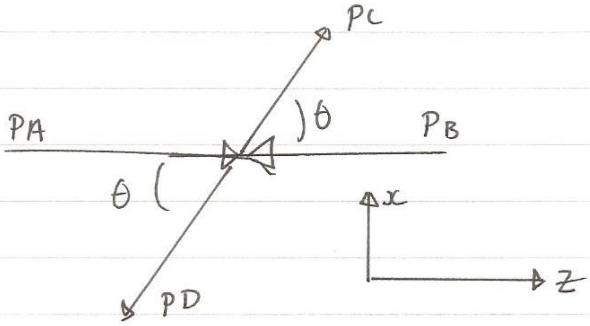
(6) Cross-section for scattering

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\left(\frac{s-u}{t}\right)^2$$



In High Energy limit,
$|E| = |P|$ (i.e neglect masses)
and can say that magnitude of momentum $|P_i| = |P_F|$
Hence
$$P_A = (P, 0, 0, P)$$
$$P_B = (P, 0, 0, -P)$$
$$P_C = (P, P\sin\theta, 0, P\cos\theta)$$
$$P_D = (P, -P\sin\theta, 0, -P\cos\theta)$$
} have angle to Z direction

Now use Mandelstam definitions

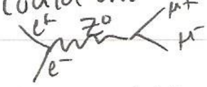
$$s = (P_A + P_B)^2 = 4p^2$$
$$t = (P_A - P_C)^2 = (p-p)^2 - (P\sin\theta)^2 - (p - P\cos\theta)^2$$
$$= -p^2\sin^2\theta - p^2 - p^2\cos^2\theta + 2p^2\cos\theta$$
$$= -p^2(\sin^2\theta + 1 + \cos^2\theta - 2\cos\theta)$$
$$= -2p^2(1-\cos\theta)$$

by a similar fashion
$$u = -2p^2(1+\cos\theta)$$

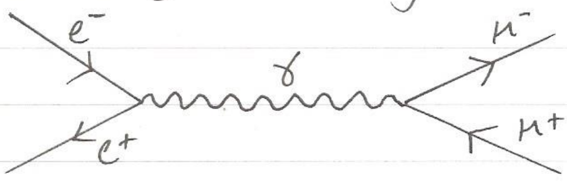
Now $$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\left[\frac{4p^2 + 2p^2(1+\cos\theta)}{-2p^2(1-\cos\theta)}\right] = \frac{e^4}{64\pi^2 s}\left[\frac{2p^2}{2p^2}\left[\frac{3+\cos\theta}{1-\cos\theta}\right]\right]^2$$

3

2) Only one possible Feynman diagram (though could also have



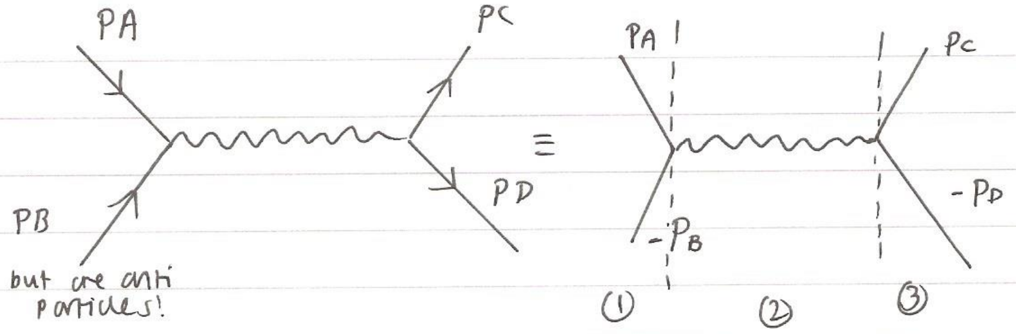
but assume we only look at QED treatment)



2

3) Can divide Feynman diagram into three regions

=>



① First vertex $= ie(P_A - P_B)_\mu$

② Propagator $\sim -ig^{\mu\nu}/q^2 \rightarrow q^2 = (P_A+P_B)^2$

③ 2nd vertex $= ie(P_C - P_D)_\nu$

Multiply in order! $-iM_i = ie(P_A-P_B)_\mu \frac{(-ig^{\mu\nu})}{(P_A+P_B)^2} ie(P_C-P_D)_\nu$

$$= -iM_i = ie^2 \frac{(P_A-P_B)_\mu \cdot (P_C-P_D)^\mu}{(P_A+P_B)^2}$$

e) Recast in term of mandlestam

So, $(P_A - P_B)\cdot(P_C-P_D) = P_A\cdot P_C - P_A\cdot P_D - P_B\cdot P_C + P_B P_D$ ⊛

$S = (P_A+P_B)^2 = (P_C+P_D)^2 = M_A^2+M_B^2+2P_A\cdot P_B = M_C^2+M_D^2+2P_C\cdot P_D$

$t = (P_A-P_C)^2 = (P_B-P_D)^2 = M_A^2+M_C^2-2P_A\cdot P_C = M_B^2+M_D^2-2P_D P_D$

$U = (P_A-P_D)^2 = (P_B-P_C)^2 = M_A^2+M_D^2-2P_A P_D = M_B^2+M_C^2-2P_B P_C$

So $P_A\cdot P_C = \frac{1}{2}[M_A^2+M_C^2-t]$ $\quad P_A\cdot P_D = \frac{1}{2}[M_A^2+M_D^2-u]$

$P_B\cdot P_D = \frac{1}{2}[M_B^2+M_D^2-t]$ $\quad P_B\cdot P_C = \frac{1}{2}[M_B^2+M_D^2-u]$

and so (⊛) $P_A\cdot P_C - P_A\cdot P_D - P_B\cdot P_C + P_B\cdot P_D$

$= \frac{1}{2}(M_A^2+m_C^2-t)+\frac{1}{2}(m_B^2+m_D^2-t) - \frac{1}{2}(M_A^2+M_D^2-u) - \frac{1}{2}(M_B^2+M_C^2-u)$

$= \frac{1}{2}(s+t+u) - t - \frac{1}{2}(s+t+u)+u$

using $M_A^2+M_B^2+M_C^2+M_D^2 = s+t+u$

Hence $|M|^2 = e^4\left[\frac{(P_A-P_B)_\mu\cdot(P_C-P_D)^\mu}{(P_A+P_D)^2}\right]^2$

$$= e^4\left[\frac{u-t}{s}\right]^2$$

$|P_C| = |P_A|$ in high E limit

Then $\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\frac{|P_C|}{|P_A|}\left[\frac{u-t}{s}\right]^2 = \frac{e^4}{64\pi^2 s}\left[\frac{u-t}{s}\right]^2 \rightarrow$ in High E, $s\rightarrow\infty$! Hence cross-section $\rightarrow 0$! as $E\rightarrow\infty$!

2+3

Problem Sheet 7

① $\gamma^0 = \beta = \begin{pmatrix} \hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix}$, $\gamma^i = \beta\alpha^i = \begin{pmatrix} \hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$

Now, $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = $ ~~...~~ as ~~...~~ zero

consider $\gamma^\mu\gamma^\nu$; note $\gamma^0\gamma^1 = 0$!

$$\gamma^0\gamma^0 = \begin{pmatrix} \hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix}\begin{pmatrix} \hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbb{I}} & 0 \\ 0 & \hat{\mathbb{I}} \end{pmatrix}$$

$$\gamma^i\gamma^i = \begin{pmatrix} 0 & \vec{\sigma}_i \\ -\vec{\sigma}_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma}_i \\ -\vec{\sigma}_i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i^2 & 0 \\ 0 & -\sigma_i^2 \end{pmatrix}$$

if $i = 1$, $\sigma_i^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\mathbb{I}}$

if $i = 2$, $\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\mathbb{I}}$

if $i = 3$, $\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\mathbb{I}}$

Hence $\gamma^0\gamma^0 + \gamma^i\gamma^i = \gamma^\mu\gamma^\nu$

$$= \begin{pmatrix} \hat{\mathbb{I}} & 0 \\ 0 & \hat{\mathbb{I}} \end{pmatrix}_0 + \begin{pmatrix} -\hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix}_1 + \begin{pmatrix} -\hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix}_2 + \begin{pmatrix} -\hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix}_3$$

$$= \hat{\mathbb{I}}_0 - \hat{\mathbb{I}}_1 - \hat{\mathbb{I}}_2 - \hat{\mathbb{I}}_3 = g^{\mu\nu}\hat{\mathbb{I}} \quad \text{as} \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Expect a similar contribution from $\gamma^\nu\gamma^\mu$; Hence get double!

Hence why $\mu \neq \nu$ is zero?

$$\{\gamma^\mu, \gamma^\nu\} = 2[g^{\mu\nu}\hat{\mathbb{I}}]$$

2

② The neater relativistic form of dirac equation $i\gamma^\mu\partial_\mu\psi - m\psi = 0$

take the Hermitian conjugate

$$-i(\partial_\mu\psi^+)(\gamma^\mu)^+ - m\psi^+ = 0$$

~~multiplying by~~ $\gamma^0\gamma^0 = 1$! $\rightarrow$ Note $(\gamma^{\mu+}) = \gamma^0\gamma^\mu\gamma^0$

$$-i(\partial_\mu\psi^+)\gamma^0\gamma^\mu\gamma^0 - m\psi^+\gamma^0\gamma^0 = 0$$

Define ADJOINT FIELD $\bar{\psi} = \psi^+\gamma^0$

$$-i(\partial_\mu(\psi^+\gamma^0))\gamma^\mu\gamma^0 \mp m\psi^+\gamma^0\gamma^0$$

$$= -i(\partial_\mu\bar{\psi})\gamma^\mu\gamma^0 \mp m\psi^+\gamma^0\gamma^0$$

$$= -i(\partial_\mu\bar{\psi})\gamma^\mu\gamma^0 \mp m\bar{\psi}\gamma^0$$

$\rightarrow$

multiply from right by γ^0

$$= -i(\partial_\mu \bar{\Psi})\gamma^\mu \underbrace{\gamma^0\gamma^0}_{=1} + m\bar{\Psi}\underbrace{\gamma^0\gamma^0}_{=1}$$

$$= -i(\partial_\mu \bar{\Psi})\gamma^\mu - m\bar{\Psi} = 0$$

$$i\partial_\mu \bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0 \qquad (\alpha)$$

3

c) The regular dirac Equation

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0$$

multiply from left by $\bar{\Psi} = \Psi^\dagger \gamma^0$

$$= \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = 0 \qquad (1)$$

and multiply from right by Ψ on adjoint dirac EQ^N

$$= \bar{\Psi}(i\partial_\mu \gamma^\mu + m)\Psi = 0 \qquad (2)$$

Here we take (α) and say that $\bar{\Psi}(i\overleftarrow{\partial}_\mu \gamma^\mu + m) = 0$

i.e ∂_μ differentiates toward the arrow; NOW ADD (1) and (2)

$$\bar{\Psi}i\gamma^\mu \partial_\mu \Psi - m\cancel{\bar{\Psi}\Psi} + \bar{\Psi}i\partial_\mu \gamma^\mu \Psi + m\cancel{\bar{\Psi}\Psi} = 0$$

$$= \bar{\Psi}i\gamma^\mu \partial_\mu \Psi + \bar{\Psi}i\partial_\mu \gamma^\mu \Psi = 0$$

$$= \bar{\Psi}\gamma^\mu \partial_\mu \Psi + \bar{\Psi}\partial_\mu \gamma^\mu \Psi = 0$$

Which looks like product rule (i.e

$$\partial_\mu(\bar{\Psi}\gamma^\mu \Psi) = 0$$

As $\partial_\mu J^\mu = 0$, the the current $J^\mu = \bar{\Psi}\gamma^\mu \Psi$

4

8) Dirac Equation : Spin + Antiparticles

1) $\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \vec{\Sigma}$

Commutator $[\vec{\Sigma}, H_0] = [\vec{\Sigma}, \alpha \cdot p + \beta m]$
but as β is just $= \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$ is a unitary, then $\vec{\Sigma}$ and βm commute, so we have

$$[\vec{\Sigma}, \alpha \cdot \vec{p}] = \Sigma_j \alpha_i p_i - \alpha_i p_i \Sigma_j$$
$$= \Sigma_j \alpha_i p_i - \alpha_i p_i \Sigma_j + \alpha_i \Sigma_j p_j - \alpha_i \Sigma_j p_j$$
$$= [\Sigma_j, \alpha_i] p_i + \alpha_i [\Sigma_j, p_j]$$

Now $[\Sigma_j, p_j] = 0$ because you have $\vec{\sigma} \cdot \vec{p} = \vec{p} \cdot \vec{\sigma}$, hence they commute; this leaves

$$= [\Sigma_j, \alpha_i] p_i$$ no, carefully compare them

~~Ap~~ Σ_j is essentially the same as α_j, so

$$= [\alpha_j, \alpha_i] p_i = -[\alpha_i, \alpha_j] p_i \quad (1)$$

the α matrices have the same commutation relation as pauli matrices so $[\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k$

Hence (1) $= -2i \varepsilon_{ijk} \alpha_k p_i$
$= -2i \alpha_j \times p_j = \underline{-2i \alpha \times p}$ 2

(2) Dirac Equation for spinors $\bar{u}(p,s)$, $v(p,s)$
$= (\gamma^\mu p_\mu - m) u(p,s) = 0$
$= (\gamma^\mu p_\mu + m) v(p,s) = 0$

Start by looking(?) at $\bar{u}(p,r) \gamma^\mu u(p,s) = \bar{u} \gamma^\mu u$
in order to manipulate, note that $(\gamma^\beta p_\beta - m) u = 0$; hence,
$\gamma^\beta p_\beta = m$, then $\frac{\gamma^\beta p_\beta}{m} = 1$

We can modify, $\bar{u} \gamma^\mu u = \frac{\bar{u} \gamma^\mu \gamma^\beta p_\beta u}{m}$

Use property of $\{\gamma^\mu, \gamma^\beta\} = \gamma^\mu \gamma^\beta + \gamma^\beta \gamma^\mu$ →

$$\bar{u}\gamma^\mu u = \frac{1}{m}\bar{u}\left(\{\gamma^\mu,\gamma^\beta\} - \gamma^\beta\gamma^\mu\right)p_\beta u$$

$$= \frac{1}{m}\bar{u}\left(2g^{\mu\beta} - \gamma^\beta\gamma^\mu\right)p_\beta u$$

$$= \frac{2}{m}\bar{u}u\,p^\mu - \frac{1}{m}\bar{u}\gamma^\beta p_\beta\gamma^\mu u \quad \rightarrow \text{using } \frac{\gamma^\beta p_\beta}{m} = 1$$

$$= \frac{2}{m}\bar{u}u\,p^\mu - \bar{u}\gamma^\mu u$$

Hence $\bar{u}\gamma^\mu u = \frac{2}{m}\bar{u}u p^\mu - \bar{u}\gamma^\mu u$

$$= \frac{\cancel{2}}{m}\bar{u}u\,p^\mu = \cancel{2}\bar{u}\gamma^\mu u \qquad u^\dagger u = 2E$$

take $\mu = 0$ so $p^0 = E$

$$= \frac{\bar{u}u\,p^0}{m} = \bar{u}\gamma^0 u = u^\dagger u$$

$$= \frac{\bar{u}u\cancel{E}}{m} = 2\cancel{E} \rightarrow \bar{u}u = 2M$$

or more reasonably, $\underline{\underline{\bar{u}(p,r)u(p,s) = 2M\delta_{rs}}}$

Do the same process for v → NOTING a small difference

$\bar{v}\gamma^\mu v$ $\qquad\qquad \frac{-\gamma^\beta p_\beta}{m} = 1$

$$\bar{v}\gamma^\mu v = -\frac{\bar{v}\gamma^\mu\gamma^\beta p_\beta v}{m} = -\frac{1}{m}\bar{v}\left(\{\gamma^\mu,\gamma^\beta\} - \gamma^\beta\gamma^\mu\right)p_\beta v$$

$$= -\frac{1}{m}\bar{v}\left(2g^{\mu\beta} - \gamma^\beta\gamma^\mu\right)p_\beta v$$

(*) $$= -\frac{2}{m}\bar{v}v\,p^\mu + \frac{1}{m}\bar{v}\gamma^\beta p_\beta\gamma^\mu v$$

$$= -\frac{2}{m}\bar{v}v\,p^\mu - \bar{v}\gamma^\mu v$$

Hence $-\frac{2}{m}\bar{v}v\,p^\mu = 2\bar{v}\gamma^\mu v$

$\mu = 0$ $\qquad -\frac{2\bar{v}vE}{m} = 2v^\dagger v = 2E$; then $\bar{v}v = -2m$

more explicitly, $\underline{\underline{\bar{v}(p,r)v(p,s) = -2m\delta_{rs}}}$

The final relation is more subtle and a little less clear for the chosen method above; the trick of using $\frac{\gamma^\beta p_\beta}{m} = 1, -1$ depending on $\bar{v}$ or $\bar{u}$

see step at (*) ; if using $\frac{\gamma^\beta p_\beta}{m}$ the first time to get to this stage, use the opposite the second ; this cancels out the LHS

$$\bar{v}\gamma^\mu v - \bar{v}\gamma^\mu v = 0, \text{ such that } \bar{v}u = 0$$

$$\underline{\bar{v}(p,r)u(p,s) = 0}$$

$\rightarrow$ $(\gamma_\mu p^\mu - m)\,u(p) = 0$
$(\gamma_\mu p^\mu + m)\,V(p) = 0$ (A)

Write $A_s(p) = \sum_s u(p,s)\,\bar{u}(p,s)$

$$A_s(p)\,U_{s'}(p) = \sum u(p,s)\left[\bar{u}(p,s)\,u_{s'}(p,s)\right]$$
$$= \sum_{s'} u(p,s')\,[2m]$$
$$= 2m\,u(p,s')$$

NOTE: $A_s(p)\,V_s(p) = 0$! because of previous relations; also noting (A), it is clear that the operation of $A_s(p)$ on a spinor, produces the same result as $\underline{\gamma_\mu p^\mu + m}$

then $\sum_s u(p,s)\,\bar{u}(p,s) = \gamma_\mu p^\mu + m$

If defining $A_s(p) = \sum_s v(p,s)\,\bar{v}(p,s)$, the same procedure gives
$\sum_s v(p,s)\,\bar{v}(p,s) = \gamma_\mu p^\mu - m$

3

there is small step you did not make clear