Eigenvalues and Eigenvectors

CIS008-2 Logic and Foundations of Mathematics

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Outline

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- Eigenvalues are sometimes called characteristic values or latent roots.

Finding the Eigenvalues

Let $\mathbf{A}$ be a linear transformation represented by a matrix $\mathbf{A}$. If there is a vector (column matrix) $\mathbf{x} \in \mathbb{R}^n \neq 0$ such that

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

for some scalar λ, then λ is called the eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{x}$.

If we let **A** be some $k \times k$ square matrix, with eigenvalue λ, then the corresponding eigenvectors satisfy

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$$

We can rearrange the matrix equation

$$\mathbf{Ax} = \lambda \mathbf{x}$$

by sutracting $\lambda \mathbf{x}$ from both sides of the equation to give

$$\mathbf{Ax} - \lambda \mathbf{Ix} = 0$$

where we have explicitly multiplied $\lambda \mathbf{x}$ by the identity matrix $\mathbf{I}$, being $k \times k$ to make the subtraction compatable with the rule for matrix addition. It should be noted that $\lambda \mathbf{Ix} = \lambda \mathbf{x}$ and you should perform this multiplication to convince yourself of it's truth. We can now factorise the left hand side of the matrix equation

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = 0$$

Outline

Cramer's rule

- Consider the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- Now multiply D by x, and use the property of determinants that multiplication by a constant is equivalent to multiplication of each entry in a single column by that constant, so

$$x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

Cramer's rule

- Another property of determinants enables us to add a constant times any column to any column and obtain the same determinant, so add y times column 2 and z times column 3 to column 1,

$$xD = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

where $d_i = a_ix + b_iy + c_iz$ is the charateristic linear equation.

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- If $d_1, d_2, d_3 = 0$ then $xD = 0$ (the case in our eigenvalue problem), and this has non-degenerate solutions (i.e. solutions other than $(0, 0, 0)$) only if $D = 0$. If $d \neq 0$ and $D = 0$ then there are no unique solutions, and if $d \neq 0$ and $D \neq 0$ then solutions exist (but are not of interest to our eigenvalue problem).

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Solution to the Eigenvalue problem

As shown in Cramer's rule, a linear system of equations has nontrivial solutions iff the determinant vanishes, so the solutions of equation $(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = 0$ are given by

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0$$

which is know as the **characteristic equation** for matrix **A**.

Eigenvalues of a 2×2 matrix

Consider a 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The characteristic equation for this matrix is given by

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

the multiplication of the identity by λ is trivial, giving a diagonal matrix with all non-zero elements being λ. since the two matrices $\mathbf{A}$ and $\lambda\mathbf{I}$ are defined to be the same size, subtraction is possible, giving a matrix similar to $\mathbf{A}$ but with different diagonal elements. The resulting determinant would be

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

Eigenvalues of a 2×2 matrix

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

can be calculated directly for such a simple determinant giving

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

which gives a quadratic equation in λ

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

which can be solved by finding the roots to this quatratic equation by iuse of the quadratic formula:

$$\lambda_{\pm} = \frac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{4a_{12}a_{21} + (a_{11} - a_{22})^2}\right]$$

giving two values for λ, termed as two eigenvalues.

Larger matrices

We notice that there are two eigenvalues to a 2×2 system. Similarly, a 3×3 matrix will produce a cubic equation from the characteristic equation, and so will have 3 eigenvalues. And a 4×4 matrix will produce a quartic equation from the characteristic equation, and so will have 4 eigenvalues. We can make the generalisation that a matrix of size d will have d eigenvalues, although some of these eigenvalues may have the same value, we still explicitly state that there are a certain number of eigenvalue that happen to have the same value.
Solution to polynomials of degree greater than two is a non-trivial problem, and generally, root finding algorithms are needed. It is possible, however, to use trial of solutions to find the eigenvalues of matrices of degree 3 or maybe 4. Polynomials will be the next subject in the unit CIS002-2.

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Eigenvectors

Each eigenvalue obtained from the method of the previous section has corresponding to it, a solution of $\mathbf{x}$ called an **eigenvector**. In matrices, the term vector indicates a row matrix or column matrix.

EXAMPLE

Consider the matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, the characteristic equation is

$$\begin{vmatrix} (4-\lambda) & 1 \\ 3 & (2-\lambda) \end{vmatrix} = 0$$

which gives the quadratic equation

$$\lambda^2 - 6\lambda + 5 = 0$$

This particular quadratic if easily factorised to

$$(\lambda - 1)(\lambda - 5) = 0$$

which gives the two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$

EXAMPLE

We can now substitute our eigenvalues back into our origional marix equation to give, for λ_1:

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which gives

$$\begin{bmatrix} 4x_1 + x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which could give two equations, both telling us that whatever the value of x_1, the value of x_2 must be -3 times it. Therefore the eigenvector $\begin{bmatrix} k \\ -3k \end{bmatrix}$ is the general form of an infinite number of such eigenvectors. The simplest eigenvector is therefore

$$x_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

EXAMPLE

We can find x_2 with a similar method, but using the other eigenvalue. Convince yourself that there are two eigenvectors, $x_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ corresponding to the eigenvalue $\lambda_1 = 1$, and $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda_1 = 5$

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Exersises

Find the eigenvaules and eigenvectors of the following Matrices

1. $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

EXERSISES

Find the eigenvaules and eigenvectors of the following Matrices

1. $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ $\lambda_1 = 2,\ \lambda_2 = 3,\ x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
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3. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ $\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 2,\ x_1 = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$
   $x_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$