An Introduction to Matrices

CIS008-2 Logic and Foundations of Mathematics

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Outline

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- Now, finds applications in fields where vast volumes of data are handled.

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- Can be looked at as a "higher-order number system".
- Denoted by bold capital letters with entries denoted by lower case italics, the array of enties is usually enclosed in square brackets, parentheses, or less common double bars.

$$\begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix} \qquad \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \qquad \begin{Vmatrix} \cdots \\ \cdots \\ \cdots \end{Vmatrix}$$

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$$\mathbf{A} = \left[a_{ij}\right]_{mn} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

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- The expression $m \times n$ is the **dimension**, or **order**, of the matrix.

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- If $m = n = 1$, the matrix is reduced to a single entry and becomes a **scalar**.

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- By definition, two matrices are said to be equal if corresponding entries throughout are equal

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- Addition of matrices is defined only for matrices of *equal dimensions* which have the same number of rows and columns, that is, they are **conformable for addition**.

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2. Addition and Subtraction
3. Multiplication
4. Transpose

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- This also means that, in reverse, we can take a common factor out of every element

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- then

$$\mathbf{A.B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} . \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}$$

Matrix Products

Each element in the top row of **A** is multiplied by the corresponding element in the first column of **B** and the products are added. Similarly, the second row of the product is found by multiplying each element in the second row of **A** by the corresponding element in the first column of **B**.

Matrix Product - example

- If $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 5 & 8 \end{bmatrix}$,

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- $\mathbf{A}.\mathbf{B} = \begin{bmatrix} 8 + 10 & 4 + 25 & 3 + 40 \\ 16 + 14 & 8 + 35 & 6 + 56 \\ 24 + 8 & 12 + 20 & 9 + 32 \end{bmatrix} = \begin{bmatrix} 18 & 29 & 43 \\ 30 & 43 & 62 \\ 32 & 32 & 41 \end{bmatrix}$

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- noting that $order(l \times m) \times order(m \times n) \rightarrow order(l \times n)$.

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- If $\mathbf{A} = -\mathbf{A}^T$, the matrix is **skew-symmetric**.

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