Introduction to Functions

CIS008-2 Logic and Foundations of Mathematics

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Outline

Recap

- Reviewed the definition of set operations:
  - Union
  - Intersection
  - Compliment
  - Difference
- Universal set
- Disjoint set
- Proper subset

What is a Function?

Definition

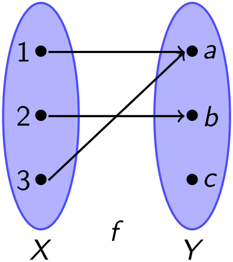
Let X and Y be sets. A **function** f from X to Y is a subset of the product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.

A function f from X to Y is sometimes denoted as $f : X \to Y$.

The set X is called the **domain** of f and the set Y is called the **codomain** of f. The set $\{y \mid (x, y) \in f\}$ is called the **range** of f.

Arrow Diagrams

Below is an example of an arrow diagram for the set $f = \{(1, a), (2, b), (3, a)\}$ being the function $f : X \to Y$ for the sets $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.



In this example the range of f is $\{a, b\}$, the domain is X and the codomain is Y.

Graph of a Function

The graph of a function f whose domain and codomain are subsets of $\mathbb{R}$ is obtained by plotting points in the plane that corresponds to the elements in f. The domain is contained in the horizontal axis and the codomain is contained in the vertical axis.

Modulus Operator

If x is an integer and y is a positive integer, we define $x \mod y$ to be the remainder when x is divided by y.

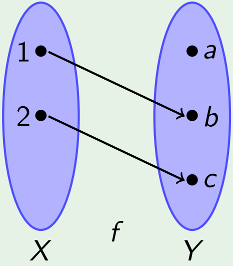
Floor and Ceiling

The **floor** of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x.
The **ceiling** of x, denoted $\lceil x \rceil$, is the least integer greater than or equal to x

One-to-one Function

A function $f : X \rightarrow Y$ is said to be **one-to-one** (or injective) if for each $y \in Y$, there is at most one $x \in X$ with $f(x) = y$.

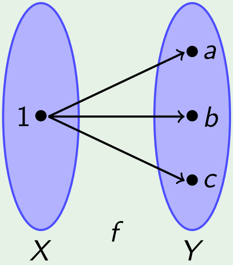
Example



Onto Function

If a function $f : X \to Y$ has the range of f being Y, f is said to be **onto** Y (or an **onto function** or a **surjective function**).

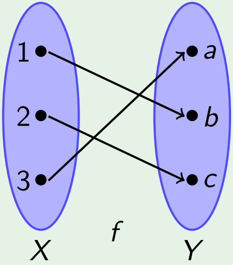
Example



Bijection

A function that is both one-to-one and onto is called a **bijection**.

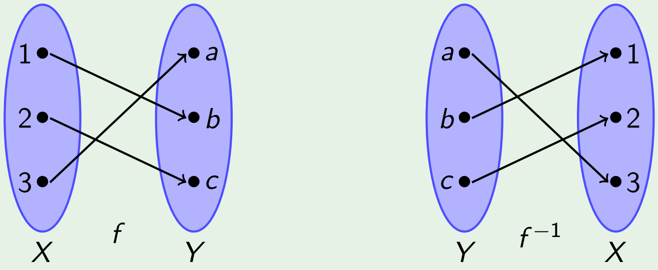
Example



Inverse Function

Suppose a function f is a one-to-one, onto function (a bijection). It can be shown that $\{(y, x) \mid (x, y) \in f\}$ is also a one-to-one, onto function, from Y to X. This new function, denoted f^{-1}, is called f **inverse**.

Example



Composition of Functions

Let g be a function from X to Y and let f be a function from Y to Z. The **composition** of f with g, denoted $f \circ g$, is the function $(f \circ g)(x) = f(g(x))$ from X to Z.

Questions

Find the domain, range and draw arrow diagrams for the following functions $f : \{1, 2, 3, 4\} \rightarrow \{a, b, c, d\}$. Also determine whether they are one-to one, onto or a bijection. If they are a bijection, give the description of the inverse function as a set of ordered pairs, draw the arrow diagram, and give the domain and range of the inverse function.

1. $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$
2. $\{(1, c), (2, d), (3, a), (4, b)\}$
3. $\{(1, b), (2, d), (4, a)\}$
4. $\{(1, b), (2, d), (3, b), (4, b)\}$

Questions (II)

Let f and g be functions from positive integers to the positive integers defined by the equations $f(n) = n^2$ and $g(n) = 2^n$. Find the compositions:

5. $f \circ f$
6. $g \circ g$
7. $f \circ g$
8. $g \circ g$

Questions (III)

Let f be the function from $X = \{0, 1, 2, 3, 4, 5\}$ to X defined by $f(x) = 4x \mod 6$.

9. Write f as a set of ordered pairs.
10. Draw the arrow diagram of f.
11. Is f a one-to-one function?
12. Is f an onto function?

Questions (IV)

13. Prove that n is an odd integer,

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}$$

14. Find a value for x for which $\lceil 2x \rceil = 2\lceil x \rceil - 1$
15. Prove that for all real numbers x and integers n, $\lceil x \rceil = n$ if and only if there exists ϵ, $0 \leq \epsilon < 1$, such that $x + \epsilon = n$.