CIS009-2, Mechatronics
Robot Coordination Systems I

David Goodwin

Department of Computer Science and Technology
University of Bedfordshire

24^{th} January 2013

Outline

Department of Computer Science and Technology
University of Bedfordshire

Maths basis

MATRIX AND OPERATIONS

A Matrix

- Matrix
  - "A group of numbers arranged in a rectangle which can be used together as a single unit to solve particular mathematical problems"
  - The numbers are called *elements*
  - Elements are arranged in *rows* and *columns*
  - Each element has two *subscripts* to describe its position in a matrix
    - First subscript – index of row
    - Second subscript – index of column

Department of Computer Science and Technology
University of Bedfordshire

MATRIX AND OPERATIONS

An Example of a Matrix

Department of Computer Science and Technology
University of Bedfordshire

- A m-row and n-column matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- 1^{st} subscript – the element is in the 2^{nd} row, 2^{nd} subscript – the element is in the 1^{st} column

- Operations
  - A + B (A and B must have the same number of rows and the same number of columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Department of Computer Science and Technology
University of Bedfordshire

- Multiply a number k with A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$k \cdot A = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

MATRIX AND OPERATIONS

Matrix Multiplication (cross product)

- Multiplication (The number of columns in A must be the same as the number rows of in B)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} \end{bmatrix}$$

$$c_{ij} = (a_{i1} \times b_{1j}) + (a_{i2} \times b_{2j}) + \cdots + (a_{in} \times b_{nj})$$

Department of Computer Science and Technology University of Bedfordshire

MATRIX AND OPERATIONS

Transpose of a Matrix

- Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} \end{bmatrix} = A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

$$a'_{ij} = a_{ji}$$

Inverse of a Matrix: The determinant

- Determinant of a matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

- A determinant has a value
- Value = sum (products starting from the first row) – sum (products starting from the last column)

$$\begin{aligned} |M| =& \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} \\ =& ((1 \times 2 \times 3) + (2 \times 1 \times 2) + (3 \times 1 \times 3)) \\ & - ((3 \times 2 \times 2) + (1 \times 1 \times 1) + (3 \times 3 \times 2)) \\ =& 19 - 31 = -12 \end{aligned}$$

Department of Computer Science and Technology University of Bedfordshire

1. The value of a determinant remains unchanged if rows are changed to columns and columns are changed to rows. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
2. If two rows (or two columns) are interchanged, the sign of the determinant is changed. $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
3. If two rows (or two columns) are identical, the value of the determinant is zero. $\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0$
4. If the elements of any one row (or column) are all multiplied by a common factor, the determinant is multiplied by that common factor. $\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
5. If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged. $\begin{vmatrix} (a_1 + kb_1) & b_1 \\ (a_2 + kb_2) & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

MATRIX AND OPERATIONS

Inverse of a Matrix

- Inverse matrix of a 2d-matrix A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$AA^{-1} = I$$

- I is called a unit matrix (or identity matrix) where all elements in the diagonal line are 1

Department of Computer Science and Technology University of Bedfordshire

The Problem

We could have the following Matrix equation to be solved:

$$\begin{bmatrix} 2 & 4 & 3 \\ 3 & 5 & 6 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

which could be represented as

$$\mathbf{Ax} = \mathbf{b}$$

Considering the interpretation, opposite, this matrix equation could also be written as

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & 3 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$$

We have changed the position of the rows, and left the $\mathbf{x}$ column matrix alone.

The interpretation of the opposite matrix equation can be a set of linear equations:

$$\begin{aligned} 2x + 4y + 3z &= 8 \\ 3x + 5y + 6z &= 7 \\ x + 3y - 2z &= 5 \end{aligned}$$

It is worth noting that each equation has no perticular hierarchical ranking, The set of equations could equally be arranged in any order. i.e.

$$\begin{aligned} x + 3y - 2z &= 5 \\ 2x + 4y + 3z &= 8 \\ 3x + 5y + 6z &= 7 \end{aligned}$$

Finding the cofactors

One solution lay in using the inverse the the matrix, $\mathbf{A}$, to find $\mathbf{x}$

$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{A^{-1}Ax} = \mathbf{A^{-1}b}$$
$$\mathbf{x} = \mathbf{A^{-1}b}$$

So this general method will need to find an inverse of a matrix and then perform a matrix multiplication. The matrix of cofactors, described as a signed version of a matrix of minors, is

$$\mathbf{C} = \begin{bmatrix} 9 & -3 & -2 \\ -28 & 12 & 4 \\ 17 & -7 & -2 \end{bmatrix}$$

The inverse of a matrix envolves finding a matrix of cofactors, $\mathbf{C}$.

$$c_{11} = (-1)^{1+1}((4 \cdot 6) - (5 \cdot 3))$$
$$c_{12} = (-1)^{1+2}((2 \cdot 6) - (3 \cdot 3))$$
$$c_{13} = (-1)^{1+3}((2 \cdot 5) - (3 \cdot 4))$$
$$c_{21} = (-1)^{2+1}((3 \cdot 6) - (5 \cdot -2))$$
$$c_{22} = (-1)^{2+2}((1 \cdot 6) - (3 \cdot -2))$$
$$c_{23} = (-1)^{2+3}((1 \cdot 5) - (3 \cdot 3))$$
$$c_{31} = (-1)^{3+1}((3 \cdot 3) - (4 \cdot -2))$$
$$c_{32} = (-1)^{3+2}((1 \cdot 3) - (2 \cdot -2))$$
$$c_{33} = (-1)^{3+3}((1 \cdot 4) - (2 \cdot 3))$$

Department of Computer Science and Technology University of Bedfordshire

Finding the adjoint and the determinant

The matrix of cofactors is

$$\mathbf{C} = \begin{bmatrix} 9 & -3 & -2 \\ -28 & 12 & 4 \\ 17 & -7 & -2 \end{bmatrix}$$

but we require the adjoint of $\mathbf{A} = adj(A) = C^T$, where the transpose is simply the reflection of the matrix about it's diagonal:

$$adj(\mathbf{A}) = \mathbf{C}^T = \begin{bmatrix} 9 & -28 & 17 \\ -3 & 12 & -7 \\ -2 & 4 & -2 \end{bmatrix}$$

The other ingredient in finding the inverse is finding the determinant of matrix $\mathbf{A}$. Here we can use the signed mionors of the top row of matrix $\mathbf{A}$ which we have already worked out, and we multiply them by the elements in the top row, then to be summed:

$$\begin{aligned} c_{11} &= (-1)^{1+1}((4 \cdot 6) - (5 \cdot 3)) \\ c_{12} &= (-1)^{1+2}((2 \cdot 6) - (3 \cdot 3)) \\ c_{13} &= (-1)^{1+3}((2 \cdot 5) - (3 \cdot 4)) \\ det(\mathbf{A}) &= (9 \cdot 1) + (-3 \cdot 3) + (-2 \cdot -2) \\ &= 4 \end{aligned}$$

Finding the inverse then using it

Department of Computer Science and Technology
University of Bedfordshire

No we have the adjoint and the determinant, we simply divide the adjoint by the determinant to find the inverse matrix:

$$\mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{det(\mathbf{A})}$$

$$= \frac{1}{4}\begin{bmatrix} 9 & -28 & 17 \\ -3 & 12 & -7 \\ -2 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.25 & -7 & 4.25 \\ -0.75 & 3 & -1.75 \\ -0.5 & 1 & -0.5 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$= \frac{1}{4}\begin{bmatrix} 9 & -28 & 17 \\ -3 & 12 & -7 \\ -2 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -15 \\ 8 \\ 2 \end{bmatrix}$$

Therefore $x = -15$, $y = 8$, and $z = 2$

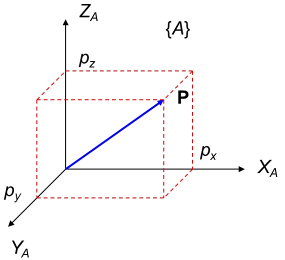
Department of Computer Science and Technology
University of Bedfordshire

Descriptions

DESCRIPTIONS

- Description of a position
- Description of a position in a coordinate system using a vector **P**

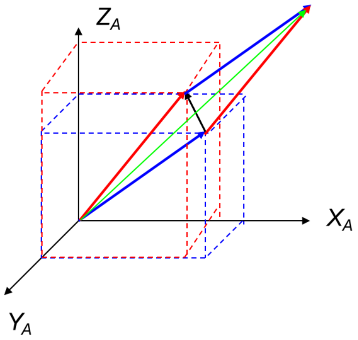
$$^{A}\mathbf{P} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



- {A} is a coordinate system
- p_x, p_y, and p_z are the coordinates of a point in X, Y and Z axes of {A}

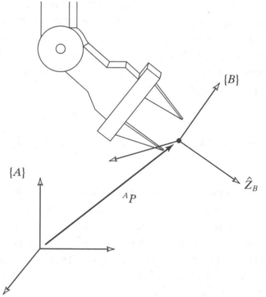
DESCRIPTIONS

- Sum of vectors
- $P = Q - R$
- $S = Q + R$



DESCRIPTIONS

- Description of an orientation
- Vector **P** also gives the orientation of an end-effector
- In robotics, we are more interested in representing the orientation of an end-effector relatively to a coordinate system $\{B\}$ called reference



DESCRIPTIONS

- The way of giving the coordinate system {B} (known as the attached coordinate system) is to write the **unit vectors** of its 3 principle axes in terms of {A}

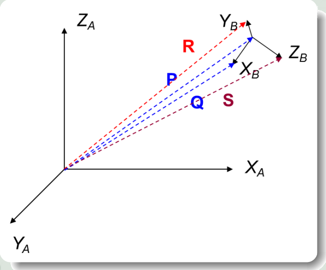
- 3 unit vectors:

$$i_B, j_B, \text{ and } k_B$$

- Using sum of vectors

$$\begin{aligned} i_B &= P - Q \\ j_B &= P - R \\ k_B &= P - S \end{aligned}$$

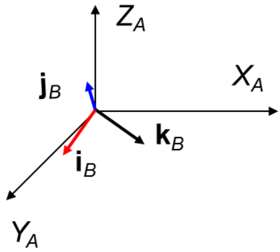
- As P, Q, R and S are vectors defined in {A}, the unit vectors of {B} can be represented with the unit vectors of {A}.



Department of Computer Science and Technology
University of Bedfordshire

DESCRIPTIONS

- Translation
- Shifting {B} towards {A} along P and aligning {B}'s origin to {A}'s origin (This is reasonable as {B} represents orientation only.)
- This process actually involves shifting along ^{A}X, ^{A}Y and ^{A}Z and the corresponding displacements can be represented in a vertical vector P
- This makes the 3 unit vectors become 3 vectors in {A}



Department of Computer Science and Technology University of Bedfordshire

DESCRIPTIONS

- Rotation
- Rotating i_B, j_B and k_B to align them with i_A, j_A and k_A, respectively
- This process yields the following vectors:

$$^A\mathbf{i}_B = (|\mathbf{i}_A||\mathbf{i}_B|\cos\alpha_i)\,\mathbf{i}_A + (|\mathbf{j}_A||\mathbf{i}_B|\cos\beta_i)\,\mathbf{j}_A + (|\mathbf{k}_A||\mathbf{i}_B|\cos\gamma_i)\,\mathbf{k}_A$$

$$^A\mathbf{j}_B = (|\mathbf{i}_A||\mathbf{j}_B|\cos\alpha_j)\,\mathbf{i}_A + (|\mathbf{j}_A||\mathbf{j}_B|\cos\beta_j)\,\mathbf{j}_A + (|\mathbf{k}_A||\mathbf{j}_B|\cos\gamma_j)\,\mathbf{k}_A$$

$$^A\mathbf{k}_B = (|\mathbf{i}_A||\mathbf{k}_B|\cos\alpha_k)\,\mathbf{i}_A + (|\mathbf{j}_A||\mathbf{k}_B|\cos\beta_k)\,\mathbf{j}_A + (|\mathbf{k}_A||\mathbf{k}_B|\cos\gamma_k)\,\mathbf{k}_A$$

Department of Computer Science and Technology University of Bedfordshire

DESCRIPTIONS

- The 3 vectors can be rewritten as:

$$ {}^A\mathbf{i}_B = \begin{bmatrix} |\mathbf{i}_A||\mathbf{i}_B|\cos\alpha_i \\ |\mathbf{j}_A||\mathbf{i}_B|\cos\beta_i \\ |\mathbf{k}_A||\mathbf{i}_B|\cos\gamma_i \end{bmatrix},\quad {}^A\mathbf{j}_B = \begin{bmatrix} |\mathbf{i}_A||\mathbf{j}_B|\cos\alpha_j \\ |\mathbf{j}_A||\mathbf{j}_B|\cos\beta_j \\ |\mathbf{k}_A||\mathbf{j}_B|\cos\gamma_j \end{bmatrix},\quad {}^A\mathbf{k}_B = \begin{bmatrix} |\mathbf{i}_A||\mathbf{k}_B|\cos\alpha_k \\ |\mathbf{j}_A||\mathbf{k}_B|\cos\beta_k \\ |\mathbf{k}_A||\mathbf{k}_B|\cos\gamma_k \end{bmatrix} $$

- Substituting elements of the vectors with r_{mn} yields

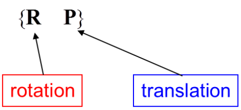
$$ {}^A\mathbf{i}_B = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix},\quad {}^A\mathbf{j}_B = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix},\quad {}^A\mathbf{k}_B = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} $$

- Matrix R (stands for rotation relation from A to B)

$$ {}^A_B\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

DESCRIPTIONS

- Description of a frame
- A frame represents both position and orientation
- It has the form of



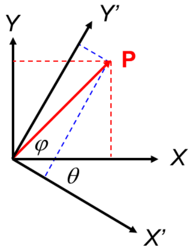
- Or more precisely, as any frame {B} is associated to another frame {A}

$$\{B\} = \{{}^{A}_{B}\mathbf{R} \quad {}^{A}\mathbf{P}_{BORG}\} = \{{}^{A}\mathbf{i}_{B} \quad {}^{A}\mathbf{j}_{B} \quad {}^{A}\mathbf{k}_{B} \quad {}^{A}\mathbf{P}_{BORG}\}$$

Department of Computer Science and Technology University of Bedfordshire

DESCRIPTIONS

- Examples
- 1. Rotation of a X-Y coordinate system to an angle θ



$$\begin{cases} x = |\mathbf{P}|\cos\varphi \\ y = |\mathbf{P}|\sin\varphi \end{cases}$$

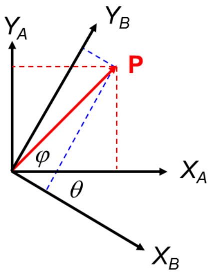
$$\begin{cases} x' = |\mathbf{P}|\sin(90-(\varphi+\theta)) = |\mathbf{P}|\cos(\varphi+\theta) \\ y' = |\mathbf{P}|\cos(90-(\varphi+\theta)) = |\mathbf{P}|\sin(\varphi+\theta) \end{cases}$$

$$\begin{cases} x' = |\mathbf{P}|(\cos\varphi\cos\theta - \sin\varphi\sin\theta) = x\cos\theta - y\sin\theta \\ y' = |\mathbf{P}|(\sin\varphi\cos\theta - \cos\varphi\sin\theta) = y\cos\theta + x\sin\theta \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

DESCRIPTIONS

- This can be extended to a frame

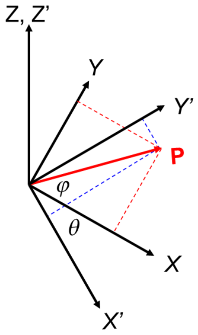


$$ {}^{A}\mathbf{i}_{B} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad {}^{A}\mathbf{i}_{B} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} $$

Department of Computer Science and Technology
University of Bedfordshire

DESCRIPTIONS

- 2. Rotating P about Z-axis by degrees of θ in a X-Y-Z coordinate system



$$\begin{cases} x = |\mathbf{P}|\cos\varphi \\ y = |\mathbf{P}|\sin\varphi \\ z = z \end{cases}$$

$$\begin{cases} x' = |\mathbf{P}|\sin(90-(\varphi+\theta)) = |\mathbf{P}|\cos(\varphi+\theta) \\ y' = |\mathbf{P}|\cos(90-(\varphi+\theta)) = |\mathbf{P}|\sin(\varphi+\theta) \\ z = z \end{cases}$$

$$\begin{cases} x' = |\mathbf{P}|(\cos\varphi\cos\theta - \sin\varphi\sin\theta) = x\cos\theta - y\sin\theta + z0 \\ y' = |\mathbf{P}|(\sin\varphi\cos\theta + \cos\varphi\sin\theta) = y\cos\theta + x\sin\theta + z0 \\ z = x0 + y0 + z \end{cases}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

SUMMARY

- Matrix
- Frames