Determinants

CIS002-2 Computational Alegrba and Number Theory

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Outline

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An introduction with simultaneous equations

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- This is almost trivial!

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- finding x to be
$$x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}$$

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give a finite value for x provided that $(a_1b_2 - a_2b_1) \neq 0$.
- The equation $(a_1b_2 - a_2b_1)$ is therefore an important one in the solution of simultaneous equations.
- There is a shorthand notation for this

$$(a_1b_2 - a_2b_1) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Notation

- For $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ to represent $(a_1 b_2 - a_2 b_1)$ then we must multiply the terms diagonally to form the product terms in the expansion:

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- i.e. $+ \searrow$ and $- \nearrow$.
- $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the **determinant** of the second order (since it has two rows and two columns), and represents $(a_1 b_2 - a_2 b_1)$

A few exercises

1. $\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix}$
2. $\begin{vmatrix} 7 & 4 \\ 6 & 3 \end{vmatrix}$
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- The final denominator, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \Delta_0$: omit the constant-terms and write the remaining coefficients in the order they stand.
- and our equations becomes

$$\frac{x}{\Delta_1} = -\frac{y}{\Delta_2} = \frac{1}{\Delta_0}$$

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- $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
- Each element in the determinant is associated with its **minor**, which is found by omitting the row and column containing the element concerned.

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- the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ obtained $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & \mathbf{b_2} & \mathbf{c_2} \\ a_3 & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix}$.

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- What is the minor of b_2?
- ... $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

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Evaluation of a third-order determinant

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$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

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- Then we already know how to expand a determinant of the second order by multiplying diagonally, $+ \searrow - \nearrow$

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Simultaneous equations in three unknowns

- consider the following set of equations:

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- If we find x, y, and z by the elimination method, we obtain results that can be expressed in determinant form:

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

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$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

- where we would remember this in the form

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0}$$

Can you see the pattern?

Considering what we have seen with simultaneous equations with two and three unknowns, what would you guess to be the results in determinant form of the following set of equations:

$$
\begin{aligned}
a_1 w + b_1 x + c_1 y + d_1 z + e_1 &= 0\\
a_2 w + b_2 x + c_2 y + d_2 z + e_2 &= 0\\
a_3 w + b_3 x + c_3 y + d_3 z + e_3 &= 0\\
a_4 w + b_4 x + c_4 y + d_4 z + e_4 &= 0
\end{aligned}
$$

Outline

Consistancy of a set of equations

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- If equations are consistent, they have a common solution.

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- Let us consider the follow set of three equations with two unknowns:

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Outline

1. The value of a determinant remains unchanged if rows are changed to columns and columns are changed to rows. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

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2. If two rows (or two columns) are interchanged, the sign of the determinant is changed. $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

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5. If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.
$$\begin{vmatrix} (a_1 + kb_1) & b_1 \\ (a_2 + kb_2) & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$