Introduction to Logic

CIS008-2 Logic and Foundations of Mathematics

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Outline

The Wire

"If you play with dirt you get dirty."

True or False?

1. The only positive integers that divide 7 are 1 and 7 itself.
2. For every positive integer n, there is a prime number larger than n.
3. $x + 4 = 6$.
4. Write a pseudo-code to solve a linear diophantine equation.

True or False?

1. The only positive integers that divide 7 are 1 and 7 itself. **True**.
2. For every positive integer n, there is a prime number larger than n. **True**
3. $x + 4 = 6$. **The truth depends on the value of** x.
4. Write a pseudo-code to solve a linear diophantine equation. **Neither true or false**.

Propositions

A sentence that is either true or false, but not both, is called a **proposition**. The following two are propositions

1. The only positive integers that divide 7 are 1 and 7 itself. **True**.
2. For every positive integer n, there is a prime number larger than n. **True**

whereas the following are not propositions

3. $x + 4 = 6$. **The truth depends on the value of** x.
4. Write a pseudo-code to solve a linear diophantine equation. **Neither true or false**.

Notation

We will use the notation

$$p : 2 + 2 = 5$$

to define p to be the proposition $2 + 2 = 5$.

The **conjunction** of p and q, denoted $p \wedge q$, is the proposition

$$p \text{ and } q$$

The **disjunction** of p and q, denoted $p \vee q$, is the proposition

$$p \text{ or } q$$

Conjuction and Disjunction

If

$$p: \text{ It is raining,}$$
$$q: \text{ It is cold,}$$

then the conjunction of p and q is

$$p \wedge q: \text{ It is raining and it is cold.}$$

The disjunction of p and q is

$$p \vee q: \text{ It is raining or it is cold.}$$

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Truth Tables

Truth tables describe the truth values of propositions such as conjunctions and disjunctions. T denotes true and F denotes false.

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

The **negation** of p, denoted $\neg p$, is the proposition

not p

Negation

p	$\neg p$
T	F
F	T

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Operator Precedence

In expressions involving some or all of $\neg$, $\wedge$, and $\vee$, in the absence of parentheses, we first evaluate $\neg$ then $\wedge$ then $\vee$.

Example

Given the proposition p is false, proposition q is true, and the proposition r is false, determine whether the proposition

$$\neg p \vee q \wedge r$$

is true or false.

Operator Precedence

In expressions involving some or all of $\neg$, $\wedge$, and $\vee$, in the absence of parentheses, we first evaluate $\neg$ then $\wedge$ then $\vee$.

Example

We first evaluate $\neg p$, which is true. We next evaluate $q \wedge r$, which is false. Finally we evaluate

$$\neg p \vee q \wedge r$$

which is true.

Conditional Propositions

If p and q are propositions, the proposition

$$\text{if } p \text{ then } q$$

is called a **conditional proposition** and is denoted

$$p \to q$$

The proposition p is called the **hypothesis** (or **antecedent**) and the proposition q is called the **conclusion** (or **consequent**). With respect to operational precedence, the conditional operator $\to$ is evaluated last.

Example

Conditional Proposition

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

Assuming that p is true, q is false, and r is true, find

A $p \wedge q \to r$

B $p \vee q \to \neg r$

C $p \wedge (q \to r)$

D $p \to (q \to r)$

Example

Conditional Proposition

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

Assuming that p is true, q is false, and r is true, find

A $p \wedge q \to r$ is true ($p \wedge q$ is false)

B $p \vee q \to \neg r$ is false ($p \vee q$ is true)

C $p \wedge (q \to r)$ is true ($q \to r$ is true)

D $p \to (q \to r)$ is true ($q \to r$ is true)

Biconditional Propositions

If p and q are propositions, the proposition

$$p \text{ if and only if } q$$

is called a **biconditional proposition** and is denoted

$$p \leftrightarrow q$$

Biconditional Proposition

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

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Logical Equivalence

Suppose that the propositions P and Q are made up of the propositions $p_1, \ldots, p_n$. We say that P and Q are **logically equivalent** and write

$$P \equiv Q$$

provided that, given any truth values of $p_1, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

De Morgan's Laws for Logic

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T

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Converse & Contrapositive

We call the proposition $q \rightarrow p$ the **converse** of $p \rightarrow q$, thus a conditional proposition can be true while its converse is false.

The **contrapositive** (or **transposition**) of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

Example

Example

Write the conditional proposition “If the network is down, then I cannot access the internet” symbolically. Then write the contrapositive and converse in words.

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Write the conditional proposition "If the network is down, then I cannot access the internet" symbolically. Then write the contrapositive and converse in words.

Let

$$p: \text{ The internet is down}$$
$$q: \text{ I cannot access the Internet}$$

Then the conditional proposition is $p \rightarrow q$.

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Write the conditional proposition "If the network is down, then I cannot access the internet" symbolically. Then write the contrapositive and converse in words.

Let

$$p: \text{ The internet is down}$$
$$q: \text{ I cannot access the Internet}$$

Then the conditional proposition is $p \to q$, the contrapositive is $\neg q \to \neg p$ "If I can access the internet, then the network is not down".

Example

Example

Write the conditional proposition "If the network is down, then I cannot access the internet" symbolically. Then write the contrapositive and converse in words.

Let

$$p: \text{ The internet is down}$$
$$q: \text{ I cannot access the Internet}$$

Then the conditional proposition is $p \to q$, the converse is $q \to p$ "If I cannot access the internet, then the network is down".

Logical Equivalence

The conditional proposition and its contrapositive are logically equivalent:

p	q	$p \to q$	$\neg q \to \neg p$	$q \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

"If the network is down, then I cannot access the internet" is logically equivalent to "If I can access the internet, then the network is not down".