ANDREW
STURGESS

(1) A length $A^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$ should be invariant under Lorentz transformations, such that

$$A'^2 = (A^{0'})^2 - (A^{1'})^2 - (A^{2'})^2 - (A^{3'})^2$$

consider L.T. in x

$$\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

Hence

$$A^{0'} = \gamma(A^0 - \beta A^1)$$
$$A^{1'} = \gamma(A^1 - \beta A^0)$$
$$A^{2'} = A^2$$
$$A^{3'} = A^{3'}$$

plug this in

$$A'^2 = \gamma^2(A^0 - \beta A^1)^2 - \gamma^2(A^1 - \beta A^0)^2 - A^{2^2} - A^{3^2}$$
$$= \gamma^2\left(A^{0^2} + \beta^2 A^{1^2} - 2\beta A^0 A^1\right) - \gamma^2(A^{1^2} + \beta^2 A^{0^2} - 2\beta A^1 A^0) - A^{2^2} - A^{3^2}$$
$$= \gamma^2 A^{0^2} + \beta^2\gamma^2 A^{1^2} - 2\gamma^2\beta A^0 A^1 - \gamma^2 A^{1^2} - \gamma^2\beta^2 A^{0^2} + 2\gamma^2\beta A^1 A^0 - A^{2^2} - A^{3^2}$$

some terms cancel

$$= \gamma^2 A^{0^2} + \beta^2\gamma^2 A^{1^2} - \gamma^2 A^{1^2} - \gamma^2\beta^2 A^{0^2} - A^{2^2} - A^{3^2}$$
$$= (\gamma^2 - \gamma^2\beta^2)A^{0^2} - (\gamma^2 - \gamma^2\beta^2)A^{1^2} - A^{2^2} - A^{3^2}$$

but $\underline{\gamma^2 - \gamma^2\beta^2 = 1}$ $\rightarrow$ from $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$; $\gamma^2 = \dfrac{1}{1-\beta^2}$

$\gamma^2(1-\beta^2) = \underline{1}$

Hence

$$A^{0^2} - A^{1^2} - A^{2^2} - A^{3^2} = A^2$$

Hence, the length is invariant under L.T. 2

this shows only special case

(2) $g_{\mu\nu} g^{\mu\nu} = 4$; in special relativity, $g_{\mu\nu} = \eta_{\mu\nu}$ the minkowski metric tensor, such that

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$; Also $g_{\mu\nu} = g^{\mu\nu}$

$\Rightarrow$

such that $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$g_{\mu\nu} g^{\mu\nu} = g_{00} g^{00} + g_{11} g^{11} + g_{22} g^{22} + g_{33} g^{33}$
(no non-zero off-diagonals)
$= (1)\cdot(1) + (-1)(-1) + (-1)(-1) + (-1)(-1) = \underline{\underline{4}}$ 2

3) $E^2 = p^2 + m^2$ in natural units, $|p| = 2\,GeV/c$
Mass of π^+ : $0.139\ GeV/c^2$
" " B^+ : $5.259\ GeV/c^2$

Work in natural units!
$E_{\pi^+} = \sqrt{(2)^2 + (0.139)^2} = 2.005\ GeV$
$E_{B^+} = \sqrt{(2)^2 + (5.259)^2} = 5.626\ GeV$

Some useful relations : $E = \gamma m$
$p = \gamma m \beta$

Hence $\beta = \frac{p}{E} \rightarrow$

$\beta_{\pi^+} = \frac{p_{\pi^+}}{E_{\pi^+}} = \frac{2.000}{2.005} = \underline{\underline{0.9975c}}$

$\beta_{B^+} = \frac{2.000}{5.626} = \underline{\underline{0.355c}}$

b) Protons have mass $\approx 1\ GeV/c^2$

$p_{TEVA} = \sqrt{(980)^2 - 1} = 979.99950\ GeV$
$p_{LHC} = \sqrt{(7000) - 1} = 6999.99993\ GeV$

Hence $\beta_{TEVA} = \frac{979.99950}{980} = 0.9999995c$
$\beta_{LHC} = 0.99999999c$ .. " " "
both are very close to 'c'

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4) The laws of physics should always have the same form (i.e covariant) regardless of the frame of reference. This frame regards uniform velocity with respect to every other; such that the laws of physics, if measured by experiment in ANY coordinate system in ANY uniform velocity frame relative to another - should be measured to be the same

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2 KLEIN GORDON EQ^N

1) the Schrodinger EQ^N $= -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$ ①

Note that $\frac{\partial\rho}{\partial t} = \frac{\partial(\psi^*\psi)}{dt} = \psi^*\frac{\partial\psi}{dt} + \psi\frac{d\psi^*}{dt}$ ②

Need to get schrodinger in this form; Hence multiply ① by ψ^* and rearrange for $\psi^*\frac{d\psi}{dt}$

$$-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi = i\hbar\psi^*\frac{d\psi}{dt}$$

$$\psi^*\frac{d\psi}{dt} = \frac{1}{i\hbar}\left[-\frac{\hbar^2\psi^*}{2m}\nabla^2\psi + V\psi^*\psi\right]$$

Do the opposite i.e via the complex conjugate of 1 + multiply by ψ

take C.C of 1 $= -\frac{\hbar^2\nabla^2\psi^*}{2m} + V\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}$

multiply by ψ $= -\frac{\hbar^2\psi\nabla^2\psi^*}{2m} + V\psi\psi^* = -i\hbar\psi\frac{\partial\psi^*}{dt}$

$$\psi\frac{d\psi^*}{dt} = -\frac{1}{i\hbar}\left[-\frac{\hbar^2\psi}{2m}\nabla^2\psi^* + V\psi^*\psi\right]$$

Substitute in ②

$$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}\left[-\frac{\hbar^2\psi^*}{2m}\nabla^2\psi + V\psi^*\psi\right] - \frac{1}{i\hbar}\left[-\frac{\hbar^2\psi\nabla^2\psi^*}{2m} + V\psi\psi^*\right]$$

Potential cancels $\frac{\partial\rho}{\partial t} = \frac{\hbar}{2im}\left[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right]$

This then means $\frac{\partial \rho}{\partial t} - \frac{\hbar}{2mi}\underbrace{\left[\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right]}_{= -\bar{\nabla} \cdot j}$

to take continuity form ; HENCE

$$\dot{j} = -\frac{\hbar}{2mi}\left[\psi^* \bar{\nabla}\psi - \psi \bar{\nabla}\psi^*\right]$$

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2) Klein Gordon Equation

$$\left[\partial_\mu \partial^\mu + \frac{m^2c^2}{\hbar^2}\right]\psi(x,t)$$

$$= \left[\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi^2 + \frac{m^2c^2}{\hbar^2}\psi\right] = 0$$

Try plane wave solution $\psi(x,t) = N\exp\left[\frac{-i}{\hbar}\left[E.t - \vec{p}\cdot\vec{x}\right]\right]$

$\frac{\partial \psi}{\partial t} = \frac{-i}{\hbar}(E)Ne^{-i(Et - \bar{p}\cdot\bar{x})}$ | $\frac{\partial \psi}{\partial x} = \frac{i}{\hbar}(p)Ne^{-i(Et - \bar{p}\cdot\bar{x})}$

$\frac{\partial^2 \psi}{\partial t^2} = \frac{(i)^2}{\hbar^2}E^2 Ne^{-i(Et - \bar{p}\cdot\bar{x})}$ | $\frac{\partial^2 \psi}{\partial x^2} = \frac{(i)^2}{\hbar^2}p^2 Ne^{-i(Et - p\cdot x)}$

plug into KG

$$= \left[\frac{-E^2}{\hbar^2c^2}Ne^{-i(Et - \bar{p}\cdot x)} + \frac{p^2}{\hbar^2}Ne^{-i(Et - \bar{p}\cdot\bar{x})} + \frac{m^2c^2}{\hbar^2}Ne^{-i(Et \cdot p\cdot x)}\right] = 0$$

Hence $\frac{-E^2}{\hbar^2c^2} + \frac{p^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2} = 0$

$\frac{E^2}{c^2} - p^2 = -m^2c^2$ hence $E = \pm\sqrt{p^2c^2 + m^2c^4}$

has positive + Negative energy solutions!

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3) Write KG in time + space notation

i.e. $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$

Note $\frac{\partial \rho}{\partial t} = \psi \frac{d\psi^*}{dt} + \psi^* \frac{d\psi}{dt}$ before

I multiply ① by ψ^*; $= \frac{1}{c^2}\psi^*\frac{\partial^2}{\partial t^2}\psi - \psi^*\nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi^*\psi = 0$

II c.c ①, multiply by ψ; $= \frac{1}{c^2}\psi\frac{\partial^2}{\partial t^2}\psi^* - \psi\nabla^2\psi^* + \frac{m^2c^2}{\hbar^2}\psi\psi^* = 0$

subtract (I − II)

$$= \frac{1}{c^2}\left[\psi^*\frac{\partial^2}{\partial t^2}\psi - \psi\frac{\partial^2}{\partial t^2}\psi^*\right] - \left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right] = 0$$

$$= \frac{1}{c^2}\frac{\partial}{\partial t}\left[\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right] - \nabla\cdot\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$$

$=$ "ρ"? $\qquad\qquad$ $j = ?$

to describe 4 momenta, must multiply by $\frac{i\hbar}{2m}$

such that $\frac{\partial \rho}{\partial t} + \nabla\cdot j = 0$

where $\rho = \frac{i\hbar}{2mc^2}\left[\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)$

and $j = \frac{-i\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$

Try plane wave solution $\psi = Ne^{(-i(\omega t - \bar{k}\cdot\bar{x})]}$

$\psi^* = Ne^{(i[\omega t - \bar{k}\cdot\bar{x})]}$

$\frac{d\psi}{dx} = ik\psi$

$\frac{d\psi^*}{dx} = -ik\psi^*$

Hence, $j = \frac{-i\hbar}{2m}\Big[(ik)N^2 \underbrace{\exp^{[i(\omega t - k\cdot x)]}\exp^{-i(\omega t - \bar{k}\cdot\bar{x})}}_{1}$

$- (-ik)N^2 \exp^{-i(\omega t - k\cdot x)}\exp^{i(\omega t - k\cdot x)}$

$j = \frac{-i\hbar}{2m}\left[2ikN^2\right] = \frac{2|N|^2}{2m}\hbar k =$ ~~[illegible]~~

Likewise

$$\rho = \frac{i\hbar}{2mc^2}\left[-i\omega N^2 - i\omega N^2\right]$$

$$= \frac{2\hbar\omega|N|^2}{2mc^2} = 2|N|^2 \frac{E}{2mc^2}$$

Not sure where factor of 'i' come in; expect $p^\mu = \left(\frac{E}{c}, \underline{p}\right)$

Hence $j^\mu_{KG} = 2|N^2|\left[\frac{E}{2mc^2}, \frac{\underline{p}}{2m}\right]$ 4

4-momentum notation makes it easier