Flavour Permutation Symmetry, Mixing and Jarlskog Invariance

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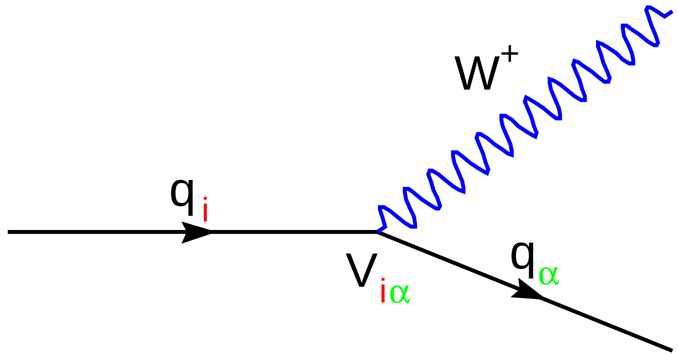
Outline of Talk

- The Flavour Problem
- Jarlskog Covariance
- Plaquette Invariance and Flavour Permutation Symmetry
- Flavour Symmetric Mixing Observables
- Applications
- Conclusions

The Flavour Problem

- SM has 20 (22) low energy flavour parameters (out of 25(27))
- They show tantalising structure, eg.

$$
|V_{CKM}| \sim \begin{array}{c} \\ u \\ c \\ t \end{array}
\begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\
\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\lambda) & \mathcal{O}(\lambda^3) \\ \mathcal{O}(\lambda) & \mathcal{O}(1) & \mathcal{O}(\lambda^2) \\ \mathcal{O}(\lambda^3) & \mathcal{O}(\lambda^2) & \mathcal{O}(1) \end{pmatrix} \end{array}
$$



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|U_{MNS}| \sim \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}
\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\
\begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & \lesssim 0.2 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \end{array}
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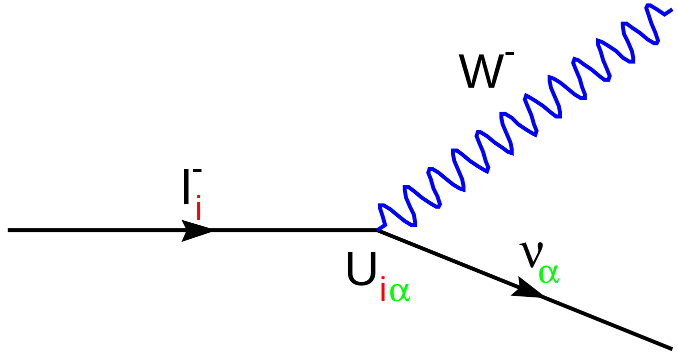

- Are not predictable in the SM - profoundly unsatisfactory.

Masses and Mixings have Common Origin

- in Quark and Lepton mass matrices: $M_{\frac{2}{3}},\ M_{-\frac{1}{3}},\ M_{\ell},\ M_{\nu}$
- Each is product of Higgs vev and Yukawa coupling matrix
- Work with Hermitian Squares.
  - eg. for neutrinos: $N = M_\nu M_\nu^\dagger$
  - for charged leptons: $L = M_\ell M_\ell^\dagger$ etc.
- Diagonalise $U_\nu N U_\nu^\dagger = D_\nu$ $\rightarrow$ eigenvalues (ie. masses-squared)
- Diagonalisation different for L and N $\Rightarrow$ mixing, ie. $U_{MNS} \equiv U_\ell U_\nu^\dagger$.

  NB. Analogous for quarks.

Jarlskog (ie. Weak Basis) Covariance

- However L and N are not observable:

$$L' = U_J L U_J^\dagger \text{ and } N' = U_J N U_J^\dagger$$

have same eigenvalues and same mixing matrix:

$$U'_{MNS} = U_\ell (U_J^\dagger U_J) U_\nu^\dagger = U_{MNS}$$

- ie. Masses and mixing angles are "Jarlskog-invariant", even though mass matrices transform.
- New basis is just as good a starting point...

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- But usually assume "special" basis, in which texture applies
- "Specialness" of this basis is not explained.

eg. H. Fritzsch:

$$M^{\frac{2}{3}} = \begin{pmatrix} 0 & a & 0 \\ a^* & 0 & b \\ 0 & b^* & c \end{pmatrix}$$

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- Since Einstein, have been alert to excessive reliance on a given basis.
- Jarlskog reiterates (hep-ph/0606050): "you should be able to formulate it in an invariant form"
- We adopt Jarlskog's prescription as a principle!

The Jarlskogian and Plaquette Invariance

Jarlskog's CP-violating invariant:

$$J = \mathrm{Im}(U_{\alpha i} U^*_{\alpha j} U^*_{\beta i} U_{\beta j})$$

Fascinating properties:

- Parameterises CP violation
- Does not depend on which plaquette is used
- May be simply related to the lepton mass matrices:

$$\begin{pmatrix} U_{e1} & U^*_{e2} & U_{e3} \\ U^*_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

$$J = -i \, \frac{\mathrm{Det}[L, N]}{2 L_\Delta N_\Delta}$$

L_Δ, N_Δ are simple polynomials in m_ℓ and m_ν.

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- Working with observables, find that, like J, FS functions of the mixing matrix can always be expressed as simple functions of the mass matrices.

- Will introduce an elemental set

  – can be used for the flavour-symmetric description of any mixing scheme.

The $S3_\ell \times S3_\nu$ Flavour Permutation Group

- 6 perms. of ℓ flavour indices and 6 of ν "flavour" labels (ie. ν mass eigenstate indices) constitute the $S3_\ell \times S3_\nu$ Flavour Permutation Group (FPG).

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- Introduce the observable P matrix:

$$P = \begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\ |U_{\tau 1}|^2 & |U_{\tau 2}|^2 & |U_{\tau 3}|^2 \end{pmatrix}$$

  – Parameterises mixing (up to the sign of J).

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  - Parameterises mixing (up to the sign of J).
  - Transforms as a (reducible) $\mathbf{3} \times \mathbf{3}$ (natural representation) of FPG.
  - Rows and columns each sum to unity (a magic square)

    $\Rightarrow$ completely specified by elements of *any* P-plaquette.
  - Each P-plaquette transforms as (irreducible) $\mathbf{2} \times \mathbf{2}$ of FPG.

Singlets Under FPG?

- J is prototype FS observable - invariant under even members of the FPG; flips sign under odd members; ie. $\overline{\mathbf{1}} \times \overline{\mathbf{1}}$.

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- Search for other singlets: $\mathbf{1} \times \mathbf{1}$, $\mathbf{1} \times \overline{\mathbf{1}}$ etc.
  - Find simple polynomials of elements of P
  - (anti-)symmetrise over flavour labels
- Simple representation theory$\longrightarrow$
  - 1st order in P: $\exists\ no$ non-trivial singlets
  - 2nd order: one each of $\overline{\mathbf{1}} \times \overline{\mathbf{1}}$ and $\mathbf{1} \times \mathbf{1}$
  - 3rd order: one each of all four singlets
  - $\geq$ 4th order: multiple instances of each
- Will stay at $\leq$ 3rd order. Clearly four are sufficient.

Elemental Set of FS Observables

Define themselves, up to normalisation (and "offset" in $\mathbf{1} \times \mathbf{1}$ case):

$$\mathcal{G} = \frac{1}{2} \left[\sum_{\alpha i} (P_{\alpha i})^2 - 1 \right] \qquad \mathcal{F} = \mathrm{Det} P$$

$$\mathcal{C} = \frac{3}{2} \left[\sum_{\alpha i} (P_{\alpha i})^3 - \sum_{\alpha i} (P_{\alpha i})^2 \right] + 1 \qquad \mathcal{A} = \frac{1}{18} \sum_{\gamma k} (L_{\gamma k})^3$$

where $L_{\gamma k} = (P_{\alpha i} + P_{\beta j} - P_{\beta i} - P_{\alpha j})$.

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where $L_{\gamma k} = (P_{\alpha i} + P_{\beta j} - P_{\beta i} - P_{\alpha j})$.

- $\mathcal{F}$ and $\mathcal{A}$ need no offset (they are anti-symmetric).
  - Reach extremum for no mixing → $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
  - 0 for trimaximal mixing → $P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$
- All normalised to maximum value $= 1$ (no mixing).
- $\mathcal{G}$ and $\mathcal{C}$ offset to zero for maximal mixing

Properties and Values

Observable Name	Order in P	Symmetry: $S3_\ell \times S3_\nu$	Theor. Range	Exptl. Range for Leptons	Exptl. Range for Quarks
$\mathcal{F}$	2	$\overline{\mathbf{1}}\times\overline{\mathbf{1}}$	$(-1, 1)$	$(-0.14, 0.12)$	$(0.893, 0.896)$
$\mathcal{G}$	2	$\mathbf{1}\times\mathbf{1}$	$(0, 1)$	$(0.15, 0.23)$	$(0.898, 0.901)$
$\mathcal{A}$	3	$\overline{\mathbf{1}}\times\overline{\mathbf{1}}$	$(-1, 1)$	$(-0.065, 0.052)$	$(0.848, 0.852)$
$\mathcal{C}$	3	$\mathbf{1}\times\mathbf{1}$	$(-\frac{1}{27}, 1)$	$(-0.005, 0.057)$	$(0.848, 0.852)$

Properties and values of FS observables. Experimentally allowed ranges estimated (90% CL) from compilations of current experimental results (neglect any correlations between the input quantities).

FSOs in Terms of Mass Matrices

Define reduced (ie. traceless) powers of mass matrices: $\widetilde{L^m} := L^m - \frac{1}{3}\mathrm{Tr}(L^m)$ (similarly for $\widetilde{N^m}$).

Now define Jarlskog-invariant:

$$\widetilde{T}_{mn} := \mathrm{Tr}(\widetilde{L^m}\widetilde{N^n}), \quad m, n = 1, 2.$$

$\widetilde{T}$ Completely equivalent to P (for known lepton masses).

Find:

$$\mathcal{F} \equiv \mathrm{Det}\, P = 3\frac{\mathrm{Det}\,\widetilde{T}}{L_\Delta N_\Delta}; \qquad \left[cf.\ J = -i\,\frac{\mathrm{Det}[L, N]}{2L_\Delta N_\Delta}\right]$$

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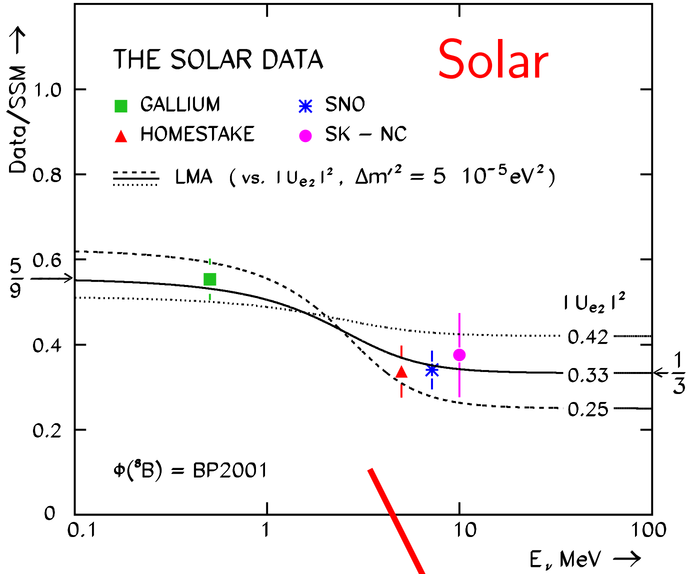
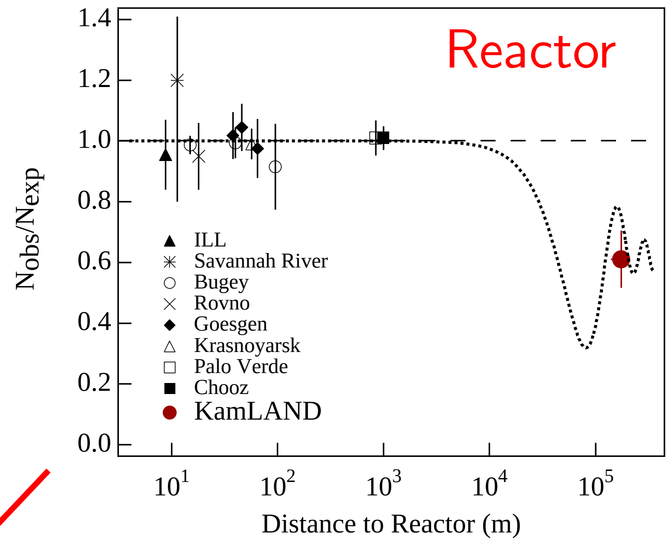
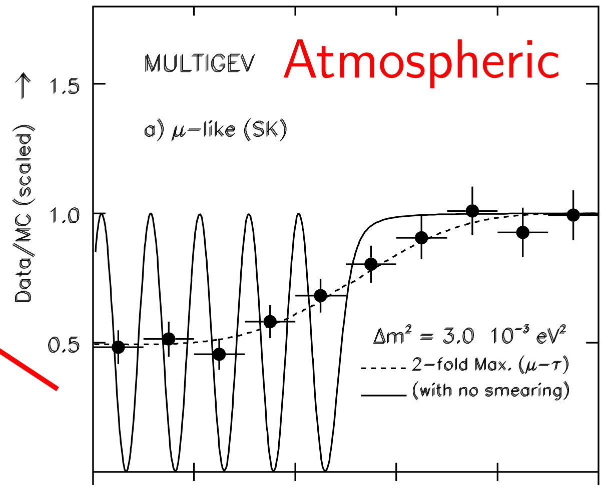
$$\mathcal{G} = \frac{\widetilde{T}_{mn}\,\widetilde{T}_{pq}\,\mathcal{L}^{mp}\,\mathcal{N}^{nq}}{(L_\Delta N_\Delta)^2}; \qquad \mathcal{C}, \mathcal{A} = \frac{\widetilde{T}_{mn}\,\widetilde{T}_{pq}\,\widetilde{T}_{rs}\,\mathcal{L}^{(mpr)}_{\mathcal{C},\mathcal{A}}\,\mathcal{N}^{(nqs)}_{\mathcal{C},\mathcal{A}}}{(L_\Delta N_\Delta)^{K_{\mathcal{C},\mathcal{A}}}}$$

The $\mathcal{L}$ ($\mathcal{N}$) are simple functions of traces of $\widetilde{L^m}$ ($\widetilde{N^m}$). $\quad K_\mathcal{C}\ (K_\mathcal{A}) = 2(3)$.

Application: Flavour-symmetric Descriptions of Mixing







$$|U| \sim \begin{array}{c} e \\ \mu \\ \tau \end{array} \overset{\begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array}}{\begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & \epsilon \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}}$$

$\Rightarrow \mathcal{F} = 0, \quad \mathcal{C} = 0, \quad \mathcal{A} = 0, \quad \mathcal{G} = \frac{1}{6}(1 - 3\epsilon^2)^2.$ Where is flavour-symmetry?

A Further Application

- What feature do quark and lepton mixing matrices have in common?
- Each has at least one "small" element.
- What is the flavour-symmetric expression of this?

$$|V_{CKM}| \sim \begin{pmatrix} \sim 1 & \sim \lambda & \sim \lambda^3 \\ \sim \lambda & \sim 1 & \sim \lambda^2 \\ \sim \lambda^3 & \sim \lambda^2 & \sim 1 \end{pmatrix}$$

$$|U_{MNS}| \sim \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & \epsilon \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

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≥ 1 zero $\Rightarrow$ CP-conservation $\implies J = 0$.

But need two constraints - find:

$$2\mathcal{A} + \mathcal{F}(\mathcal{F}^2 - 2\mathcal{C} - 1) = 0 \; and \; J = 0 \implies \text{at least one zero}$$

A Further Application (Cont.)

- NB. Consider K-matrix: $K_{\gamma k} = \mathrm{Re}(U_{\alpha i}U^*_{\alpha j}U^*_{\beta i}U_{\beta j})$ (cf. J).
- Noticed that $2\mathcal{A} + \mathcal{F}(\mathcal{F}^2 - 2\mathcal{C} - 1) \equiv \mathrm{Det}K$
- Both MNS and CKM mixing matrices satisfy $\mathrm{Det}K = 0$ (within errors)
- Conjecture: MNS *and* CKM constrained according to the same FS JI condition:

$$\mathrm{Det}K = 0; \quad J \text{ small.}$$

- $\Longrightarrow$ FS prediction: if $\mathrm{Det}K = 0$, then as $J \to 0$, at least one UT angle $\to 90^\circ$.

  For quarks: $\mathrm{Det}K = 0 \implies (90^\circ - \alpha) = \overline{\eta}\lambda^2 = 1^\circ \pm 0.2^\circ$

  cf. $(90^\circ - \alpha) = 0^{\circ}{}^{+3^\circ}_{-7^\circ}$ experimentally.

  For leptons: $\mathrm{Det}K = 0 \implies (90^\circ - \delta) = \frac{|U_{e3}|}{\sqrt{2}} \lesssim 8^\circ$ (good for CPV).

Summary

- Models of masses and mixings should be weak-basis invariant
- Have defined flavour-symmetric mass and/or mixing observables.
- "Simplest" set defines itself up to normalisation
- Remarkably, leptonic mixing is consistent with 3 of these $= 0$!
- Can use them to implement FS and JI constraints on flavour observables, and make testable conjectures