# STOCHASTIC PROCESSES: Introducing Differential Equations

## CIS002-2 Computational Alegrba and Number Theory

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# Outline

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# Introducing Differential Equations

- A differential equation is one that involves one or more derivatives of a function

### Example

Say we have a toy train on a straight track, and $x$ is the position of the train along the track. If the train is moving then $x$ will be a function of time, $x(t)$. If we apply a constant force (from constant electric power), $F$, to the train, then its accelaration, being the second derivative of $x$, is equal to $F/m$, where $m$ is the mass of the train. Thus we have a simple differential equation:

$$\frac{d^2x}{dt^2} = \frac{F}{m}$$

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# Introducing Differential Equations

- Although an integral can be seen as the area under a plotted curve between two limits, it can also be viewed as the inverse of differentiation, where differentiation can be seen as the gradient of a plotted curve at any single point.
- The pneumonic is "add one to the power and divide by the new power" for the integral of simple powers. Integration is accurate up to an arbitrary constant of integration, which can be thought of as comming from the y-intercept of the plot.

### Example

To find how $x$ varies with time, we need to find the function $x(t)$ that satisfies the previous differential equation. Here we can integrate both sides of the equation twice to recover $x$:

$$x(t) = \frac{F}{2m}t^2 + v_0 t + x_0$$

where $v_0$ and $x_0$ come from the constants of integration, and represent the initial velocity and initial position of the toy train.

# Outline

# Vector Differential Equations

- We can change a second order differential equation into two first order differential equations.
- To do this we need to introduce a second variable, and set this equal to the first derivative.

## Example

We can define the velocity of the toy train as the derivative of the position of the train, so we have two first order differential equations.

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = \frac{F}{m}$$

We can now write this set of first order differential equations in "vector form".

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ F/mx & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

# Vector Differential Equations

## Example

Defining $\mathbf{x} = (x, v)^T$ and $A$ as the matrix

$$A = \begin{bmatrix} 0 & 1 \\ F/mx & 0 \end{bmatrix}$$

We can now write the set of equations in compact form

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

If the elements of the matrix $A$ do not depend on $\mathbf{x}$, then this equation would be a linear first-order vector differential equation.

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# Vector Differential Equations

- We can consider a differential equation as involving the change in $x$ at an infinitesimal time-step $dt$

$$dx = \frac{dx}{dt} dt$$

- We can write differential equations in terms of $dx$ and $dt$ instead of using the derivatives previously.

### Example

$$d \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ F/mx & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} dt$$

or in the more compact notation

$$\mathbf{dx} = A\mathbf{x}dt$$

The infinitesimal increments $dx$, $dt$, etc., are called "differentials", and so writing differential equations in this way is often referred to as writing them in "differential form".

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# Asside: The Exponential fucntion

The exponential function is the entire function defined by

$$exp(z) = e^z$$

where $e$ is the solution of the equation

$$\int_1^x \frac{dt}{t}$$

so that $e = x = 2.718\ldots$. The exponential function has Maclaurin series

$$exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and satisfies the limit

$$exp(x) = \lim_{(n\to\infty)} \left(1 + \frac{x}{n}\right)^n$$

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- $x(t + dt) = x(t) - \gamma x(t)dt = (1 - \gamma)x(t)$
- To solve this we note that to first order in $dt$ (that is, when $dt$ is very small) $e^{\gamma dt} \approx 1 + \gamma dt$. This comes from the definition of $e^x$ ($e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.)

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- Let us say that $dt = \tau/N$ for some large $N$.
  $x(t + \tau) = \left(e^{-\gamma dt}\right)^N x(t) = e^{-\gamma N dt} x(t) = e^{-\gamma \tau} x(t)$

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  $x(t + \tau) = \left(e^{-\gamma dt}\right)^N x(t) = e^{-\gamma N dt} x(t) = e^{-\gamma \tau} x(t)$
- The equation solved above is the simplest linear differential equation.

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# Outline



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# A Linear Differential Equation with Driving

- Consider the simple differential linear equation with the addition of a "driving" term $\frac{dx}{dt} = -\gamma x + f(x)$ where $f$ is any function of time.

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- To solve this we must first transform to a new variable $\gamma(t)$, defined as $y(t) = x(t)e^{\gamma t}$. We have chosen this definition so that if $x(t)$ was a solution to $dx = -\gamma x dt$, then $y$ would be constant.

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- We now calculate the differential equation for $y$, giving $dy = e^{\gamma t} f(t) dt$.
- The solution is obtained by integratinf both sides of this equation giving

$$y(t) = y_0 + \int_0^t e^{\gamma s} f(s) ds$$

where we have defined the value of $y$ at time $t = 0$ as $y_0$.

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- We can now find $x(t)$

$$x(t) = x_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} f(s) ds$$

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# A Linear Differential Equation with Driving

- We can just as easily solve a linear equation when the coefficient $\gamma$ is a function of time also. In this case we transform $y(t) = x(t)e^{\Gamma(t)}$ where we define

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# A Linear Differential Equation with Driving

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$$\Gamma(t) \equiv \int_0^t \gamma(s)ds$$

- The solution to this is then

$$x(t) = x_0 e^{-\Gamma(t)} + \int_0^t e^{\Gamma(s)-\Gamma(t)} f(s)ds$$

# Outline

# Solving Vector Linear Differential Equations

- We can usually solve a linear differential equation with more than one variable

$$\dot{\mathbf{x}} = A\mathbf{x}$$

- This is done by transforming to a new set of variables, $\mathbf{y} = U\mathbf{x}$, where $U$ is a matrix chosen so the equations for the new variables are "decoupled" from each other.
- The equation for $\mathbf{y}$ is

$$\dot{\mathbf{y}} = D\mathbf{y}$$

  where $D$ is a diagonal matrix. For many square matrices $A$, there exists a matrix $U$ so that $D$ is diagonal.
- This is the case when $A^{\dagger}A = AA^{\dagger}$, where $A^{\dagger}$ is called the Hermitian conjugate of $A$ (if $A$ were real, then $A^{\dagger} = A^{T}$), defined as the transpose of a the complex conjuagate of $A$.
- If $U$ exists then it is unitary, which means $U^{\dagger}U = UU^{\dagger} = I$.
- The diagonal elements of $D$ are called the eigenvalues of $A$.

# Solving Vector Linear Differential Equations

- If $D$ is diagonal then for each element of $y$, $y_n$ we have the simple equation $\dot{y}_n = \lambda_n y_n$, where $\lambda_n$ are the diagonal elements of $D$.
- This has the solution $y_n(t) = y_n(0)e^{\lambda_n t}$ so the solution for $\mathbf{y}$ is

$$\mathbf{y}(t) = e^{Dt}\mathbf{y}(0)$$

- To get the solution for $\mathbf{x}(t)$ we use the fact that $U^{\dagger}U = I$, from which if follows immediately that $\mathbf{x} = U^{\dagger}\mathbf{y}$, which leads us to the equation

$$\mathbf{x}(t) = U^{\dagger}e^{Dt}U\mathbf{x}(0)$$

- Further, it makes sense to define the exponential of any square matrix $A$ as

$$e^{At} = U^{\dagger}e^{Dt}U$$

- Therefore, the natural definition of any function of a square matrix $A$ is

$$f(A) \equiv U^{\dagger}f(D)U$$

# Summary

To summarise the above results, the solution to the vector differential equation

$$\dot{\mathbf{x}} = A\mathbf{x}$$

is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0)$$

where

$$e^{At} = U^{\dagger} e^{Dt} U$$

We can also solve any linear vector differential equation with driving, just as we did for thwe single variable linear equation.

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