Experimental Aspects of Amplitude Analysis

[Most of what I say will be generic for any amplitude analysis.
But to give concrete examples I will stick to discussing Dalitz plot analyses, mainly of B decays.
Moreover, I will mostly follow the approach used in Laura++ (arXiv:1711.09854).]

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Starting point

- Previous lectures have described the "physics model"

$$\mathcal{P}_{\rm phys}\left(m_{13}^2, m_{23}^2\right) = \frac{\left|\mathcal{A}\left(m_{13}^2, m_{23}^2\right)\right|^2}{\iint_{\rm DP} \left|\mathcal{A}\left(m_{13}^2, m_{23}^2\right)\right|^2 dm_{13}^2\, dm_{23}^2}$$

- $A(m_{13}^2, m_{23}^2)$ is the amplitude as a function of Dalitz plot position, typically built as a sum of interfering contributions
- The PDF, $P_{\rm phys}$, is normalised to unity when integrated over the Dalitz plot
- "Experimental effects" are the differences between $P_{\rm phys}$ and the Dalitz plot density that we actually observe
- [In practice, it is necessary to develop the model in parallel with treatment of experimental effects. But let's ignore that.]

Efficiency

- Not all decays will be reconstructed due to
  - geometrical acceptance
  - trigger requirements
  - selection efficiency

experiment dependent!

- Efficiency function, $\varepsilon(m_{13}^2, m_{23}^2)$, must be accounted for

$$\mathcal{P}_{\rm sig}\left(m_{13}^2, m_{23}^2\right) = \frac{\left|\mathcal{A}\left(m_{13}^2, m_{23}^2\right)\right|^2 \, \epsilon\left(m_{13}^2, m_{23}^2\right)}{\iint_{\rm DP} \left|\mathcal{A}\left(m_{13}^2, m_{23}^2\right)\right|^2 \, \epsilon\left(m_{13}^2, m_{23}^2\right) \, dm_{13}^2 \, dm_{23}^2}$$

- Due to normalisation, we only need care about efficiency variation across the phase space
  - phase-space independent effects cancel out
  - [not true if we simultaneously determine the inclusive 3-body decay branching fraction]

Resolution

- Determination of momenta is not perfect
  - Reconstructed Dalitz plot position will differ from true position – need to account for this resolution

$$\mathcal{P}^{\rm r}_{\rm sig}\left(m^2_{13\,{\rm r}}, m^2_{23\,{\rm r}}\right) = \int\!\!\int_{\rm DP} R\left(m^2_{13\,{\rm r}}, m^2_{23\,{\rm r}}; m^2_{13\,{\rm t}}, m^2_{23\,{\rm t}}\right)\, \mathcal{P}^{\rm t}_{\rm sig}\left(m^2_{13\,{\rm t}}, m^2_{23\,{\rm t}}\right)\, dm^2_{13\,{\rm t}}\,, dm^2_{23\,{\rm t}}$$

  - "r" and "t" indicate "reconstructed" and "true", respectively
  - $P_{\rm sig}^{\rm t}$ includes efficiency
    - [aside: don't conflate efficiency and resolution!]
  - *R* is the resolution function
    - for any given true Dalitz plot position ($m_{13\,t}^2$,$m_{23\,t}^2$), it is the probability of the decay being reconstructed at ($m_{13\,r}^2$,$m_{23\,r}^2$)
    - completely generic description of *R* is tricky: aim for simplification

experiment dependent!

Background

- Most likely some of the selected events are not from the signal process of interest

experiment dependent!

- Can have various categories of background
  - Typically need to know the Dalitz plot distribution of each

$$\mathcal{P}_{\text{tot}}\left(m_{13}^2, m_{23}^2\right) = \sum_k f_k \mathcal{P}_k\left(m_{13}^2, m_{23}^2\right)$$

  - f_k is the fraction of each category k (signal and all backgrounds)
  - P_k is the Dalitz plot PDF for category k
  - P_{tot} is the observed Dalitz plot density
    - Often do extended maximum likelihood fits

$$\mathcal{L} = e^{-N} \prod_j^{N_c} \left[\sum_k N_k \mathcal{P}_k^j\right]$$

product over N_c candidates
N_k = yield for category k
$N = \Sigma_k N_k$

Experimental effects

- The three main experimental effects are those introduced above, which I will focus on today: efficiency, resolution, background
- In studies of CP violation need to allow that these can differ for B and $\overline{B}$ decays
  - may also need to account for production asymmetry
    - negligible for $B\overline{B}$ production at Υ(4S)
    - for $D\overline{D}$ may need to account for dependence on polar angle
- Depending on analysis details, may also have additional effects
  - e.g. flavour tagging, decay-time resolution for decay-time-dependent analyses
    - if decay time uncorrelated with Dalitz plot position these factorise (i.e. can treat in same way as for 2-body decay)
    - if not, things become very complicated ...

Lepton vs. hadron colliders

- Essentially two categories of collider experiments
  - (at least, those relevant here)
- e^+e^- colliders (CLEOc, BESIII, BaBar, Belle, Belle II, etc.)
  - produce meson-antimeson pair in coherent state
- hadron colliders (CDF, D0, ATLAS, CMS, LHCb, etc.)
  - produce hadrons from various mechanisms, such as gluon splitting
- What are relative advantages and disadvantages of the two approaches?
  - consider: yield, efficiency, resolution, background

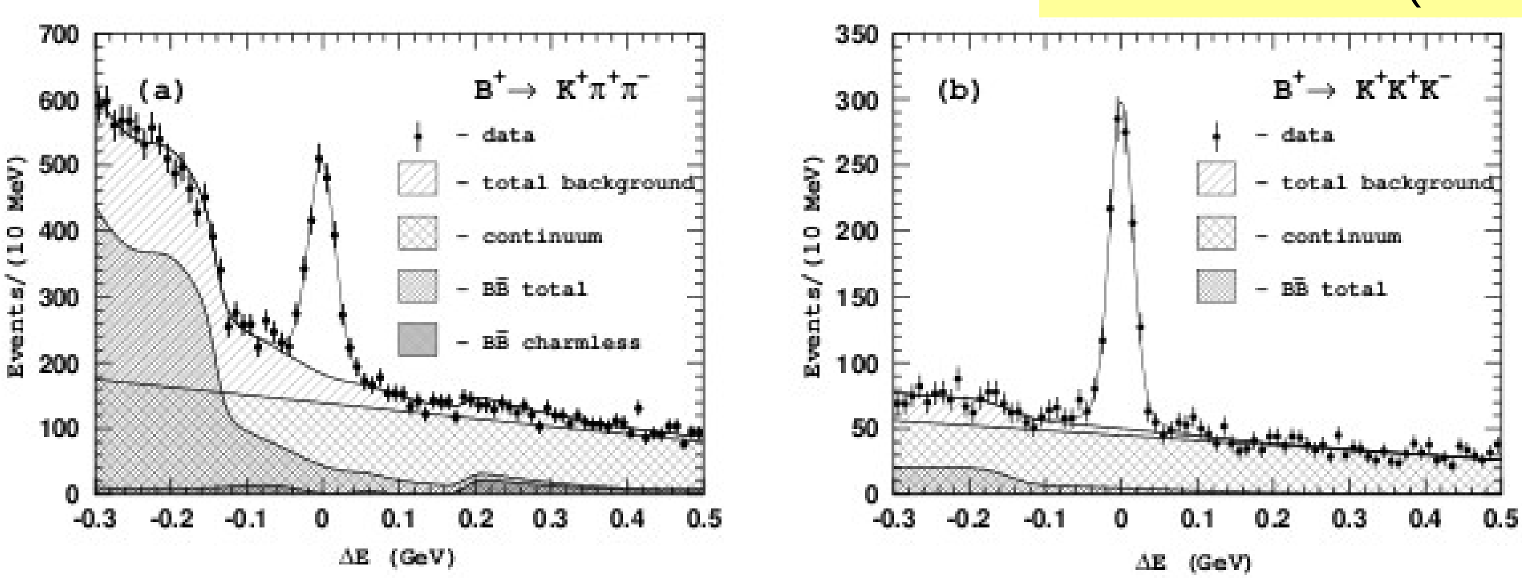
Backgrounds

- Do you expect the background to be lower in lepton or hadron colliders?
  - It depends (of course …)
  - Overall multiplicity much lower in e^+e^- collisions
    - very low backgrounds if you reconstruct everything in the event
    - but if signal is, e.g., B meson from Υ(4S) decay, still have background from “the rest of the event”
  - Particles produced in hadron collisions have high momenta
    - can efficiently reduce background using variables related to flight distance and transverse momenta
    - extreme example: charged kaon beams

Background fractions and distributions

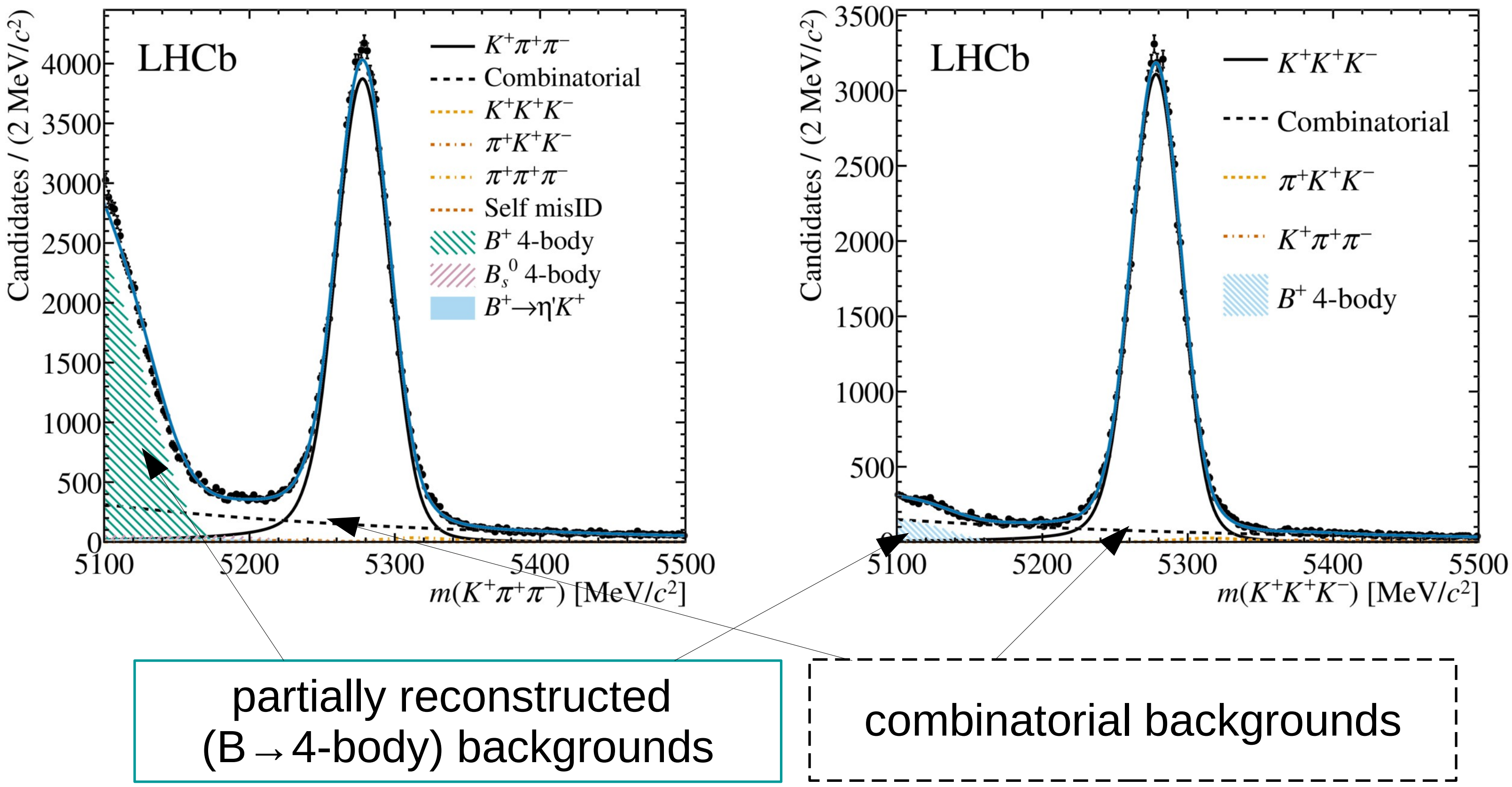
- It is usually possible to determine the background fraction by fitting some kinematic variable (e.g. invariant mass)
  - Can be done prior to, or simultaneously with, the fit to the Dalitz plot
- The background distribution can then be studied from sidebands of this variable
  - Care needed: background composition may be different in the signal and sideband regions

Belle PRD71 (2005) 092003



B→Kππ & B→KKK at LHCb

LHCb arXiv:2010.11802

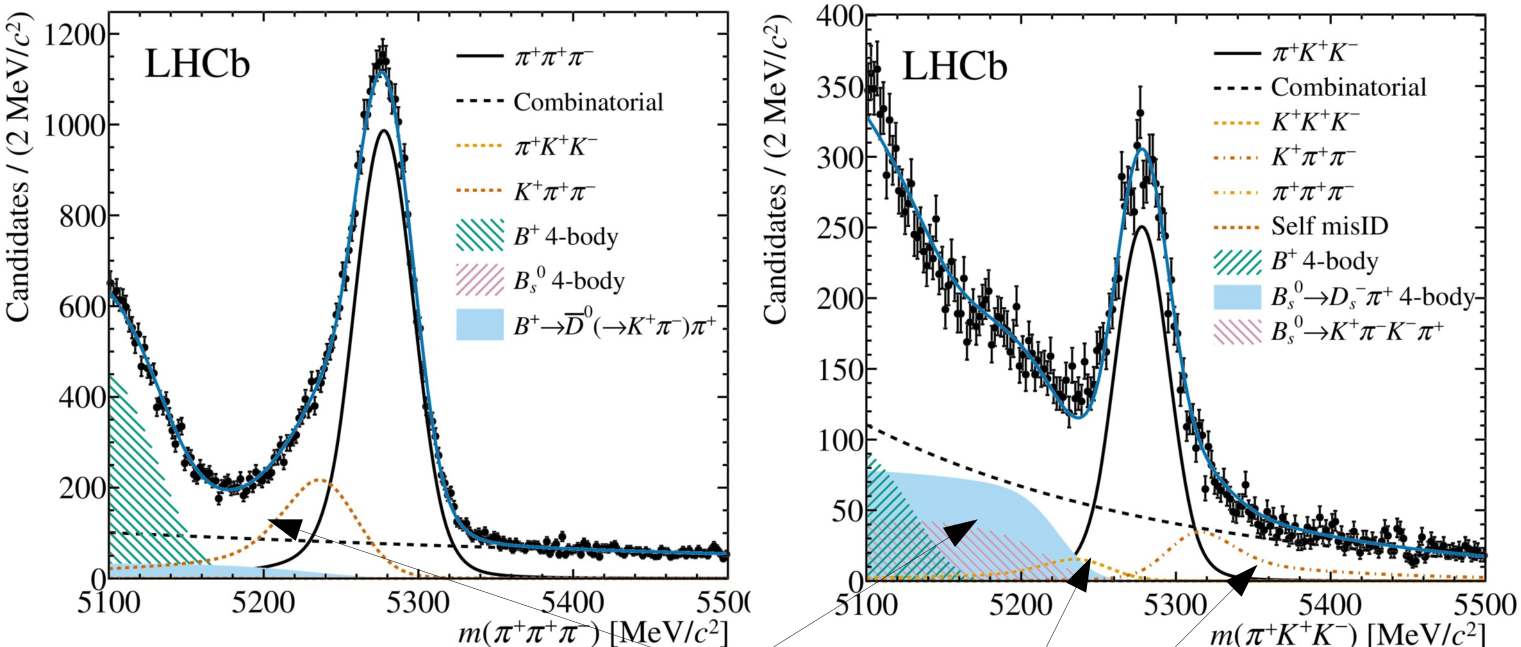


These channels are almost ideal for LHCb
Not everything is as clean as this ;(

Aside: note that the ΔE variable used by B factory experiments corresponds to reconstructed mass used by hadron collider experiments

B → πππ & B → πKK at LHCb

LHCb arXiv:2010.11802

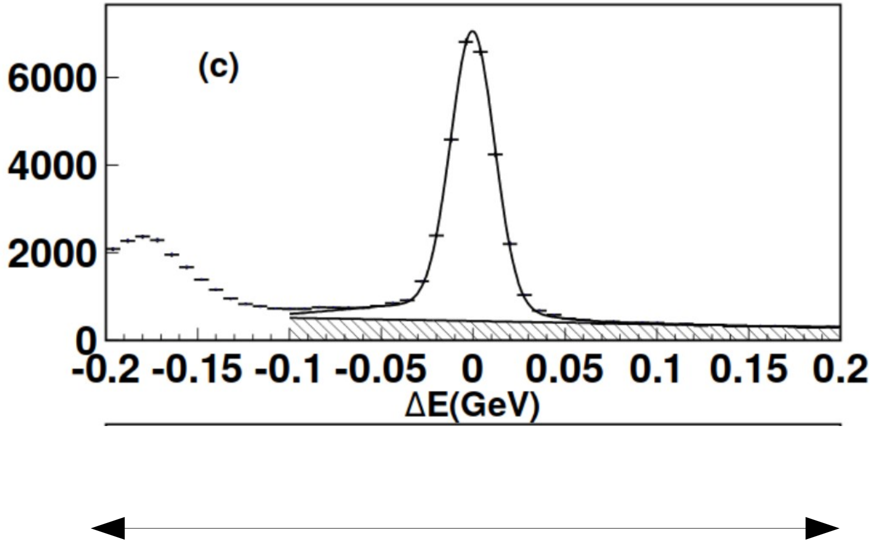


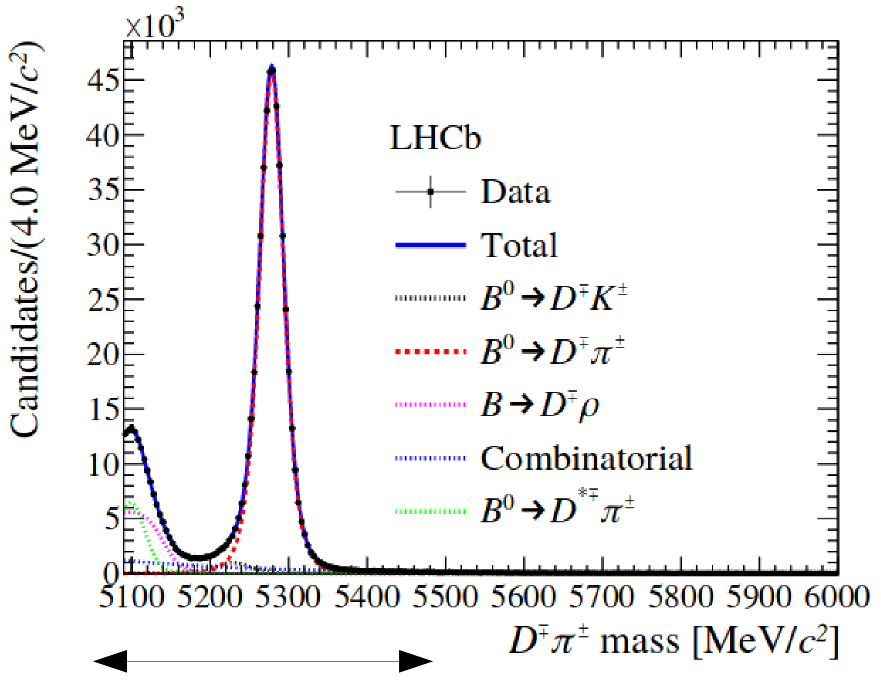
$B_s \to$ 4-body background

misidentified backgrounds

Vertexing kills background

Comparison of (left) Belle and (right) LHCb signals for $B^0 \rightarrow D^- \pi^+$



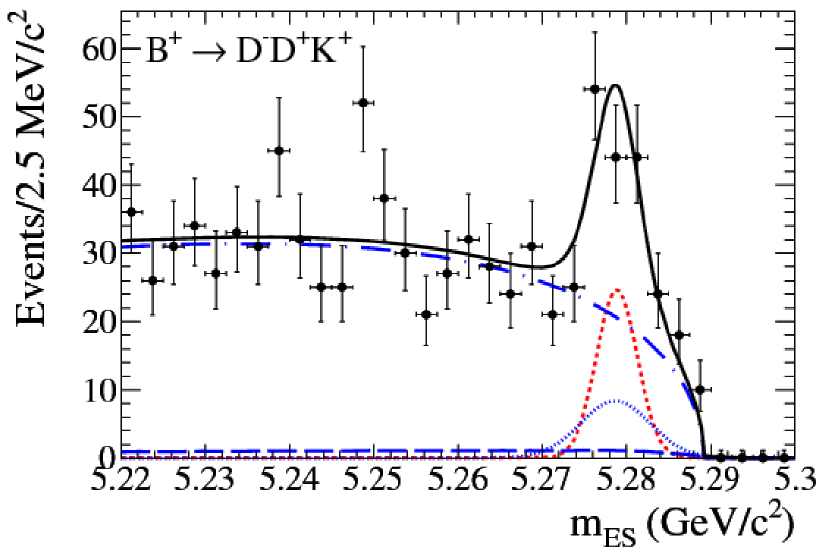


Note difference x-axis ranges.

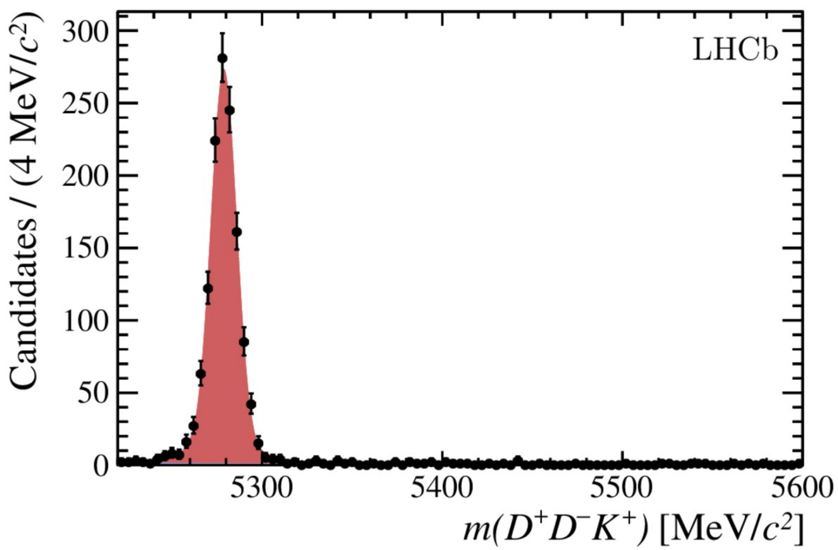
Multiple vertices ↔ better background killing

Comparison of (left) BaBar and (right) LHCb signals for $B^+ \to D^+ D^- K^+$

BaBar arXiv:1011.3929



LHCb arXiv:2009.00026

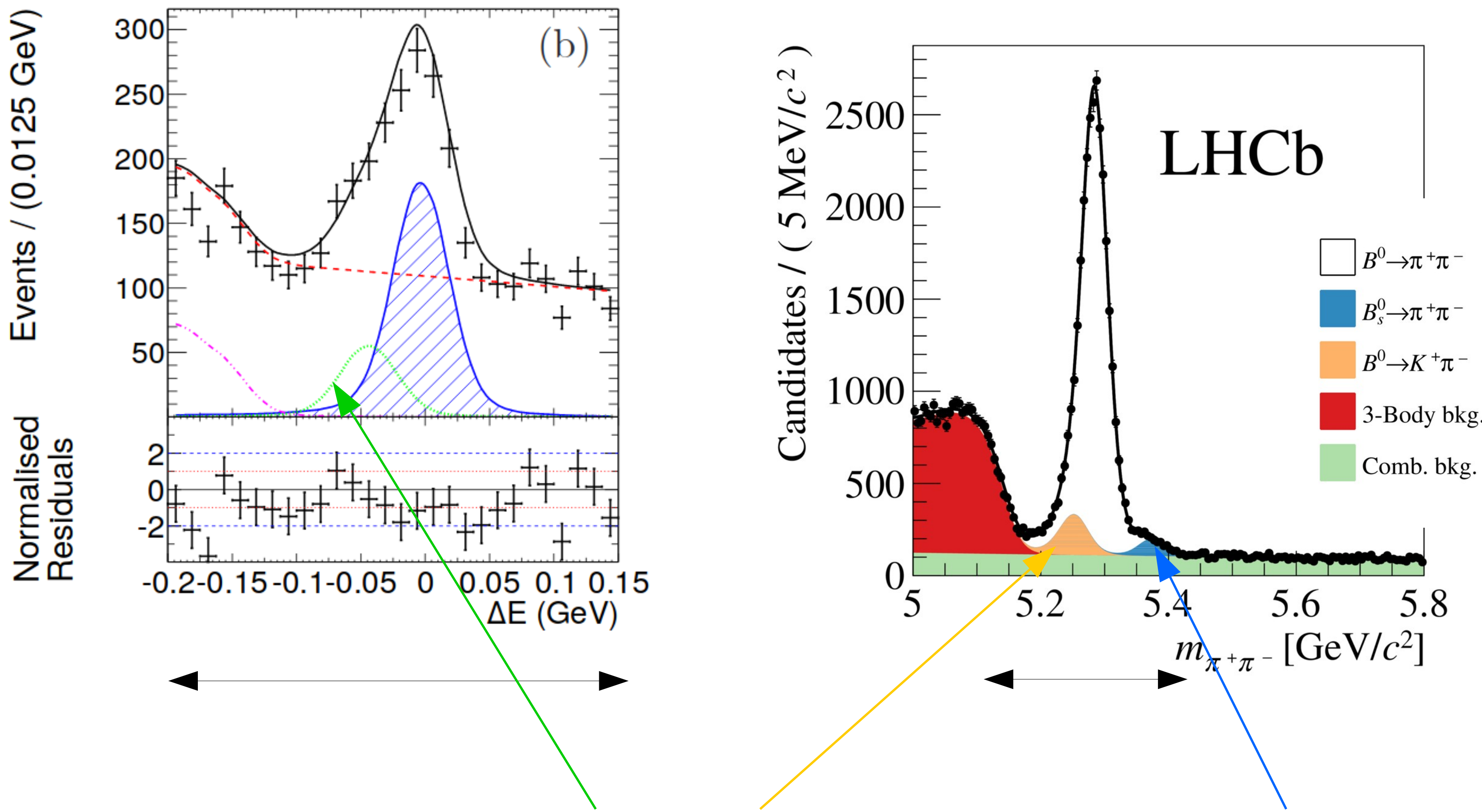


m_{ES} rather than ΔE plotted

99% purity!

Particle ID kills other backgrounds

Comparison of (left) Belle and (right) LHCb signals for $B^0 \to \pi^+\pi^-$



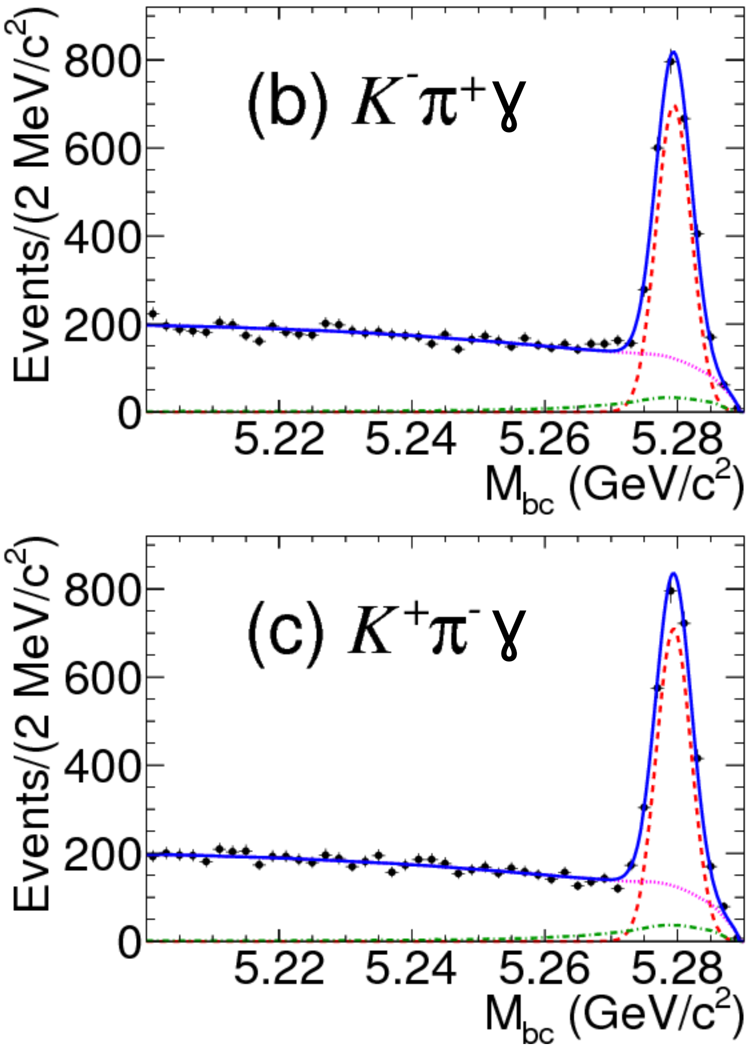
$B^0 \to K^+\pi^-$ background

$B_s^0 \to \pi^+\pi^-$ not present in Belle

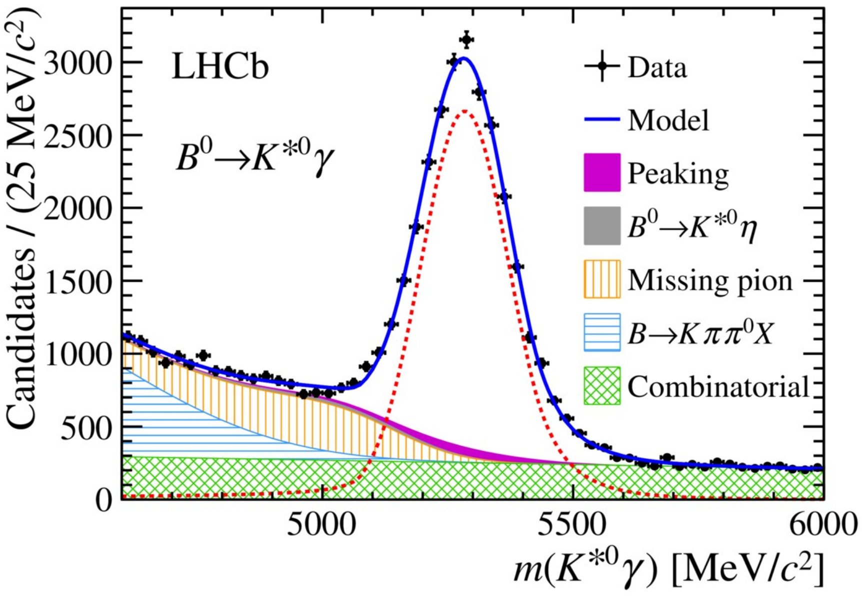
Modes with neutrals

worse resolution (particularly for LHCb)

Belle arXiv:1707.00394



LHCb arXiv:1609.02032

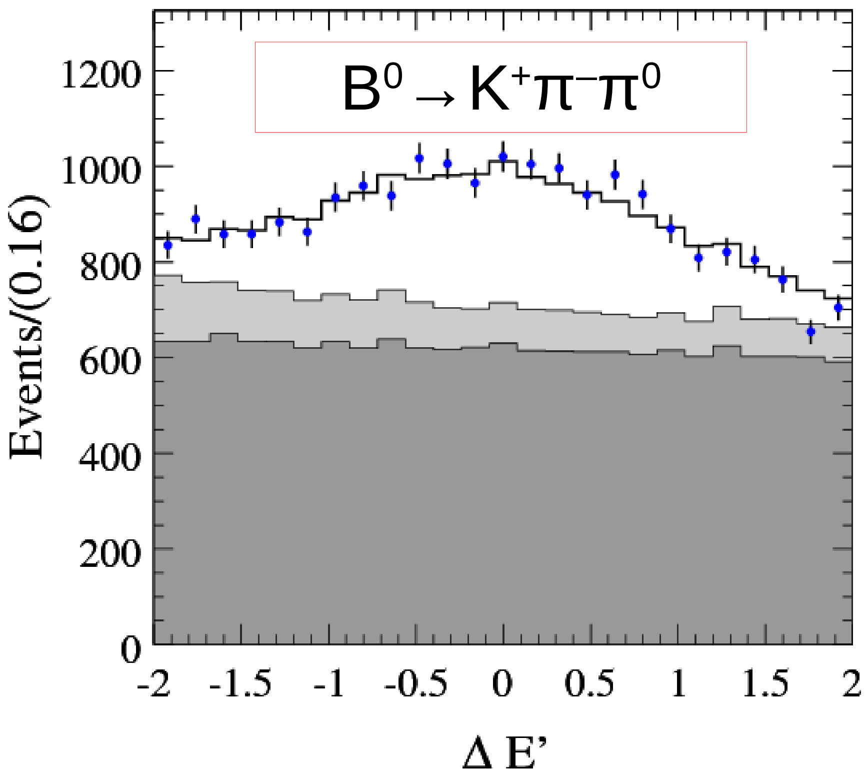


Modes with neutrals

worse resolution (particularly for LHCb)

BaBar arXiv:1105.0125

LHCb … nothing



Note: ΔE' instead of ΔE

Comments on mass fitting

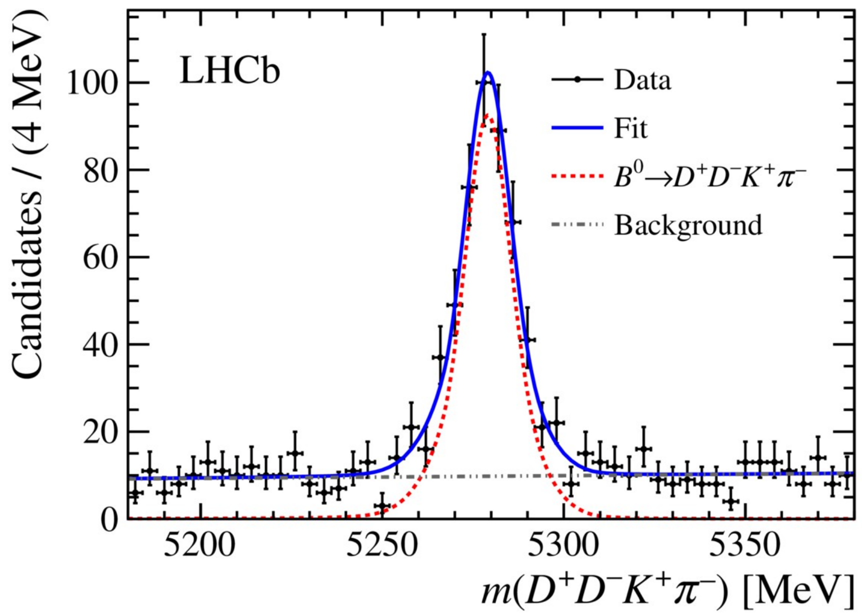
- In most amplitude analyses, the B candidate mass fit is done separately to the Dalitz plot fit
  - The yields of signal and background components, within a defined signal window, are then inputs to the Dalitz plot fit
- Why not do a simultaneous fit to both?
  - Advantage: better signal/background separation
    - Event-by-event information instead of per-category yields
  - Disadvantage: impact of correlations between variables
    - May require wider window, containing more background
- Same arguments hold when other variables that discriminate between signal and background are included
- Correlations may be unavoidable in some cases
  - e.g. if signal B mass resolution depends on Dalitz plot position
    - may be possible to transform to a decorrelated variable (e.g. $\Delta E'$)

Avoiding background modelling

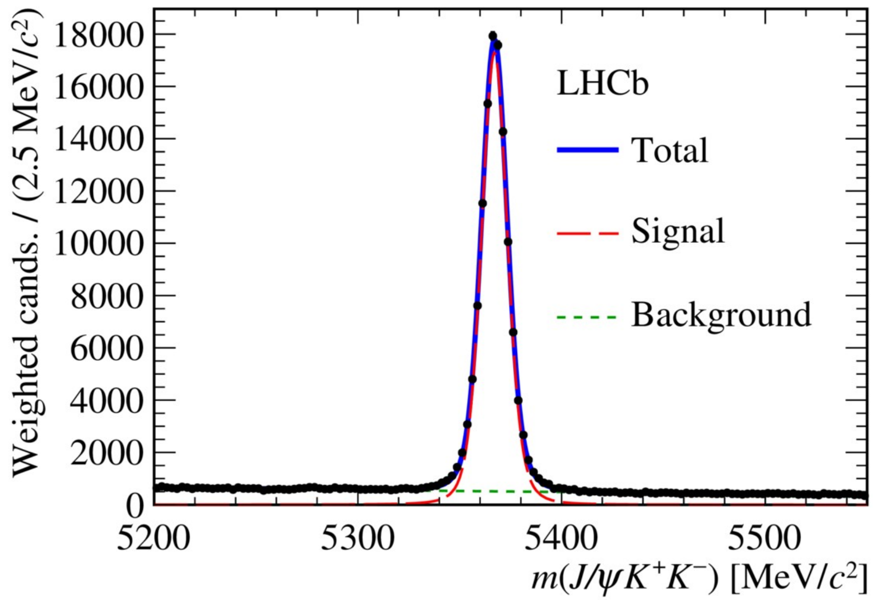
- In unbinned fits typically need to model explicitly the Dalitz plot distributions of background components
- Can avoid this in certain cases with statistical subtraction
  e.g. sFit (arXiv:0905.0724)
  - apply weights obtained from fit that determines signal and background yields (e.g. B candidate mass fit)
  - method requires that B candidate mass be uncorrelated with Dalitz plot position for all components
  - only true in simple cases
    - e.g. combinatorial background only (still an approximation, not perfect)
    - e.g. not true for misidentified background components
    - range of B candidate mass fit becomes an important consideration
- Also: weighted likelihood fits are not likelihood fits!
    - care needed over uncertainties (see, e.g., arXiv:1911.01303)

Examples where sFit works

$B^0 \to DDK\pi$ arXiv:2011.09112



$B_s \to J/\psi KK$ arXiv:1906.08356



A “cheat” in this analysis: $\Lambda_b \to J/\psi pK$ background removed by adding negatively weighted MC

Note symmetrical mass fit ranges

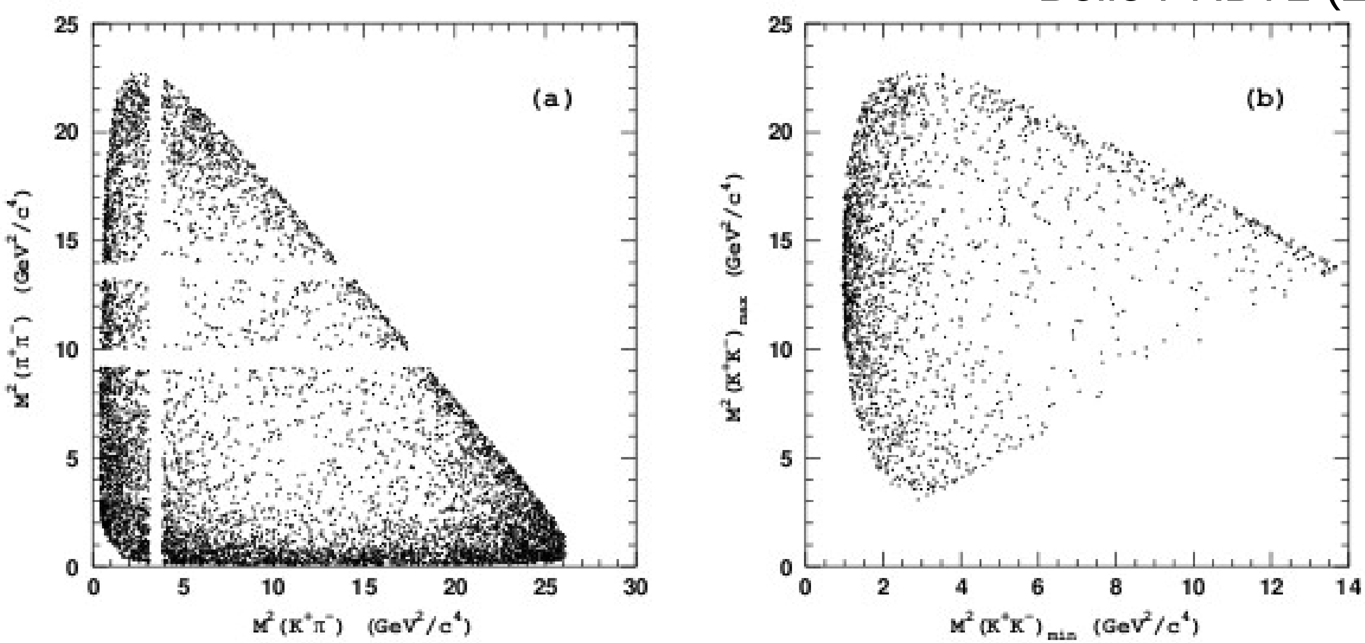
Avoiding background modelling

- In unbinned fits typically need to model explicitly the Dalitz plot distributions of background components
- Can avoid this in certain cases with statistical subtraction
- Other (crude) way to avoid modelling: tight cuts
  - optimisation of cuts should account systematic as well as statistical uncertainties
    - "significance x purity" figure of merit often appropriate for Dalitz plot analyses
    - i.e. $[S/\sqrt{(S+B)}] \times [S/(S+B)] = S^2/(S+B)^{3/2}$

Background distribution issues

- In a binned fit, the background can be subtracted
- In an unbinned fit, the background PDF must be described, either
  - parametrically (usually some smooth function plus incoherent sum of narrow states)

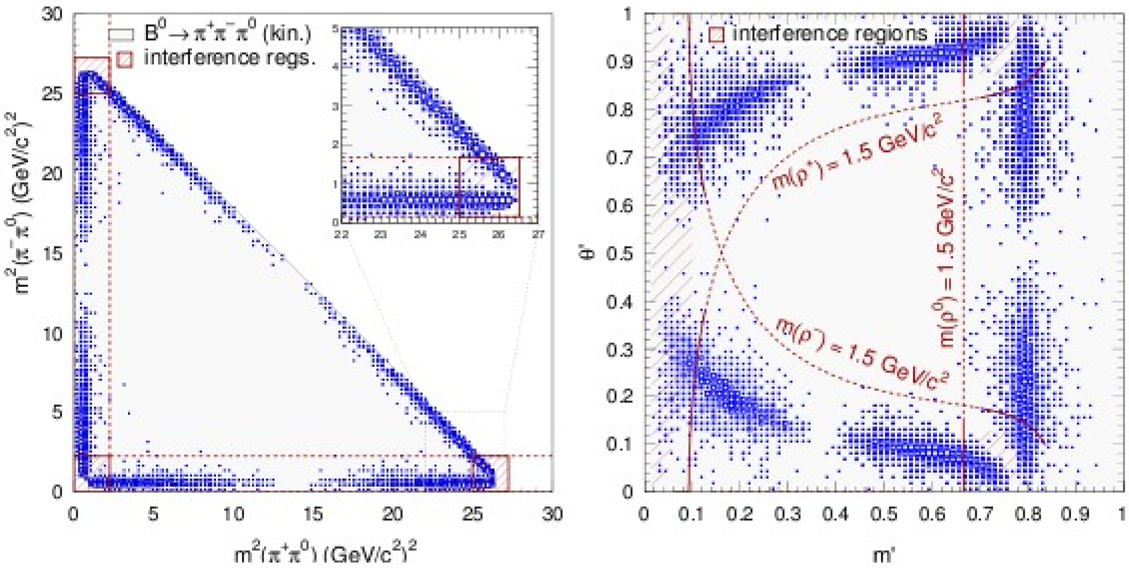
Belle PRD71 (2005) 092003



Background distribution issues

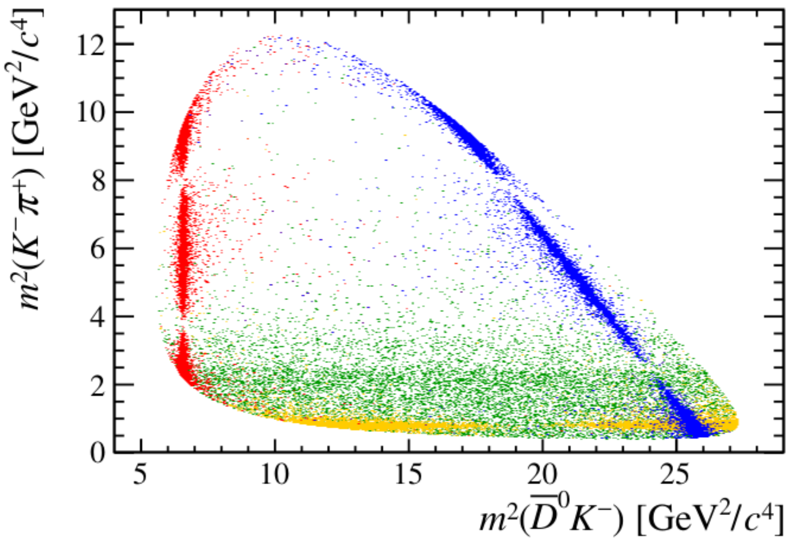
- In a binned fit, the background can be subtracted
- In an unbinned fit, the background PDF must be described, either
  - nonparametrically (usually as a histogram)
    - since background tends to cluster near DP boundaries, advantageous to use “square Dalitz plot”

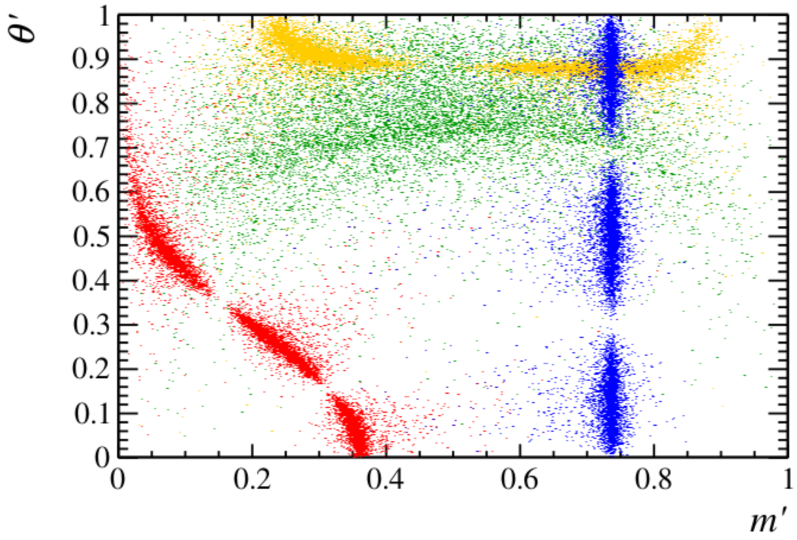
BaBar PRD 76 (2007) 012004



Square Dalitz plot

A less symmetrical example



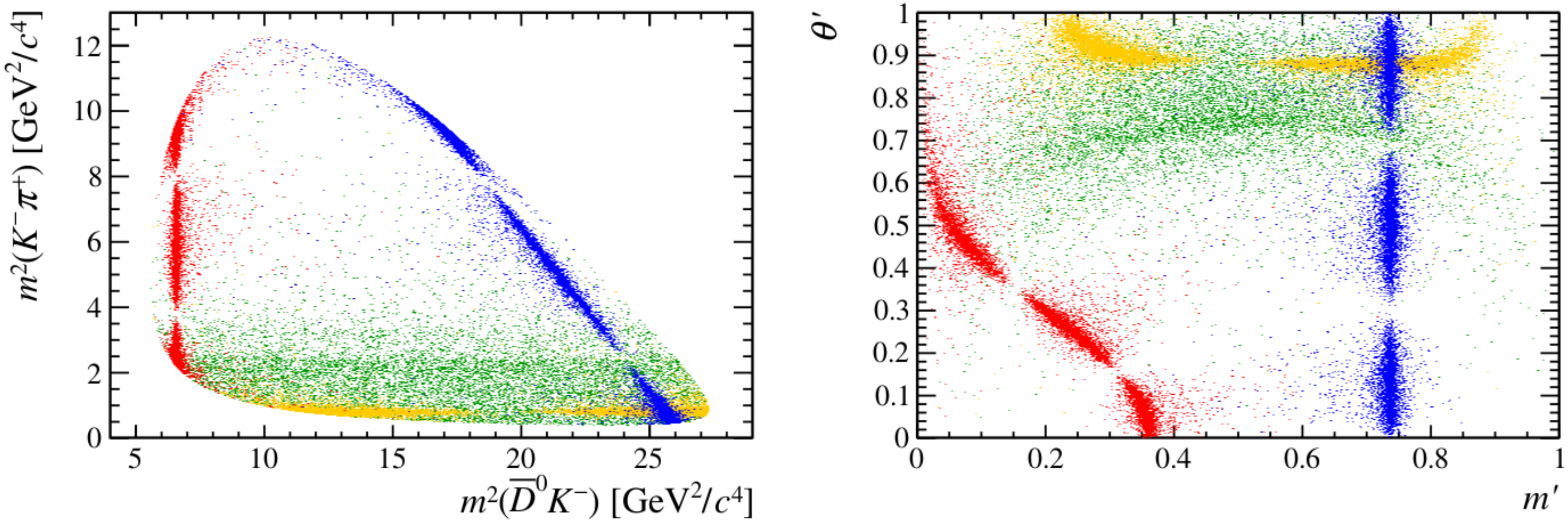


- phase-space of the Dalitz plot is flat for any choice of axes
- but that of the square Dalitz plot is not
  - moreover, it differs (in general) for the 6 possible choices of axes (m’,θ’)
  - appropriate and clear SDP definition is important

$$m' = \frac{1}{\pi}\arccos\left(2\frac{m_{ij}-(m_i+m_j)}{m_B-(m_i+m_j+m_k)}-1\right),$$

$$\theta' = \frac{1}{\pi}\left(\frac{m_{ij}^2(m_{jk}^2-m_{ij}^2)-(m_j^2-m_i^2)(m_B^2-m_k^2)}{\sqrt{(m_{ij}^2+m_i^2-m_j^2)^2-4m_{ij}^2m_i^2}\sqrt{(m_B^2-m_k^2-m_i^2)^2-4m_{ij}^2m_k^2}}\right)$$

Square Dalitz plot



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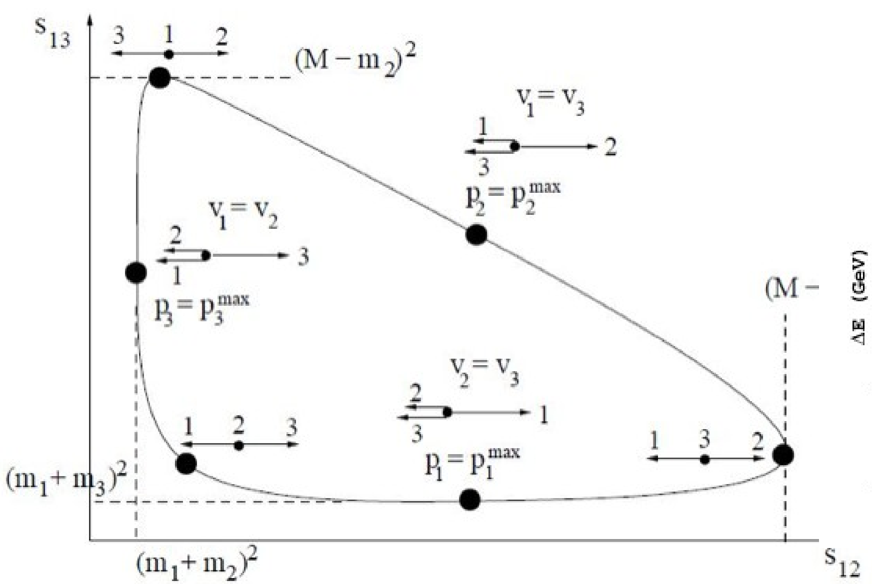
- but that of the square Dalitz plot is not
  - moreover, it differs (in general) for the 6 possible choices of axes (m’,θ’)
  - appropriate and clear SDP definition is important

Looks complicated but
m’ is just a transform of m_{ij} & θ’ is just a transform of θ_{ij}

Choice of which particles are “i” and “j” defines which phase space regions are zoomed into

Background distribution issues

- Boundary of Dalitz plot depends on 3-body invariant mass
  - To have a unique DP, and to improve resolution for substructure, apply 3-body mass constraint
  - This procedure distorts the background shape
    - noticeable if narrow resonances are present in the sideband
    - can be alleviated by averaging upper and lower sidebands (not always possible)
    - alternative: smart choice of sidebands (not always possible)



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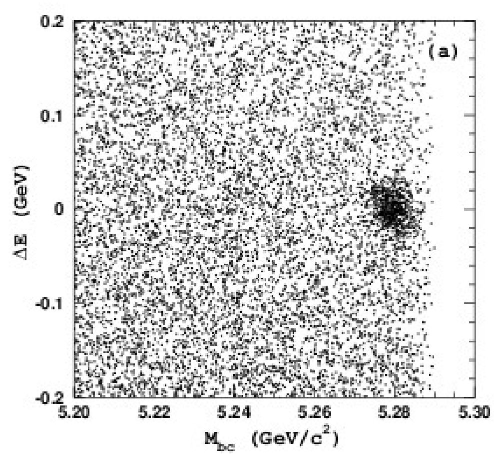

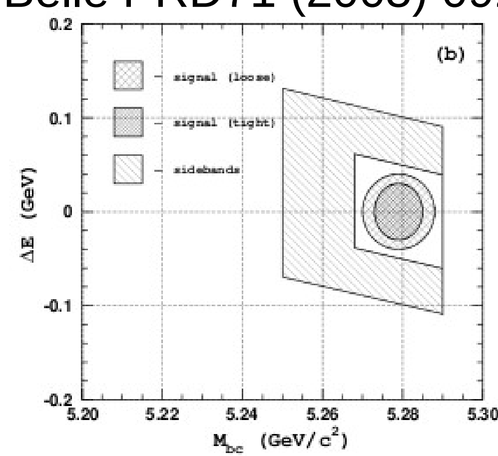

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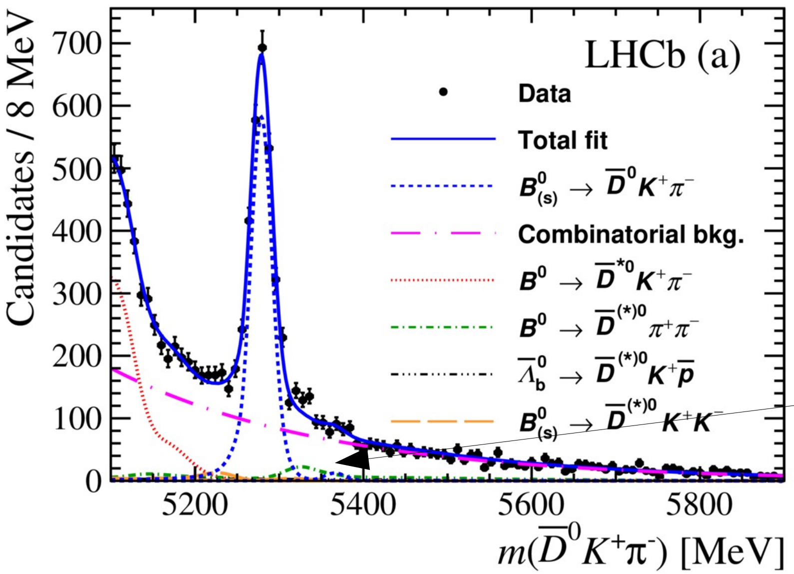
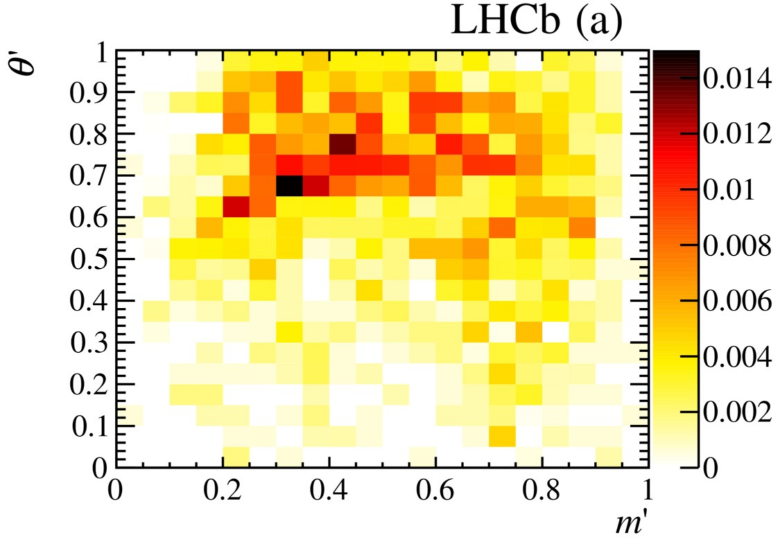
    - noticeable if narrow resonances are present in the sideband
    - can be alleviated by averaging upper and lower sidebands (not always possible)
    - alternative: smart choice of sidebands (not always possible)
    - alternative: account for distortion (arXiv:1902.01452)

Background modelling example

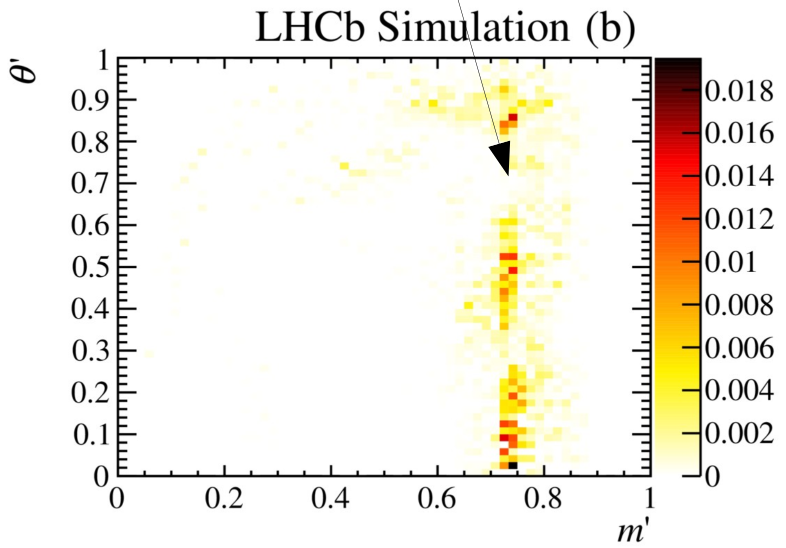
$B^0 \to DK\pi$ arXiv:1505.01505



Small background can have important impact if clustered in certain DP regions



Combinatorial background from high m(DKπ) sideband



Misidentified $B^0 \to D\pi\pi$ from weighted MC

Efficiency

Efficiency

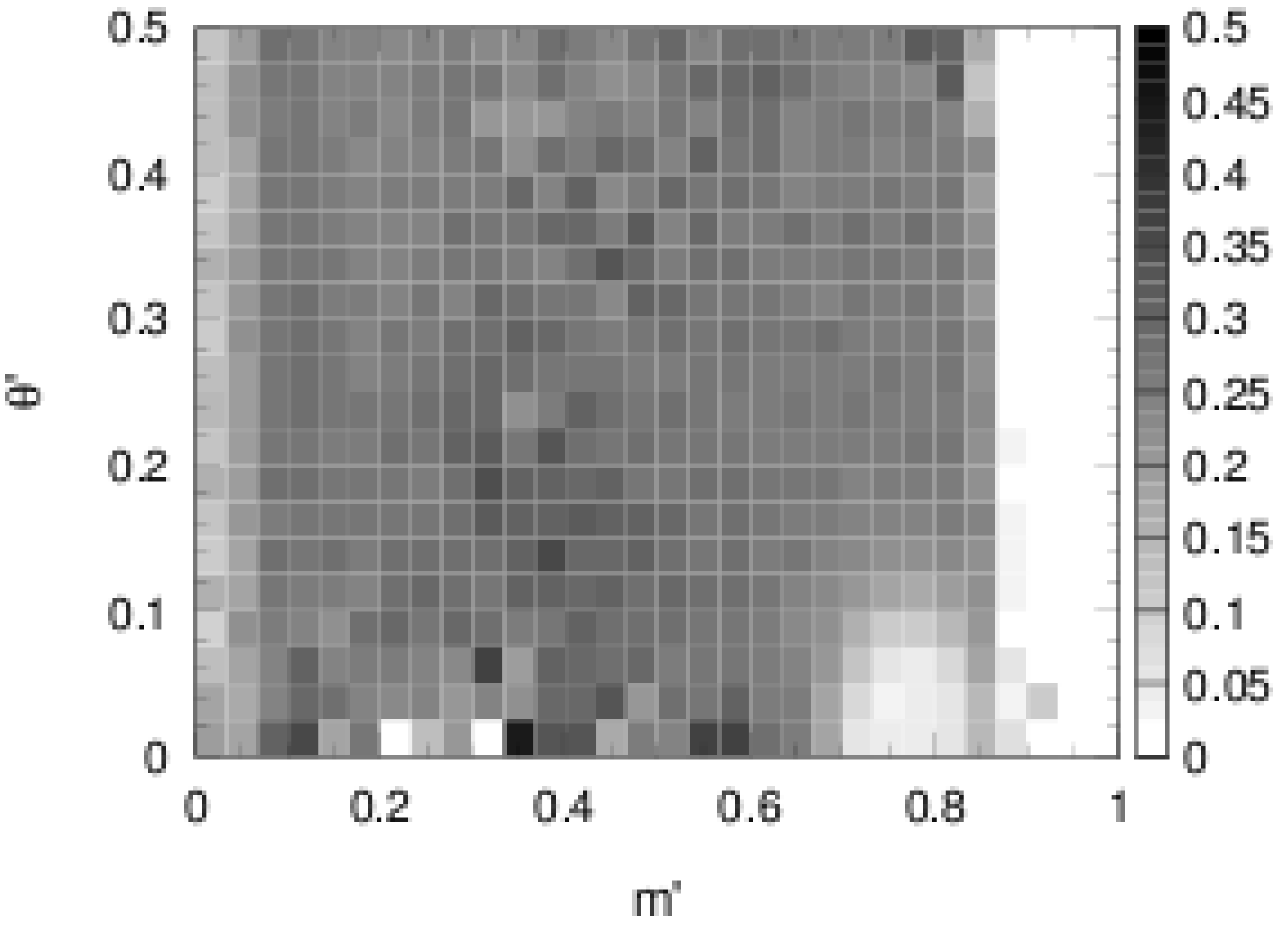
- Key point:
  - If the efficiency is uniform across the phase-space, we can ignore it in the maximum likelihood fit
- Efficiency non-uniformity must be accounted for
  - Choose selection variables to minimise effect
    - can design MVA to do this (e.g. μBoost arXiv:1305.7248)
    - may not help if dominant effect due to acceptance or trigger
  - Determine residual variation from Monte Carlo simulation (validated/corrected using data where possible)
  - Can either
    - explicitly correct for efficiency (event-by-event)
      - usually implemented as a histogram (using square DP or otherwise)
    - determine overall effect from MC simulation with same model parameters
      - only viable approach for high-dimensional problems

Breaking down efficiency

- For LHCb, conventional to consider separately different contributions
  - geometrical acceptance
    - can be evaluated without detector simulation
    - relatively small effect for Belle II
  - trigger (especially hardware trigger) efficiency
    - difficult to determine reliably from simulation → use data-driven methods
    - n.b. hardware trigger will be removed for Run 3 and beyond
  - offline selection
    - use full simulation (data driven corrections applied in some cases)
  - charged hadron particle identification
    - difficult to determine reliably from simulation → use data-driven methods
- For Belle II probably everything can be determined from MC
    - will still require careful calibration and validation of simulation
    - vetoes should still be considered separately

Example of efficiency variation

BaBar $B \to \pi^+\pi^-\pi^0$ PRD 76 (2007) 012004



Choice of binning scheme is important:

require sufficient granularity to be sensitive to variation

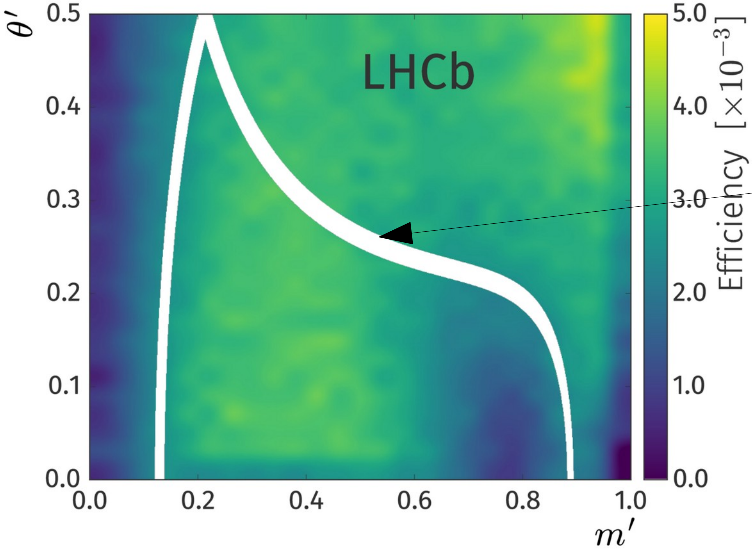
but want to avoid sparsely populated bins

smoothing is also possible

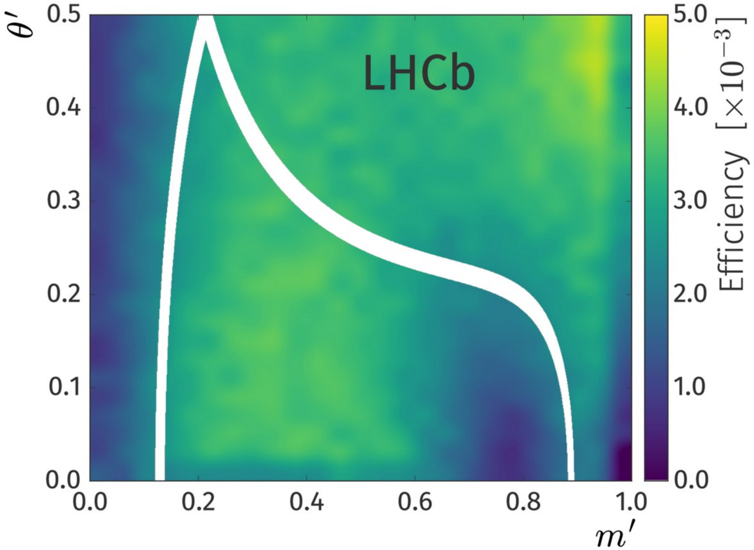
Example of efficiency variation

LHCb $B^+ \to \pi^+\pi^-\pi^+$ arXiv:1909.05212

B⁺



B⁻



Vetoes of specific backgrounds (here $B^+ \to \overline{D}^0\pi^+$) may be necessary

Better to treat separately as these do not vary smoothly across the Dalitz plot

Visual effect of square Dalitz plot can exaggerate efficiency variation, due to zoom into certain phase-space regions

Resolution

Resolution and misreconstruction

- Key point:
  - If resolution is << width of narrowest structure on the Dalitz plot, we can ignore it
- Applying 3-body mass constraint helps, but
  - Some Dalitz plots contain narrow structures (ω, φ, D^*)
  - Misreconstruction effects ("self-cross-feed") can lead to significant non-Gaussian tails
    - complicated smearing of events across the Dalitz plot
    - hard to model
    - relies on Monte Carlo simulation – hard to validate with data
    - significant for states with multiple soft particles at B factories

Parametric resolution

- Simple case
  - a single narrow state, far from edges of Dalitz plot, and far from any other peaking structure
  - can convolve with a Gaussian (in relevant range of m_{ij}^2)
- Cannot do this when
  - narrow state close to some other peaking structure
    - will get discontinuity at boundary where resolution is included/not included
  - narrow state is close to edge of Dalitz plot
    - resolution will be asymmetric and depend on true position
    - (kinematic constraint → cannot smear position outside DP boundary)
- Often simplest to veto narrow states that do not significantly overlap with other structures (e.g. D*, J/ψ)
    - interference effects may be negligible in any case
  - or ignore resolution and account for as systematic uncertainty

Self-cross feed

- Separate signal into well reconstructed and misreconstructed components
  - Resolution negligible for the former

$$\mathcal{P}_{\mathrm{sig-scf}}(s_{\mathrm{reco}}) = [1 - f_{\mathrm{scf}}(s_{\mathrm{reco}})]\, \mathcal{P}_{\mathrm{sig}}(s_{\mathrm{reco}}) + \int f_{\mathrm{scf}}(s_{\mathrm{true}})\, w_{\mathrm{scf}}(s_{\mathrm{reco}}, s_{\mathrm{true}})\, \mathcal{P}_{\mathrm{sig}}(s_{\mathrm{true}})\, ds_{\mathrm{true}}$$

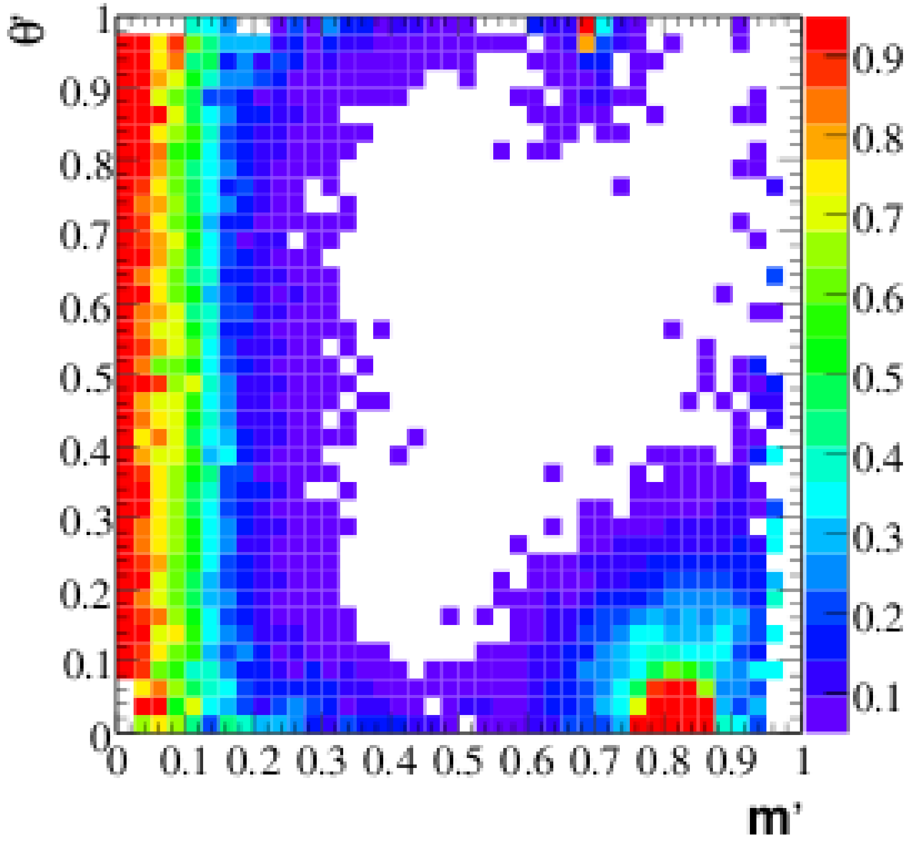
self-cross-feed fraction as function of Dalitz plot position

well reconstructed component so $s_{reco} = s_{true}$ (resolution is delta function)

smearing function from true to reconstructed Dalitz plot position

Example SCF fraction

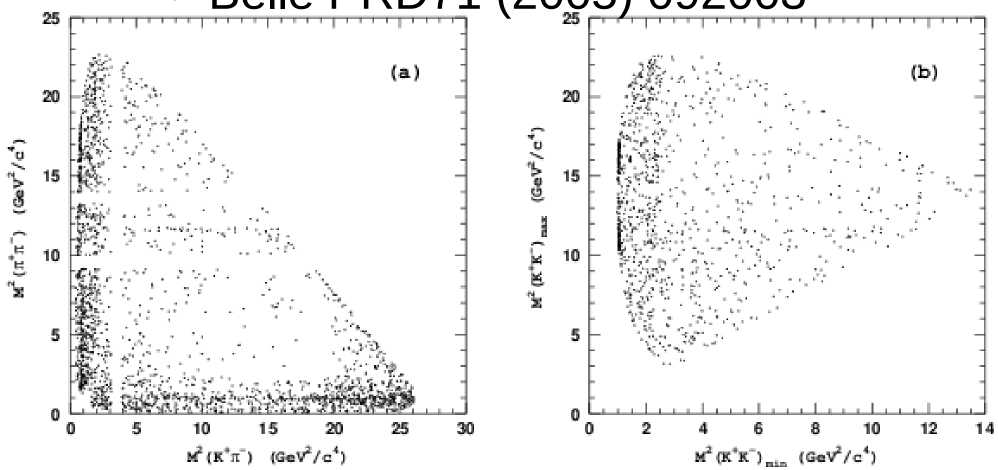
BaBar $B \to K^+\pi^-\pi^0$ PRD 78 (2008) 052005



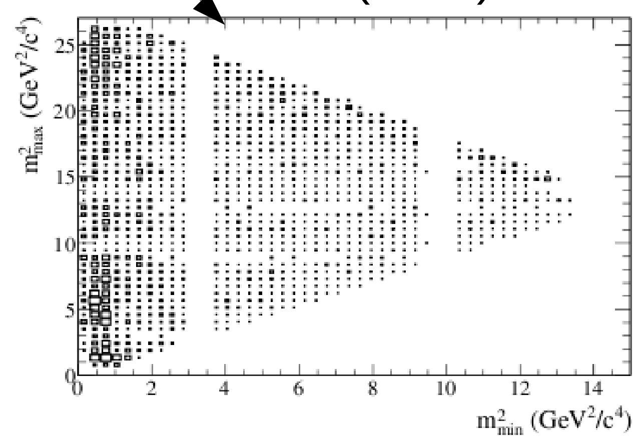
Visualisation of the Dalitz plot

- Obviously important to present the data to the world
- How to present it?
  - 2D scatter plot of events in the signal region
    - unbinned, hence most information
    - but contains background and not corrected for efficiency
  - Binned 2D (or 1D) projections
    - can correct for background and efficiency
      - sPlots is a useful tool
    - but tend to wash out some of the fine structure

Belle PRD71 (2005) 092003



BaBar PRD 79 (2009) 072006



Summary

- Amplitude analysis can require detailed understanding of many subtle effects
  - includes not only the physics model …
  - … but also the experimental effects
- As size of available samples increases, need to continually refine understanding of these effects
  - new and improved methods continually under development
  - speed of fitting code also becomes critical
    - many packages now using GPU
- Looking forward to many amplitude analyses from Belle II!

THE END

Parametrisations

- Fit parameters are complex coefficients of the contributing amplitudes
  - allowing for CP violation, 4 parameters for each
    - usually necessary to fix (at least) two reference parameters
  - many possible parametrisations
    - $r \exp(i\delta) \rightarrow (r \pm \Delta r) \exp(i(\delta \pm \Delta\delta))$
    - $r \exp(i\delta) \rightarrow r \exp(i\delta)\,(1 \pm \Delta\rho \exp(i\Delta\varphi))$
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      - #parameters explodes for >3 resonances

Maximum likelihood fit

$$L = \prod_{i=1}^{N} P_i$$

$$-2\ln L = -2\sum_{i=1}^{N} \ln(P_i)$$

$$P_i = P_{i,sig} + P_{i,bkg}$$

$$P_{i,sig} = P_{i,phys} * R_{det}$$

likelihood can also be “extended” to include Poisson probability to observe N events

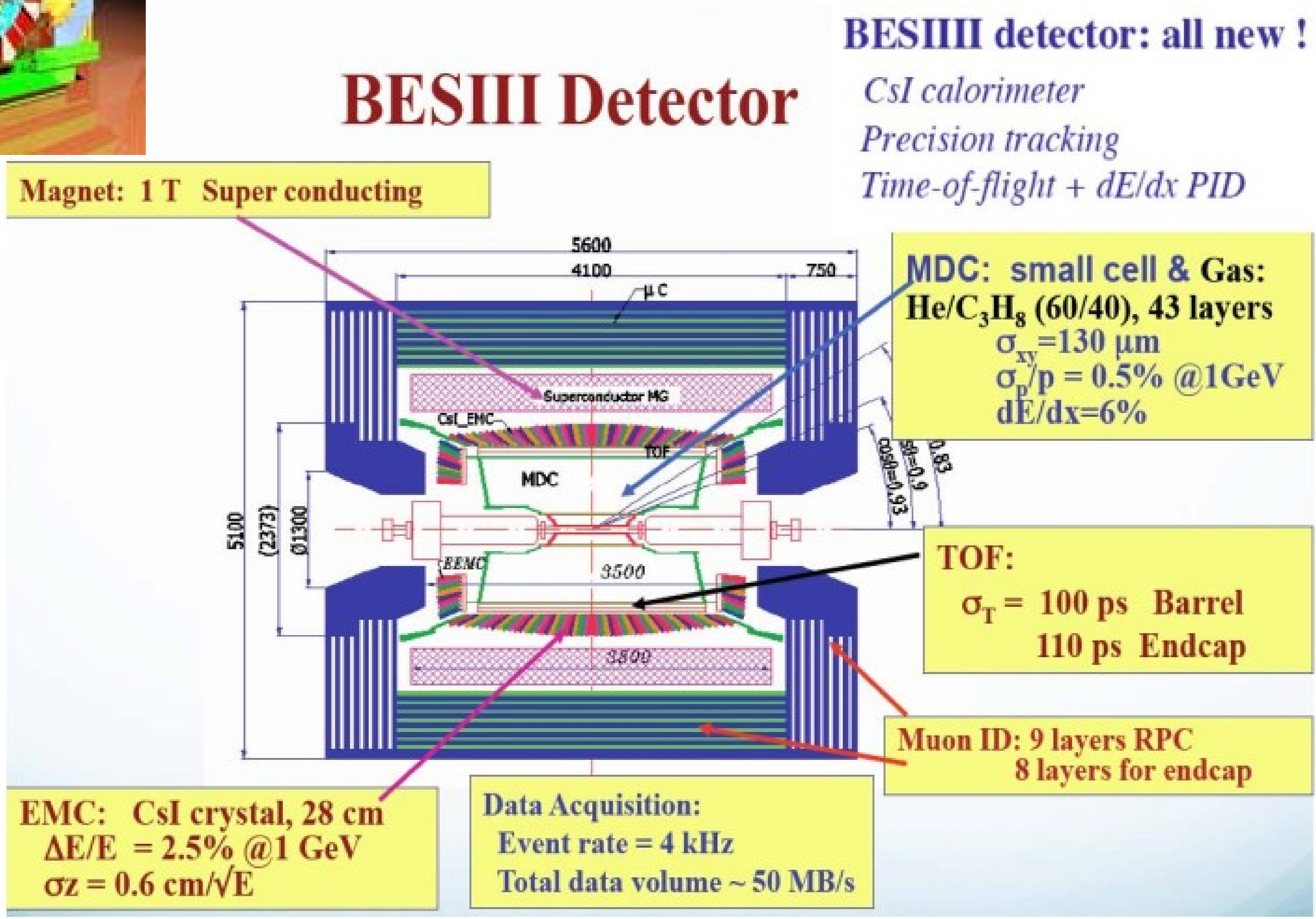
need to obtain background distributions and to known background fraction (or event-by-event background probability)

convolution with detector response: includes efficiency and resolution

$P_{i,phys}$ contains the physics …
but most be coded in a way that allows reliable determination of the model parameters

In the case of a binned fit to data, sum over events is replaced by sum over bins

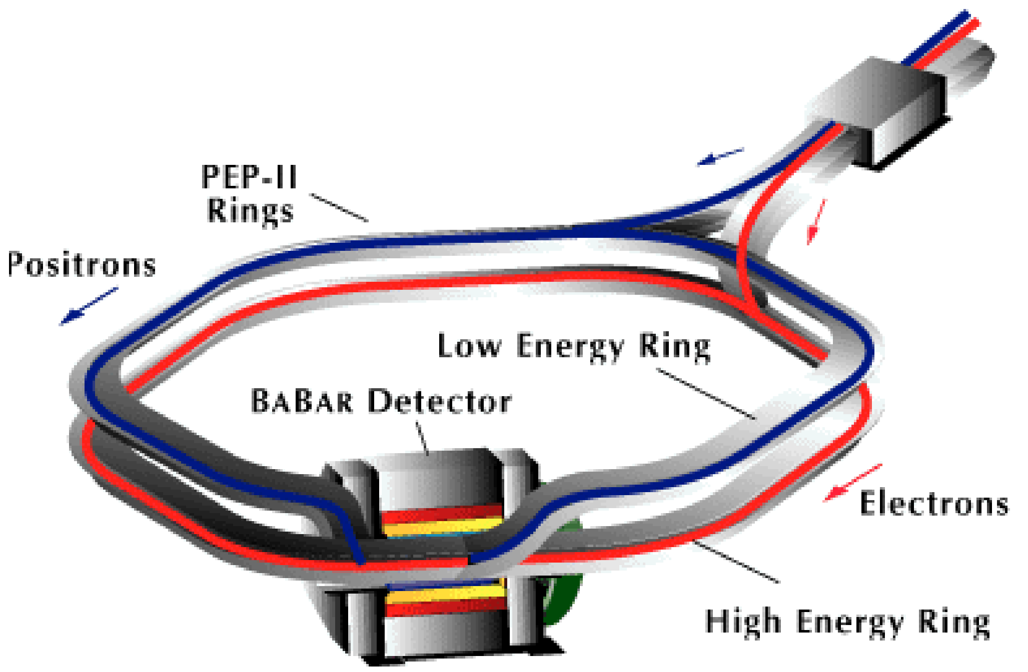
But first, let's look at some experiments



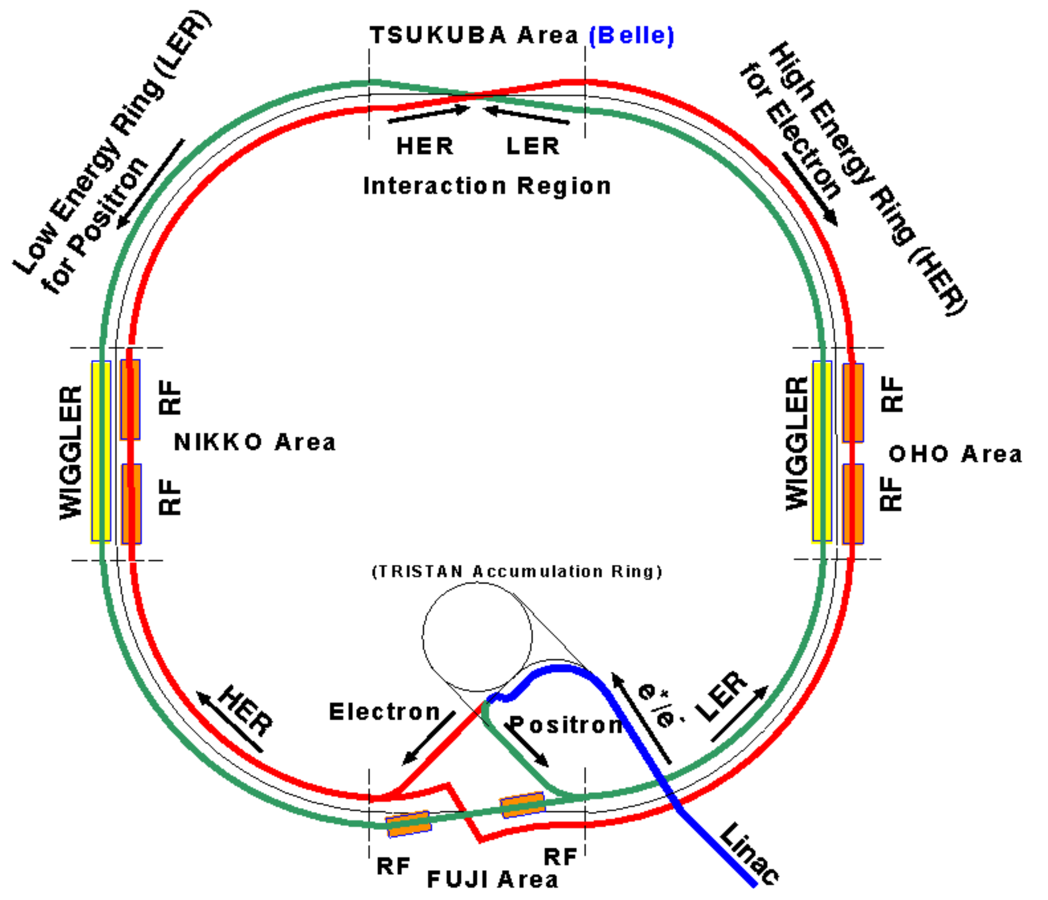
BESIII Detector
BESIII detector: all new !
CsI calorimeter
Precision tracking
Time-of-flight + dE/dx PID
Magnet: 1 T Super conducting
MDC: small cell & Gas:
He/C3H8 (60/40), 43 layers
σxy=130 μm
σp/p = 0.5% @1GeV
dE/dx=6%
TOF:
σT = 100 ps Barrel
110 ps Endcap
Muon ID: 9 layers RPC
8 layers for endcap
EMC: CsI crystal, 28 cm
ΔE/E = 2.5% @1 GeV
σz = 0.6 cm/√E
Data Acquisition:
Event rate = 4 kHz
Total data volume ~ 50 MB/s
5600
4100
750
μC
Superconductor MG
CsI_EMC
TOF
MDC
EEMC
3500
3809
5100
(2373)
Ø1300
cosθ=0.93

The Asymmetric B Factories

PEPII at SLAC
9.0 GeV e^- on 3.1 GeV e^+

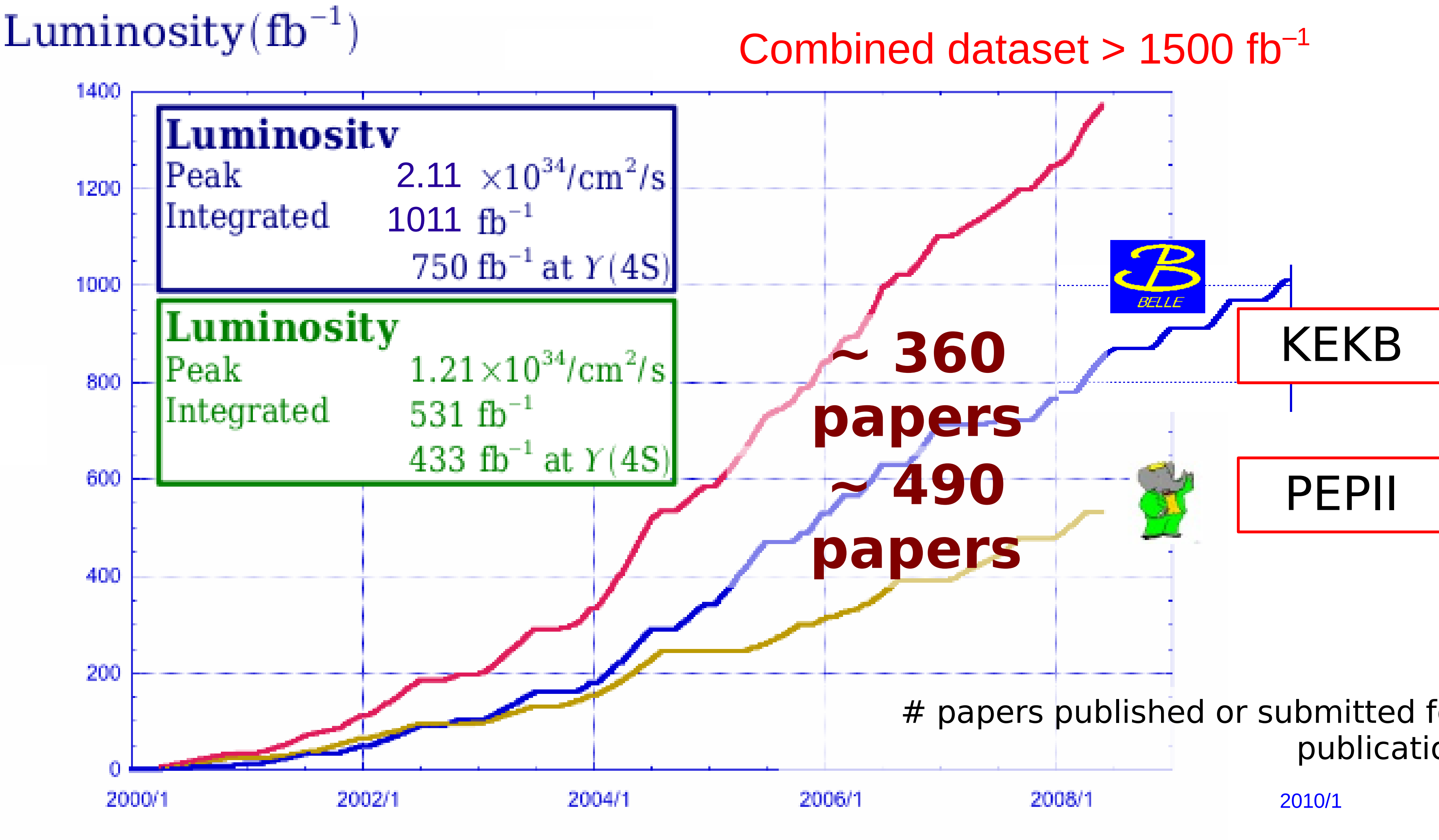


KEKB at KEK
8.0 GeV e^- on 3.5 GeV e^+



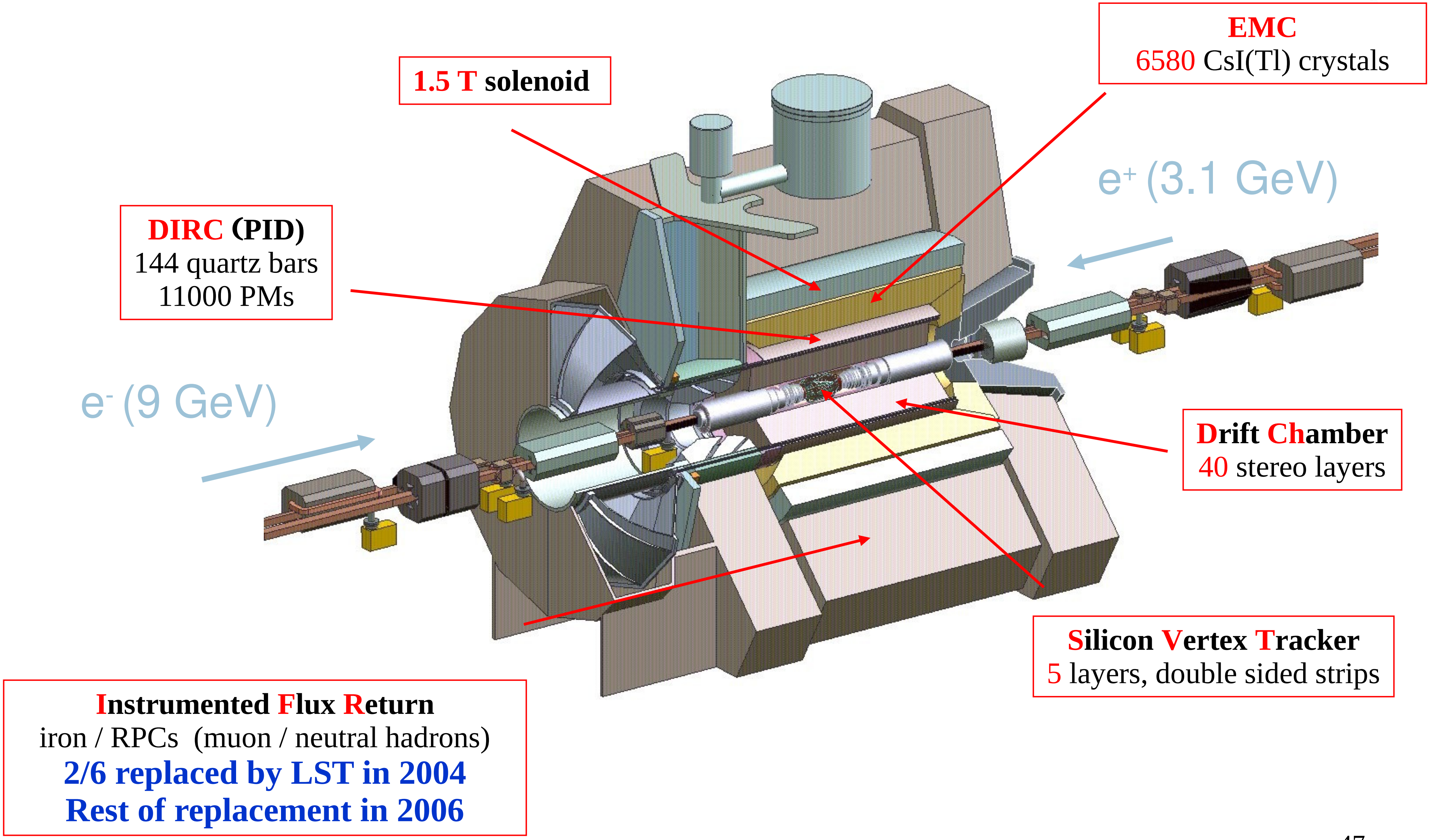
B factories – World Record Luminosities

Combined dataset > 1500 fb^{-1}

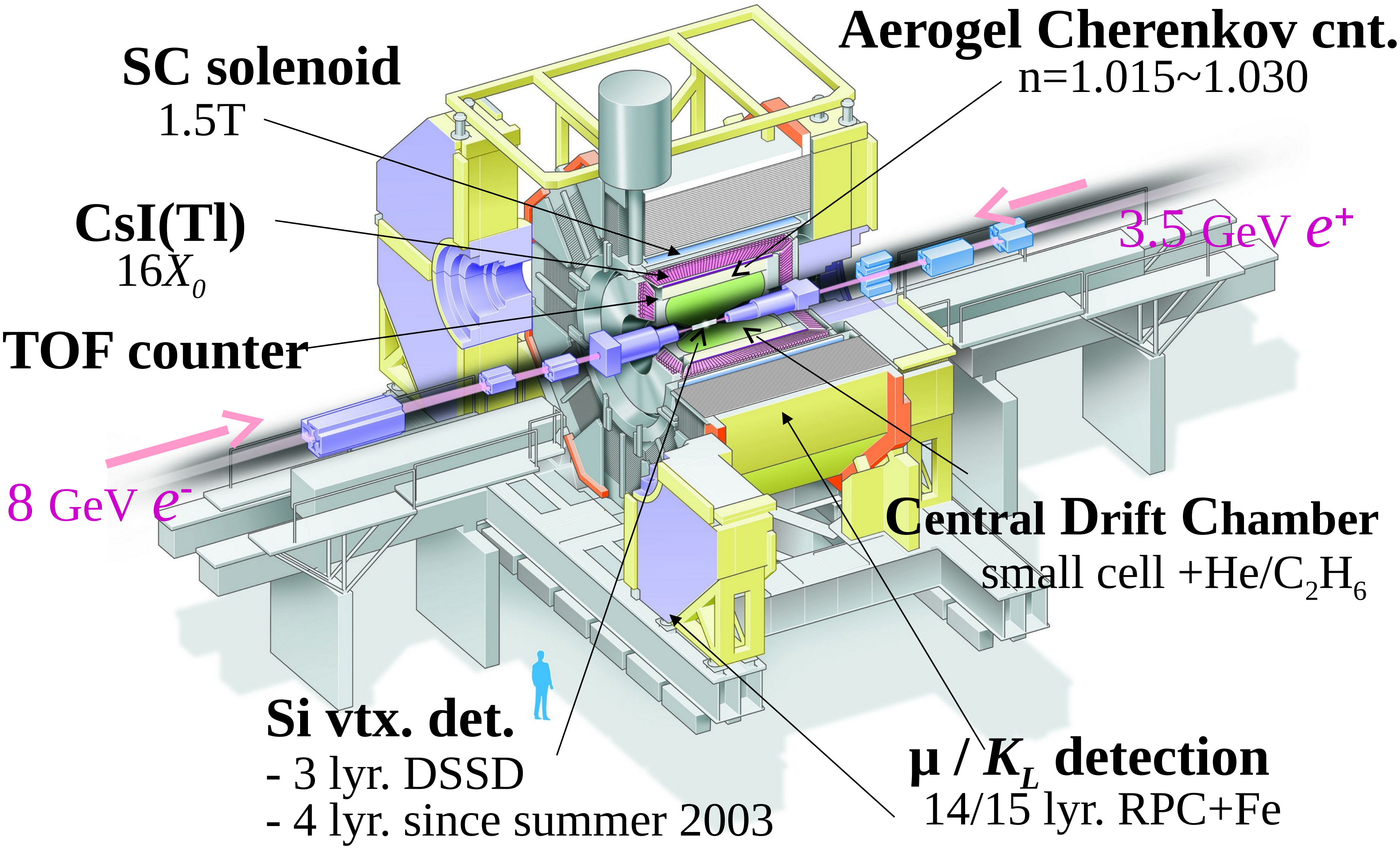


papers published or submitted for publication

BABAR Detector

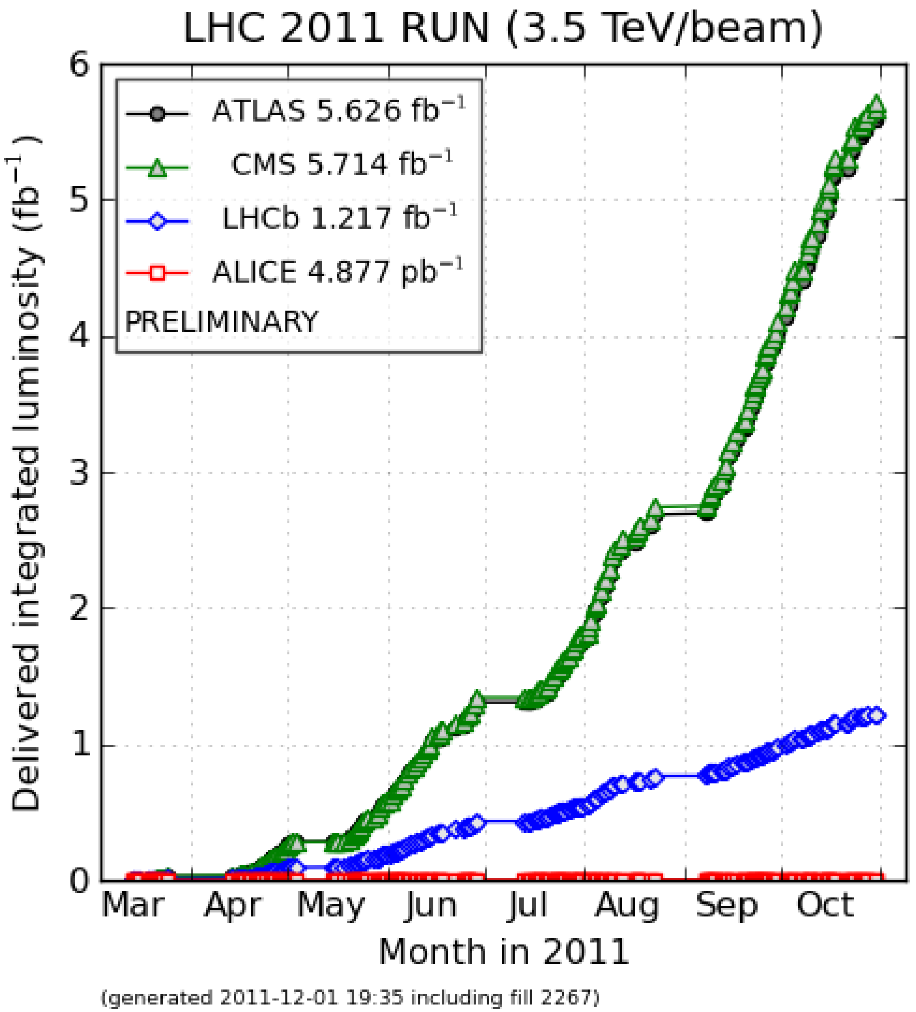


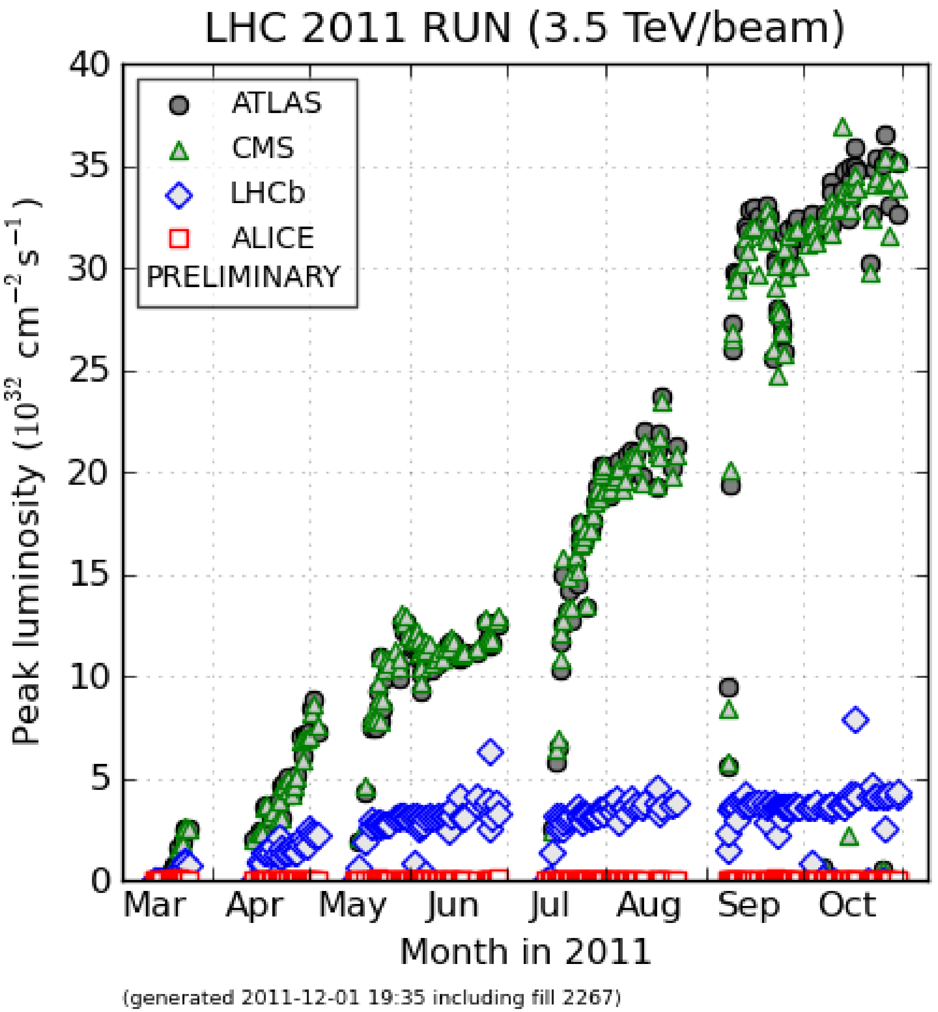
Belle Detector



The LHC

LHC performance 2011





LHCb design luminosity: 2 10^{32}/cm^2/s

What does $\int L\,dt$ = 1/fb mean?

- Measured cross-section, in LHCb acceptance

$$\sigma(pp \rightarrow b\bar{b}X) = (75.3 \pm 5.4 \pm 13.0)\ \mu b$$

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- So, number of $b\bar{b}$ pairs produced

$$10^{15} \times 75.3\ 10^{-6} \sim 10^{11}$$

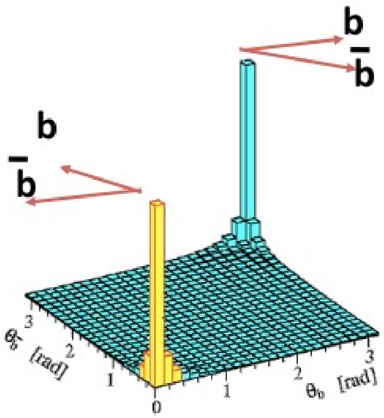
- Compare to combined data sample of e^+e^- “B factories” BaBar and Belle of ~ 10^9 $B\bar{B}$ pairs

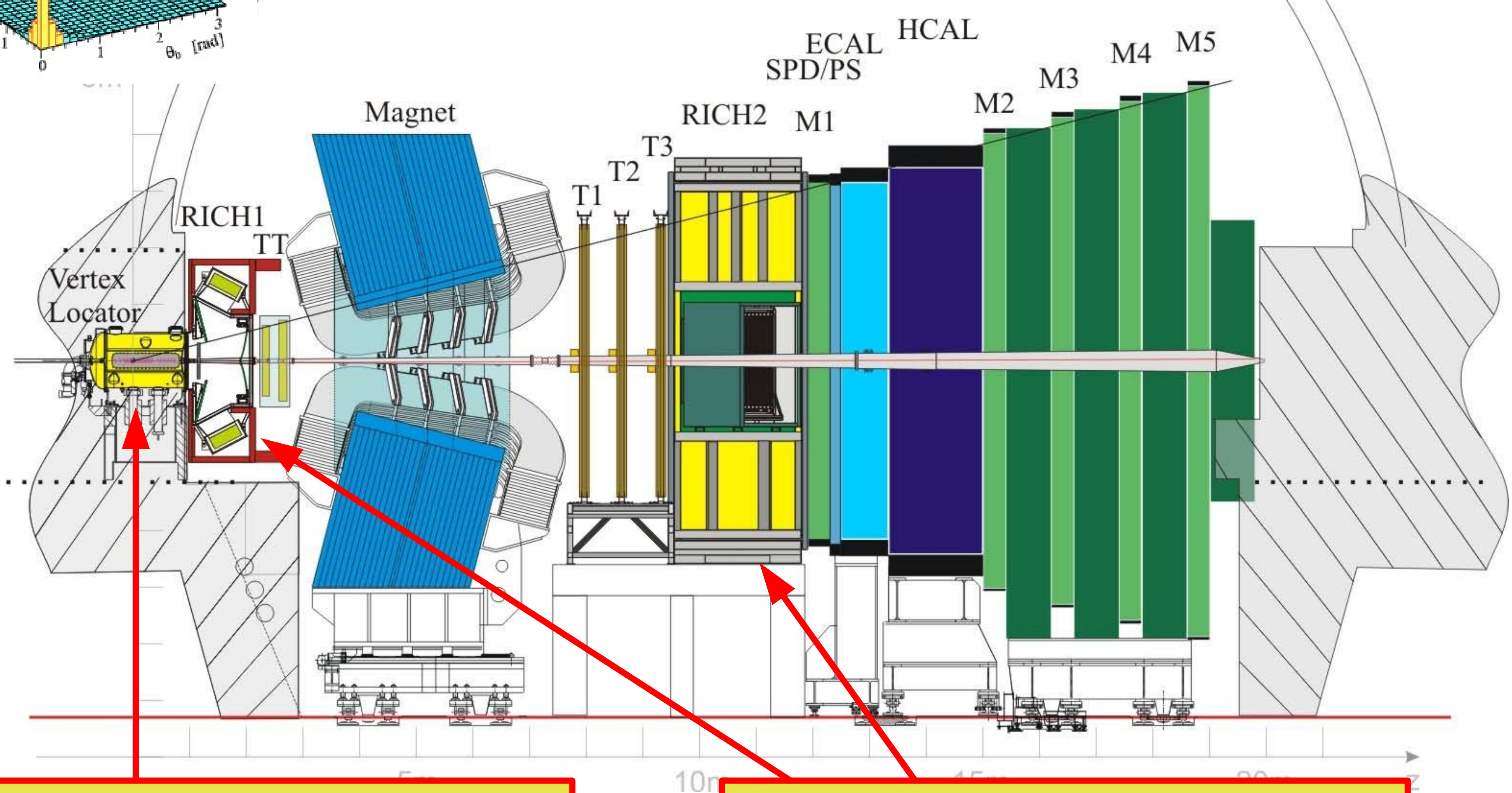
for any channel where the (trigger, reconstruction, stripping, offline) efficiency is not too small, LHCb has world's largest data sample

- p.s.: for charm, $\sigma(pp \rightarrow c\bar{c}X) = (6.10 \pm 0.93)$ mb

LHCb-CONF-2010-013

The LHCb detector





Precision primary and secondary vertex measurements

Excellent K/π separation capability

Parametrisations

- Fit parameters are complex coefficients of the contributing amplitudes
  - allowing for CP violation, 4 parameters for each
    - usually necessary to fix (at least) two reference parameters
  - many possible parametrisations
    - $r \exp(i\delta) \rightarrow (r \pm \Delta r) \exp(i(\delta \pm \Delta\delta))$
    - $r \exp(i\delta) \rightarrow r \exp(i\delta) (1 \pm \Delta\rho \exp(i\Delta\varphi))$
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      - #parameters explodes for >3 resonances

Conventions

- There are many different ways to write the lineshapes, spin factors, etc.
  - choice of normalisation is important
- Even if all code is bug-free, it is very hard to present unambiguously all information necessary to allow the Dalitz plot model to be reproduced
- Important to present results in convention-independent form (as well as other ways)
  - e.g. fit fractions and interference fit fractions

Example fit fraction matrix

TABLE I: Fit fractions matrix of the best fit. The diagonal elements F_{kk} correspond to component fit fractions shown in the paper in Table I. The off-diagonal elements give the fit fractions of the interference terms defined as $F_{kl} = 2\Re\int \mathcal{M}_k\mathcal{M}_l^* ds_{23}ds_{13}/\int|\mathcal{M}|^2 ds_{23}ds_{13}$.

$F_{kl} \times 100\%$	ϕ	$f_0(980)$	$X_0(1550)$	$f_0(1710)$	χ_{c0}	NR
ϕ	**11.8 ± 0.9 ± 0.8**	−0.94 ± 0.18 ± 0.11	−1.71 ± 0.36 ± 0.24	0.01 ± 0.10 ± 0.03	0.11 ± 0.02 ± 0.05	3.54 ± 0.38 ± 0.40
$f_0(980)$		**19 ± 7 ± 4**	53 ± 12 ± 7	−4.5 ± 2.9 ± 1.2	−0.9 ± 0.2 ± 0.5	−85 ± 21 ± 14
$X_0(1550)$			**121 ± 19 ± 6**	−30 ± 11 ± 4	−1.1 ± 0.3 ± 0.5	−140 ± 26 ± 7
$f_0(1710)$				**4.8 ± 2.7 ± 0.8**	−0.10 ± 0.07 ± 0.07	4 ± 6 ± 3
χ_{c0}					**3.1 ± 0.6 ± 0.2**	3.9 ± 0.4 ± 1.9
NR						**141 ± 16 ± 9**

Goodness of fit

- How do I know that my fit is good enough?
- You don't (sorry) … but some guidelines can tell you if there are serious problems
  - Is your fit model physical?
    - sometimes there may be little choice but to accept this
  - Do you get an acceptable χ^2/n.d.f. for various projections (1D and 2D)?
    - if no, is the disagreement localised in the Dalitz plot?
    - with high statistics it is extremely difficult to get an acceptable p-value; check if the disagreement is compatible with experimental systematics
    - some unbinned goodness-of-fit tests are now becoming available
  - Do you get an excessive sum of fit fractions?
    - values >100% are allowed due to interference, but very large values are usually indicative of unphysical interference patters (possibly because the model is not physical)
  - Do you think you have done the best that you possibly can?
    - eventually it is better to publish with an imperfect model than to suppress the data

Summary

- It must be clear by now that Dalitz plot analyses are extremely challenging
  - both experimentally and theoretically
- So let's recall that the motivation justifies the effort
  - hadronic effects: improved understanding of QCD, including possible exotic states
  - CP violation effects: potential sensitivity to discover new sources of matter-antimatter asymmetry
- We have an obligation to exploit the existing and coming data to the maximum of our abilities
  - and if that is not enough, we will have to improve our abilities!