$$\frac{\rho^\gamma}{\gamma - 1} \frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = \frac{p \rho^\gamma}{\gamma - 1} \frac{d \rho^{-\gamma}}{dt} = -\frac{\gamma}{\gamma - 1} \frac{p}{\rho} \frac{d\rho}{dt} \tag{1.14}$$

Using the expression for c_p, we rewrite the left hand side of Eq. (1.13)

$$\rho c_p \frac{\mathrm{d} T}{\mathrm{d t}} = \frac{\gamma \rho}{\gamma - 1} \frac{d}{dt} \left(\frac{k_B T}{m} \right). \tag{1.15}$$

Using the ideal gas law,

$$\frac{p}{\rho} = \frac{k_B T}{m}, \tag{1.16}$$

we have

$$\frac{\gamma \rho}{\gamma - 1} \frac{d}{dt} \left(\frac{k_B T}{m} \right) = \frac{\gamma \rho}{\gamma - 1} \frac{d}{dt} \left(\frac{p}{\rho} \right) = \frac{\gamma \rho p}{\gamma - 1} \frac{d}{dt} \left(\frac{1}{\rho} \right) = -\frac{\gamma p}{(\gamma - 1)\rho} \frac{d\rho}{dt}. \tag{1.17}$$

Problem 5 *There is a dimensionless number, R_m, called the magnetic Reynold's number.*

$$R_m = \frac{LV}{\eta}, \tag{1.18}$$

where V is a typical plasma velocity, L is a typical scale and η is the magnetic diffusivity. The magnetic diffusivity is defined as

$$\eta = \frac{1}{\mu\sigma}, \tag{1.19}$$

where σ is the electrical conductivity in ohm m^{-1}.

According to Braginski, in a strongly magnetised plasma η can be estimated as

$$\eta = 10^9 T^{-3/2} m^2 s^{-1}$$

Estimate the magnetic Reynolds number in the corona, in sunspots and in the solar wind, making appropriate assumptions about the typical spatial scales and times.

Solution In the corona, the typical parameters are $T = 10^6$ K, $V = 1000$ km/s (e.g., the Alfvén speed) and $L = 1$ Mm (e.g., the loop cross-section diameter), which gives $\eta = 1$ m^2/s and $R_m \approx 10^{12}$.

In sunspots, $T \approx 6 \times 10^3$ K, $V \approx 1$ km/s (e.g., the Alfvén speed) and $L \approx 10$ Mm (e.g., the sunspot umbra diameter), giving $\eta \approx 2000$ m^2/s and $R_m \approx 5 \times 10^6$.

In the solar wind, $T \approx 10^5$ K (e.g., near the Earth's orbit), $V \approx 500$ km/s (the Alfvén and the sound speeds, the solar wind speed) and $L = 1R_\odot = 6.96 \times 10^8$ m, giving $\eta \approx 30$ m^2/s and $R_m \approx 10^{13}$.

In all the cases, the magnetic Reynolds number is much greater than unity.

Problem 6 *If the radius of a sunspot is $l = 10^7 m$ and $\eta = 10^3 m^2 s^{-1}$, estimate the time of the sunspot magnetic field will diffuse away.*

Solution In this answer, we take $R_m \ll 1$ and study pure magnetic diffusion by considering

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \tag{1.20}$$

This is a diffusion equation and so indicates that any irregularities in an initial magnetic field will diffuse away and be smoothed out. The field will tend towards a simpler uniform field. This process of smoothing out variations will occur on a timescale given by the diffusion timescale, $\tau_{diffusion} \approx L^2/\eta$,

$$\tau_{\text{diffusion}} \approx 10^{11} \text{seconds} \approx 3,000 \text{years}.$$

Problem 7 *A chromospheric magnetic element has the magnetic field of 1000 G, and the radius 100 km. Determine the radius of the coronal loop which is a continuation of this element if the field is estimated to be 15 G.*

Solution The magnetic flux must be conserved. Assuming both the cross-sections of the chromospheric magnetic element and the coronal loop to be circular, we have

$$\pi r^2 B = \text{const}, \tag{1.21}$$

where r is the radii of the circles.

Consequently, the magnetic field B_p at the chromospheric level and B_c in the corona and the radii of the magnetic flux tube in the chromosphere and in the corona, r_p and r_c, respectively, are connected with each other by the relation

$$\frac{B_p}{B_c} = \left(\frac{r_c}{r_p}\right)^2. \tag{1.22}$$

It gives us

$$r_c = \sqrt{B_p/B_c}\, r_p \approx 800\text{km}. \tag{1.23}$$

Problem 8 *A neutral slab of a non-magnetized plasma of width l is confined by two regions of a magnetized plasma penetrated by the magnetic field of strength B_0, parallel to the slab boundaries. The field reverses its direction from one side of the slab to the other. Neglecting the magnetic diffusivity η, show that the slab can be in equilibrium. What is the thermodynamic pressure in the slab?*

Solution The magnetostatic condition of the equilibrium in the configuration is

$$\frac{d}{dx}\left(p + \frac{B^2}{2\mu}\right) = 0, \tag{1.24}$$

which is the continuity of the total pressure across the slab (here, x is the coordinate perpendicular to the slab). We can see that this condition is independent of the orientation of the magnetic field lines, provided the field is parallel to the slab boundaries and straight.

Assuming that the gas pressure is p_0 and p_s in the external and in the internal media, respectively, we obtain the equilibrium condition,

$$p_0 + \frac{B^2}{2\mu} = p_s. \tag{1.25}$$

In the case of the cold plasma (or, the low-β plasma, $p_0 \ll B_0^2/2\mu$), $p_s = B_0^2/2\mu$.

This condition can be rewritten through the characteristic velocities. Assuming that outside the slab, the density is ρ_0 and the sound and Alfvén speeds are $C_{s0} = \sqrt{\gamma p_0/\rho_0}$ and $C_{A0} = B_0/\sqrt{\mu\rho_0}$, and in the slab the density is ρ_s and the sound speed is $C_{ss} = \sqrt{\gamma p_0/\rho_0}$, we rewrite the equilibrium condition as

$$\rho_0 \left(\frac{C_{s0}^2}{\gamma} + \frac{C_{A0}^2}{2} \right) = \frac{\rho_s C_{ss}^2}{\gamma}. \tag{1.26}$$

(It was assumed that γ is the same inside and outside the slab).

Problem 9 *Sketch the magnetic field lines for* $\mathbf{B} = 1\mathbf{e_x} + 2x\mathbf{e_y}$, *and determine the* $\mathbf{j} \times \mathbf{B}$ *force at the point* $x = 0, y = 1$.

Solution The field lines are parallel to the y-axis, and consequently the magnetic field lines are described by these two equations

$$\frac{dx}{B_x} = \frac{dy}{B_y}. \tag{1.27}$$

Substituting the components of the magnetic field vector, we get

$$\frac{dx}{1} = \frac{dy}{2x}, \tag{1.28}$$

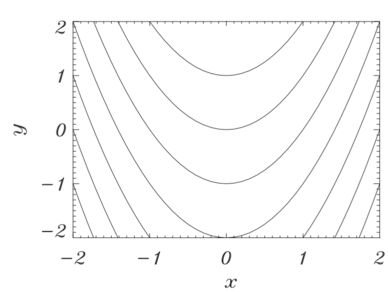
or

$$2x\ dx = dy, \tag{1.29}$$

which is a separable ODE. Integrating, we get

$$\int 2x\ dx = \int dy \qquad \Rightarrow \qquad y = x^2 + C. \tag{1.30}$$

This equation describes a family of parabolae.



The Lorentz force is

$$F_L = \mathbf{j} \times \mathbf{B} = \frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{1.31}$$

Calculating $\nabla \times \mathbf{B}$ we find

$$\nabla \times \mathbf{B} = 0\mathbf{e_x} + 0\mathbf{e_y} + 2\mathbf{e_z}, \tag{1.32}$$

and, finally, for the Lorentz force,

$$\mathbf{j} \times \mathbf{B} = -4\mathbf{e_x} + 2\mathbf{e_y} + 0\mathbf{e_z}. \tag{1.33}$$

At the point $x = 0, y = 1$, the force is directed in y-direction and its absolute value is 2.

Problem 10 *Investigate the polytropic model of the solar wind. Consider a steady spherically symmetric outflow from the surface of the Sun. Show that the MHD equations in spherical coordinates reduce to*

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\rho v) = 0, \tag{1.34}$$

$$\rho v\frac{\mathrm{d}\,v}{\mathrm{d}\,r} = -\frac{\mathrm{d}\,p}{\mathrm{d}\,r} - \frac{GM_\odot \rho}{r^2}, \tag{1.35}$$

$$v\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p}{\rho^\gamma}\right) = 0. \tag{1.36}$$

Show that the Euler equation may be rewritten as

$$v\frac{\mathrm{d}\,v}{\mathrm{d}\,r} = -\frac{\gamma}{\gamma - 1}\left(\frac{p}{\rho^\gamma}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho^{\gamma-1}\right) - \frac{GM_\odot}{r^2}, \tag{1.37}$$

and get the expression

$$\frac{v^2}{2} + \frac{C_s^2}{\gamma - 1} - \frac{GM_\odot}{r} = \text{const.} \tag{1.38}$$

Solution The mass continuity equation gives us

$$r^2\rho v = \text{const} \equiv \mathcal{M} \quad \Rightarrow \rho = \frac{\mathcal{M}}{r^2 v}. \tag{1.39}$$

The adiabatic equation gives us

$$p\rho^{-\gamma} = \text{const} \equiv \mathcal{S} \quad \Rightarrow \quad p = \mathcal{S}\rho^\gamma. \tag{1.40}$$

Differentiating this equation, we obtain

$$\frac{dp}{dr} = \gamma\mathcal{S}\rho^{\gamma-1}\frac{d\rho}{dr}. \tag{1.41}$$

The Euler equation becomes

$$v\frac{\mathrm{d}\,v}{\mathrm{d}\,r} = -\gamma\mathcal{S}\rho^{\gamma-2}\frac{\mathrm{d}\,\rho}{\mathrm{d}\,r} - \frac{GM_\odot}{r^2}, \tag{1.42}$$

and then

$$v\frac{\mathrm{d}\,v}{\mathrm{d}\,r} = -\frac{\gamma\mathcal{S}}{\gamma-1}\frac{d}{dr}\left(\rho^{\gamma-1}\right) + GM_\odot\frac{d}{dr}\left(\frac{1}{r}\right). \tag{1.43}$$

Integrating the equation, we get

$$\frac{v^2}{2} + \frac{\gamma\mathcal{S}}{\gamma-1}\rho^{\gamma-1} - \frac{GM_\odot}{r} = \text{const.} \tag{1.44}$$

With the use of the definition of the sound speed, $C_s = \sqrt{\gamma p/\rho}$, and $\mathcal{S} = p\rho^{-\gamma}$ we obtain the expression sought.

Problem 11 *Observations by GOES spacecraft show that the integrated soft X-ray flux of a solar flare varies with time as $F_{\mathrm{soft}}(t) \propto \exp(-t^2/t_f^2)$, where t_f is about 500 s. Determine the expected time dependence of the hard X-ray and microwave fluxes generated by the flare.*

Solution According to the Neupert effect, during the rise phase of the flare, the time derivative of the soft X-ray light curve resembles the hard X-ray curve F_{hard}. Hence,

$$F_{\mathrm{hard}} \propto \frac{d}{dt}\exp(-t^2/t_f^2) = -\frac{2t}{t_f^2}\exp(-t^2/t_f^2). \tag{1.45}$$

You may also like to make sketches of these two curves.

As the microwave emission is produced by the same non-thermal electrons as the hard X-ray emission, the microwave light curve would look similarly to the hard X-ray curve

Problem 12 *The vertical dependence of the absolute value of the magnetic field in an active region is estimated as $B = B_0(z/a)^{-4}$, where a and B_0 are constant. Take that the density profile is given by the hydrostatic equilibrium, $\rho = \rho_0\exp(-z/\Lambda)$, where ρ_0 and Λ are constant. Determine the heights of the minima of the Alfvén speed and the fast speed.*

Solution The Alfvén speed C_A is proportional to $B/\rho^{1/2}$. Hence

$$C_A \propto \left(\frac{z}{a}\right)^{-4}\exp(z/2\Lambda). \tag{1.46}$$

Looking for the extreme of the function, we get

$$\frac{dC_A}{dz} = -\frac{4}{a}\left(\frac{z}{a}\right)^{-5}\exp(z/2\Lambda) + \frac{1}{2\Lambda}\left(\frac{z}{a}\right)^{-4}\exp(z/2\Lambda) = \tag{1.47}$$

$$= \left(\frac{z}{a}\right)^{-5}\exp(z/2\Lambda)\left(-\frac{4}{a} + \frac{1}{2\Lambda}\frac{z}{a}\right) = 0. \tag{1.48}$$

Hence

$$-\frac{4}{a} + \frac{z}{2\Lambda a} = 0, \qquad z = 8\Lambda. \tag{1.49}$$

This is the height of the minimum of the Alfvén speed.

The fast speed is $C_F = \sqrt{C_A^2 + C_s^2}$. The sound speed C_s is proportional to the root of the temperature. As the scale height Λ is constant, hence the temperature is constant. Hence the minimum of the fast speed coincides with the minimum of the Alfvén speed.

Problem 13 *The Extreme Ultraviolet Imager onboard SOHO spacecraft observes an off-limb loop. The emission intensity in the loop is measured to have the scale height two times larger than outside it. Under the assumption that the loop cross-section does not change with height, determine the ratio of the plasma temperatures inside and outside the loop.*

Solution Intensity of the optically thin emission is proportional to the density squared multiplied to the column depth of the emitting object. Take that the plasma in the loop and outside it is gravitationally stratified, $\rho_{e,i} \propto \exp(-z/\Lambda_{e,i})$, where the indices i and e refer to the physical parameters inside and outside in the loop, respectively. Hence, the dependence of the intensity of the emission from the loop of constant minor radius (and observed off the limb, so its plane can be taken perpendicular to the line-of-sight) on the height, is $I_{e,i} \propto \rho_{e,i}^2 \propto \exp(-2z/\Lambda_{e,i})$. Thus, the intensity is stratified with $\Lambda_{e,i}/2$. Taking that the scale height is proportional to the temperature, obtain

$$\frac{T_e}{T_i} = \frac{I_e}{I_i} = \frac{\Lambda_e/2}{\Lambda_i/2}. \tag{1.50}$$

Hence, $T_e = T_i/2$.

Problem 14 *Spectroscopic measurements show that outside the loop the plasma is cooler and less dense than inside. What information it gives you about the ratio of the magnetic field inside and outside the loop?*

Solution Across the loop the total pressure balance should be kept:

$$p_e + \frac{B_e^2}{2\mu} = p_i + \frac{B_i^2}{2\mu} \tag{1.51}$$

Using the ideal gas law, $p \propto \rho T$, we get that $B_e > B_i$.

Problem 15 *Imaging observations in EUV show that the global kink mode of a coronal loop has the phase speed of 1000 km/s. The loop appears to be 100 times brighter than the background plasma. Show that in the low-β plasma the kink speed can be estimated as $C_k \approx C_{Ai}\sqrt{2/(1+\rho_e/\rho_i)}$. Using this expression or otherwise estimate the Alfvén speed in the loop. (Here the indices i and e refer to the physical parameters inside and outside in the loop, respectively.)*

Solution Using the definitions of the Alfvén speed and the kink speed we obtain

$$C_K = \frac{1}{\mu^{1/2}} \left(\frac{B_e^2 + B_i^2}{\rho_e + \rho_i} \right)^{1/2}. \tag{1.52}$$

If $\beta \ll 1$, magnetic pressure dominates across the loop, hence $B_e \approx B_i$. Then

$$C_K \approx \frac{B_i}{\sqrt{\mu\rho_i}}\sqrt{\frac{2}{1+\rho_e/\rho_i}} = C_{Ai}\sqrt{2/(1+\rho_e/\rho_i)}. \tag{1.53}$$

As intensity of the optically thin emission is proportional to the density, we obtain that $\rho_e \ll \rho_i$, as the loop is much brighter than the background plasma. Hence $C_k \approx 2^{1/2} C_{Ai}$, which gives $C_{Ai} \approx 707$ km/s.

Problem 16 *The SOHO/SUMER spectral instrument discovered quasi-periodic oscillations of the intensity and Doppler shift in the coronal emission line Fe_{xix}. The period of the oscillations is in the range of 5-15 min. It is suggested that these oscillations can be interpreted as standing sound waves in coronal loops.*

(a) Consider linear sound waves propagating strictly along the magnetic field, parallel to the z-axis, neglect dissipation and gravity, and show that the Euler equation and continuity equations may be reduced to the equations,

$$\rho_0 \frac{\partial \tilde{V}}{\partial t} = -\frac{\partial \tilde{p}}{\partial z}, \quad \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \frac{\partial \tilde{V}}{\partial z}$$

where ρ_0 is the constant equilibrium density, and the tilde denoted perturbations of the physical value.

(b) Show that with the use of the expression $\tilde{p} = C_s^2 \tilde{\rho}$ where C_s is the sound speed, the above equations combine to the wave equation

$$\frac{\partial^2 \tilde{V}}{\partial t^2} - C_s^2 \frac{\partial^2 V}{\partial z^2} = 0.$$

(c) Apply the "rigid wall" boundary conditions ($V = 0$) at the loop footpoints ($z = 0$ and $z = L$) and determine the resonant frequency of the global acoustic (also called "longitudinal") mode of the loop.

Solution (a) The sound (or acoustic, or longitudinal, in solar plasma structures all these names describe the same wave mode) waves do not perturb the magnetic field, hence the only non-zero force on the right hand side of the Euler equation is the gradient of the pressure. Consider weak perturbation of the equilibrium given by the constant density ρ_0, gas pressure p_0 and temperature T_0. Linearising the equation we obtain

$$\rho_0 \frac{\partial \tilde{\mathbf{V}}}{\partial t} = -\nabla \tilde{p}. \tag{1.54}$$

Taking that the flows in the wave are parallel to the z-axis we get $\nabla \to \partial/\partial z$. Thus we get the first of the equations given in this subquestion.

Similarly linearising the continuity equation and considering the flows along the z-axis only we obtain the second given equation. Here we should also use that the equilibrium density is constant.

(b) Differentiating the first giving equation with respect to time, and the second given equation with respect to z and excluding $\tilde{\rho}$ and $\tilde{p}$ we get the wave equation.

(c) Taking that in a harmonic wave all physical quantities vary in time as e.g. $\cos \omega t$, we get that

$$\partial^2 \tilde{V}/\partial t^2 = -\omega^2 V_0 \cos \omega t, \tag{1.55}$$

where V_0 is the amplitude of the wave. In general the amplitude is a function of the position along the loop and should satisfy the boundary conditions at the footpoints. Hence, the wave equations becomes

$$\frac{d^2 V_0}{dz^2} + \frac{\omega^2}{C_s^2} V_0 = 0. \tag{1.56}$$

This equations is a simple harmonic oscillator equation with the general solution

$$V_0 = A\cos(\omega z/C_s) + B\sin(\omega z/C_s), \tag{1.57}$$

where A and B are arbitrary constants. Applying the given boundary conditions, $V = 0$ at the loop footpoints ($z = 0$ and $z = L$), we get

$$V_0 = B\sin(\omega z/C_s), \tag{1.58}$$

where the frequency ω satisfies the condition $\omega = \pi C_s/L$. (Here we used that the oscillation is the global acoustic mode that corresponds to the spatial harmonic of the lowest frequency.)

In a more rigorous description we have to take into account that the actual speed of the longitudinal mode of a coronal loop is not the sound speed C_s, but the tube speed C_T. Also, an interesting problem would be to look at the phase relation between the oscillations of the velocity and the perturbations of the density and the pressure in this mode.

Problem 17 *A global coronal wave is observed to propagate in a coronal hole in the horizontal direction. The speed of the wave at the height 50 Mm is estimated to be 2 times higher than near the bottom of the corona. Consider the atmosphere to be plane, the magnetic field vertical and uniform, and the temperature constant, and estimate the temperature.*

Solution The wave propagates in the horizontal direction, hence across the magnetic field. In a uniform plasma (as it is stated in the problem) the only MHD wave that propagates across the field is the fast wave, propagating at the speed $C_F = \sqrt{C_A^2 + C_s^2}$. In a low-$\beta$ plasma of the coronal holes, we can neglect the sound speed in comparison with the Alfvén speed in the expression: $C_F \approx C_A = B_0/\sqrt{\mu\rho_0}$, where B_0 is the equilibrium magnetic field and ρ_0 is the equilibrium density. As the temperature is constant, the density is exponentially stratified, $\rho_0(z) = \rho_{00}\exp(-z/\Lambda)$, where Λ is the scale height and ρ_{00} is the density at the bottom of the corona. The scale height is determined by the temperature, T_0, $\Lambda/\text{Mm} \approx 50T_0/\text{MK}$.

Comparing the fast speeds at the two given heights we get

$$\frac{C_F(z_0)}{C_F(0)} = 2 \approx \sqrt{\frac{\rho_0(0)}{\rho_0(z_0)}}, \tag{1.59}$$

where $z_0 = 50$ Mm. Substituting in that expression $\rho_0(0) = \rho_{00}$ and $\rho_0(z_0) = \rho_{00}\exp(-z_0/\Lambda)$, we get

$$\Lambda \approx \frac{z_0}{2\log 2}. \tag{1.60}$$

Hence, $T_0 \approx 720,000$ K.

Problem 18 *The temperature of the exponentially stratified plasma in a coronal hole is 1 MK. The absolute value of the magnetic field is assumed to depend on height as* $B_0 = B_{00}\left(\frac{z}{z_0}\right)^{\alpha}$, *where* $z_0 = 50$ *Mm and* α *is a constant. Observations show that the global coronal compressive wave propagates at the heights* $z = 10$ *Mm and* $z = 50$ *Mm at speeds 0.5 Mm* s^{-1} *and 0.6 Mm* s^{-1}*, respectively. Assuming that the wave propagates perpendicularly to the field, and the plasma* β *is about 0, determine the index* α*.*

Solution The density is stratified as

$\rho = \rho_0\exp(-z/\Lambda)$, with $\Lambda(\text{Mm}) \approx 50 \times T(\text{MK}) \approx 50$ Mm.

The global coronal wave is a fast magnetoacoustic wave (compressive, perpendicular) hence its speed is $C_F = (C_A^2(z) + C_s^2)^{1/2}$.

As $\beta \approx 0$, we get $C_F = C_A(z)$.

Thus, using the observed ratio of the global wave speeds,
$0.6/0.5 \approx C_F(50\text{ Mm})/C_F(10\text{ Mm}) \approx C_A(50\text{ Mm})/C_A(10\text{ Mm})$,
with
$C_A(50\text{ Mm})/C_A(10\text{ Mm}) =$
$= B_0(50/50)^{\alpha}(\mu\rho_0)^{-1/2}\exp(50/100)/B_0(10/50)^{\alpha}(\mu\rho_0)^{-1/2}\exp(10/100) =$
$= \exp(0.5)/(0.2)^{\alpha}\exp(0.1)$

Hence,
$6/5 = \exp(0.5)/(0.2)^{\alpha}\exp(0.1)$, or $\alpha = \log_{0.2}(5\exp(0.4)/6) \approx -0.13$.