PX358 **Plasma Physics, 2006.** Problem Sheet 1

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Question 1.

Consider a slab of neutral plasma of concentration n_0 and thickness L. The ions are displaced by x_i in the $+\hat{x}$ direction whilst the electrons are displaced by x_e in $+\hat{x}$. Find the electric field within the bulk of the slab in terms of these displacements, and combine the separate equations of motion for the ion and electron slabs to show that the difference between the two displacements is oscillatory with angular frequency:

$$\omega_p^2 = \frac{n_0 e^2}{\epsilon_0}\left(\frac{1}{m_e} + \frac{1}{m_i}\right).$$

Comment on the approximations that were made in the lecture to estimate ω_p^2.

Question 2.

A single particle with charge $+e$ and mass m has equations of motion

$$m\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = e(\mathbf{E} + \mathbf{V}\times\mathbf{B}), \quad \text{and} \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{V}$$

a) Show that the particle motion in zero electric field and constant background field $\mathbf{B} = B\hat{\mathbf{b}}$ is circular motion about the field direction $\hat{\mathbf{b}}$ and straight line motion along $\hat{\mathbf{b}}$. Show that the frequency of gyration about the field is

$$\omega_c = \frac{eB}{m}$$

and the gyroradius of the circular motion is

$$R_c = \frac{v_\perp}{\omega_c}.$$

Show that the kinetic energy is conserved.

b) A constant electric field $\mathbf{E}_\perp$ that is perpendicular to $\mathbf{B}$ is now added. Show that the particle motion is now cycloidal, with guiding center drift:

$$\mathbf{V}_D = \frac{\mathbf{E}\times\mathbf{B}}{B^2}.$$

c) Show that the above motion is consistent with a non-relativistic frame transformation of the fields.

Question 3.

Neglecting radiation losses, find the motion of a charged particle with relativistic energy in a constant $\mathbf{B}$ field. What is the gyrofrequency?

Which of the following are relativistic: a 1 keV proton (planetary magnetosphere), a 1 MeV proton (high energy solar flare), a 1 GeV proton (cosmic ray)?

PX358 **Plasma Physics, 2006.** Problem Sheet 2

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Question 1.

Cosmic ray protons are typically in the energy range of a few tens of MeV to a few GeV. If the proton is released from a supernova explosion, what would the supernova shock expansion speed need to be to accelerate the proton to this energy by simple 1st order Fermi acceleration (one collision between the proton and the shock)?

Assuming that the proton is reflected many times between the shock and field structures at rest of surrounding plasma what is the fractional increase in energy as a function of energy? Comment on your result.

Question 2.

The Earth's magnetic field can be approximated as a first approximation as a dipole to distances of a few earth radii (R_E).

a) If the field at one of the poles is 5.1×10^{-5} T calculate the magnetic dipole moment.

b) Give expressions for the gyrofrequency and gyroradius for an electron of energy W at radius $r > R_E$ in the equatorial plane.

c) Assuming that the electron is confined to the equatorial plane, obtain expressions for its gradient and curvature drift velocities.

d) The Van Allen radiation belts are populated by isotropic plasma number density 10^{-7} m^{-3} at about $4R_E$, composed of 1 MeV protons and 100 keV electrons. Estimate the time it takes the electrons to drift once around the Earth.

What would is the ring current carried by these trapped particles if they occupy a region between 3-5 R_E from the Earth?

Question 3.

Consider a gas composed of identical particles of number density $n(\mathbf{r})$ and velocity $\mathbf{v}(\mathbf{r})$ at position $\mathbf{r}$. Particles are neither created nor destroyed.

a) What is the flux of particles across surface S (give integral form)?

b) If S encloses a volume V show that the gas obeys a conservation equation

$$\nabla \cdot (n\mathbf{v}) = -\frac{\partial n}{\partial t}.$$

Hence obtain an expression for conservation of charge in terms of charge density ρ and current density $\mathbf{j}$.

PX358 **Plasma Physics, 2006.** Problem Sheet 3

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Question 1.

A lightning stroke on earth leads to the generation of a whistler mode pulse in the ionosphere. The pulse contains many frequencies (a wide spectrum pulse) which are in the range $\omega_{ci} < \omega \ll \omega_{ce}$. The pulse propagates through the ionosphere, along the earth magnetic field, to the point where the magnetic field returns back to the ionosphere.

a) Determine the dependences of the whistler mode group and phase speeds upon the frequency ω.

b) Derive the dependence of the delay time involved in the arrival of different frequencies ω, upon the frequency.

c) How can the dependence be used as a tool for plasma diagnostics (e.g. for determination of electron concentration)?

Question 2.

Consider high-frequency electromagnetic waves in a cold magnetized plasma in the case $\omega_{pe} > \omega_{ce}$. Obtain a plot of ω versus $|\mathbf{k}|$ for the case of parallel propagation $\mathbf{k} \parallel \mathbf{B}$. Determine the cut-off frequencies.

Question 3.

For low frequency wave motions in a cold magnetized plasma it is necessary to take into account the motion of the ions. In this case, the elements of the dielectric tensor $\hat{\varepsilon}$ are

$$\varepsilon_1 = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2}, \qquad \varepsilon_2 = \frac{\omega_{ce}}{\omega}\frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} - \frac{\omega_{ci}}{\omega}\frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2},$$

$$\varepsilon_3 = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}.$$

Consider the dispersion relation for parallel modes:

$$\varepsilon_3[(\varepsilon_1 - N_z^2)^2 - \varepsilon_2^2] = 0.$$

a) For left circularly polarized waves, show there are modes in the band $0 < \omega < \omega_{ci}$ (ion cyclotron waves).

b) In the limit $\omega \ll \omega_{ci}$, assuming that $\omega_{pi} \ll \omega_{pe}$, show that the mode satisfies

$$\frac{\omega}{k} = \frac{C_A}{(1 + C_A^2/c^2)^{1/2}},$$

where $C_A = B/(\mu_0 n_i M)^{1/2}$ is the Alfvén speed.

c) For right polarized waves, derive the approximate dispersion relation in the limit $\omega \ll \omega_{ci}, \omega_{ce}$. Show that the right circularly polarized waves are described by the same dispersion relation as the left circularly polarized waves in this limit.

PX358 **Plasma Physics, 2006.** Problem Sheet 4

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Question 1.

Consider a plasma slab of width d confined by two parallel rigid planes $x = 0$ and $x = d$ (a plasma waveguide). The plasma is magnetized and the magnetic field $\mathbf{B} = B_0\mathbf{e_y}$ is uniform and parallel to the planes. Density of the plasma is ρ_0. Consider MHD waves propagating in the waveguide along the z-axis (perpendicular to the field) in the cold plasma limit.

a) Linearize the ideal MHD equations near the equilibrium, taking into account $\partial/\partial y = 0$.

b) Assume that the waves are proportional to $\exp(ikz - i\omega t)$ and depend on the x-coordinate. Show that they are described by the equation

$$\frac{\mathrm{d}^2 V_x}{\mathrm{d}x^2} + \left(\frac{\omega^2}{C_A^2} - k^2\right) V_x = 0,$$

where C_A is the Alfvén speed.
c) By supplementing this equation with the rigid wall boundary conditions

$$V_x(x = 0) = V_x(x = d) = 0,$$

obtain the dispersion relation connecting the frequency ω and the wave number k. What are the modes described by the dispersion relation? Plot the frequency $\omega(k)$, and the phase and group speeds as a function of k. Compare the phase and group speeds of the modes with the Alfvén speed.

d) Prove that the mode is compressive.

Question 2.

Consider a coronal loop with $B_0 = 10$ G (10^{-3} T), of length $L = 50$ Mm (5×10^7 m) and concentration $n = 5 \times 10^{14}$ m^{-3}. (Effects of the loop curvature and gravity are neglected.

a) What are the MHD modes which propagate along the loop?

b) Assume that, at the loop footpoints, the transversal perturbations of the magnetic field are zero ("line-tied boundary conditions") and determine the frequencies of the two lowest frequency standing Alfvén modes of the loop.

Question 3.

The fluid equations for conservation of mass and momentum for a single species (ions or electrons, assume that the ion charge is e) in a collisionless plasma are:

$$\frac{\partial n_\alpha}{\partial t} + \nabla(n_\alpha \mathbf{V}_\alpha) = 0,$$

$$m_\alpha n_\alpha \left[\frac{\partial \mathbf{V}_\alpha}{\partial t} + (\mathbf{V}_\alpha \nabla)\mathbf{V}_\alpha\right] = -\nabla p + q_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}),$$

where the indices $\alpha = i, e$ correspond to either ions or electrons. An idealized uniform current carrying plasma is initially comprised of an ion and an electron fluid of equal number density. Both populations are cold, that is, initially all the ions are at rest and all the electrons move with the same speed V_e relative to the ions. There are no background electromagnetic fields.

For electrostatic waves only (assume no magnetic fields are generated), and for small amplitude waves of the form:

$$\mathbf{E} = \mathbf{E_1} \exp(ikz - i\omega t)$$

show that the dispersion relation for the plasma is

$$\left[\left(\frac{\omega_{pi}}{\omega} \right)^2 + \frac{1}{\left(\frac{\omega}{\omega_{pe}} - \frac{kV_e}{\omega_{pe}} \right)^2} \right] = 1$$

where ω_{pi}, ω_{pe} are the ion and electron plasma frequencies. (Assume that the waves are longitudinal and propagate parallel or anti-parallel to the direction of the electron current and that the perturbations of the electric field are described by Poisson's equation $\varepsilon_0 \nabla \mathbf{E}_1 = e(n_i - n_e)$.

By sketching the curve

$$F(x, y) = \frac{M}{x^2} + \frac{1}{(x - y)^2}$$

as a function of x, or otherwise, determine how many roots the dispersion relation has for ω, and whether they are real or imaginary. Explain what happens to the wave as a function of time for each case.