Katie Graham
13/10/14

RQM Assessment 3: Perturbation Theory for Particle Scattering.

1) Show that solution of the Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = \left[H_0(\vec{x}) + \lambda V(t,\vec{x})\right]\Psi$$

written in form of linear superposition of stationary states $\phi_n(\vec{x})$

$$\Psi = \sum_n c_n(t)\phi_n(\vec{x})e^{-iE_nt}$$

yields to system of differential equations [6]

$$\frac{dc_f(t)}{dt} = -i\lambda\sum_n c_n(t)e^{i(E_f-E_n)t}\int\phi_f^* V(t,\vec{x})\phi_n(\vec{x})d^3\vec{x}.$$

$$i\frac{\partial}{\partial t}\left[\sum_n c_n(t)\phi_n(\vec{x})e^{-iE_nt}\right] = \left[H_0(\vec{x}) + \lambda V(t,\vec{x})\right]\left[\sum_n c_n(t)\phi_n(\vec{x})e^{-iE_nt}\right]$$

$$= i\sum_n c_n(t)\frac{\partial}{\partial t}\left[\phi_n(\vec{x})e^{-iE_nt}\right] + i\sum_n\phi_n(\vec{x})e^{-iE_nt}\frac{\partial c_n(t)}{\partial t}$$

Using the free ($V=0$) Schrödinger equation,

$$\sum_n i\frac{\partial}{\partial t}\left(\phi_n(\vec{x})e^{-iE_nt}\right) = H_0(\vec{x})\sum_n\left(\phi_n(\vec{x})e^{-iE_nt}\right),$$

$$\therefore\ i\sum_n\phi_n(\vec{x})e^{-iE_nt}\frac{\partial c_n(t)}{\partial t} = \lambda V(t,\vec{x})\sum_n c_n(t)\phi_n(\vec{x})e^{-iE_nt}$$

Multiplying by $\phi_f^*(\vec{x})$, then integrating over space,

$$i\phi_f^*(\vec{x})\sum_n\phi_n(\vec{x})e^{-iE_nt}\frac{\partial c_n(t)}{\partial t} = \lambda\phi_f^*(\vec{x})V(t,\vec{x})\sum_n c_n(t)\phi_n(\vec{x})e^{-iE_nt}$$

$$i\sum_n e^{-iE_nt}\frac{\partial c_n(t)}{\partial t}\underbrace{\int\phi_f^*(\vec{x})\phi_n(\vec{x})d^3\vec{x}}_{=\delta_{fn}\ \text{(normalisation)}} = \lambda\sum_n c_n(t)e^{-iE_nt}\int\phi_f^*(\vec{x})V(t,\vec{x})\phi_n(\vec{x})d^3\vec{x}$$

$$\therefore\ ie^{-iE_ft}\frac{dc_f(t)}{dt} = \lambda\sum_n c_n(t)e^{-iE_nt}\int\phi_f^*(\vec{x})V(t,\vec{x})\phi_n(\vec{x})d^3\vec{x}$$

$$\therefore\ \frac{dc_f(t)}{dt} = -i\lambda\sum_n c_n(t)e^{i(E_f-E_n)t}\int\phi_f^*(\vec{x})V(t,\vec{x})\phi_n(\vec{x})d^3\vec{x}$$

6

KG

2) System of equations from problem 1 can be solved approximately by expanding c_n as

$$c_n(t) = c_n^0(t) + \lambda c_n^1(t) + \lambda^2 c_n^2(t) + \dots$$

Derive recursive formulas to calculate c_f.

$$\frac{dc_f^0(t)}{dt} + \lambda \frac{dc_f^1(t)}{dt} + \lambda^2 \frac{dc_f^2(t)}{dt} + \dots = -i\lambda \sum_n \left(c_n^0(t) + \lambda c_n^1(t) + \lambda^2 c_n^2(t) + \dots \right) e^{i\omega_{nf}t} V_{fn}(t)$$

using the definitions $(E_f - E_n) \equiv \omega_{nf}$, $\int \phi_f^*(\vec{x}) V(t,\vec{x}) \phi_n(\vec{x}) d^3\vec{x} \equiv V_{fn}(t)$.

Comparing powers of λ,

$$\frac{dc_f^0(t)}{dt} = 0$$

$$\frac{dc_f^1(t)}{dt} = -i \sum_n c_n^0(t) e^{i\omega_{nf}t} V_{fn}(t)$$

$$\frac{dc_f^2(t)}{dt} = -i \sum_n c_n^1(t) e^{i\omega_{nf}t} V_{fn}(t)$$

...

$$\frac{dc_f^{s+1}(t)}{dt} = -i \sum_n c_n^s(t) e^{i\omega_{nf}t} V_{fn}(t)$$

4

$$c_f^2(t) = -i \sum_n \int c_n^1(t) e^{i\omega_{nf}t} V_{fn}(t)\, dt$$

Katie
Graham
13/10/14

RQM Assessment: Lecture 4

1) Using recursion formulae for coefficients in linear superposition of general solution of Schrödinger equation, show that the second order term is

$$c_f^2(t) = -2\pi i\,\delta(E_f - E_i)\sum_{n\neq i} V_{fn}\frac{1}{E_n - E_i}V_{ni},$$

where E_f and E_i is energy of final and initial state. [4]

$$C_f^2(t) = -i\sum_n \int_{-\infty}^{t} dt' V_{fn} e^{i(E_f - E_n)t'} C_n^1(t)$$

$$= (-i)^2 \sum_n \int_{-\infty}^{t} dt' V_{fn} e^{i(E_f-E_n)t'} \int_{-\infty}^{t'} dt'' V_{ni} e^{i(E_n - E_i)t''}$$

$$= (-i)^2 2\pi i\,\delta(E_f - E_i)\sum_{n\neq i} V_{fn}\frac{1}{E_n - E_i}V_{ni}.$$

4

2) Extend calculation from previous question to obtain coefficient $C_f^3(t)$ at third order in expansion. [4]

$$C_f^3(t) = -i\sum_n \int_{-\infty}^{t} dt' V_{fn} e^{i(E_f - E_n)t'} C_n^2(t')$$

$$= +i\sum_n \int_{-\infty}^{t} dt' V_{fn} e^{i(E_f-E_n)t'}\, 2\pi i\,\delta(E_f - E_i)\sum_{n\neq i} V_{fn}\frac{1}{E_n - E_i}V_{ni}$$

$$= -\sum_{n\neq f} (2\pi i)^2 \frac{1}{E_f - E_n} V_{fn}\cdot V_{fn}\frac{1}{E_n - E_f}\cdot V_{nf}$$

3

$$= -4\pi^2 \cancel{\frac{1}{(E_f - E_n)^2}} |V_{fn}|^2 \cdot V_{nf}$$

should be careful about labeling states

$i \to n \to m \to f$

KG

3) Using minimal substitution derive interaction potential for spin-0 particle interacting with electromagnetic field. [2]

Minimal substitution: $i\partial^\mu \to i\partial^\mu + eA^\mu \therefore \partial^\mu \to \partial^\mu - ieA^\mu$

Klein-Gordon equation: $\left[\partial^\mu \partial_\mu + m^2\right]\phi = 0$

$$\therefore \left[(\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) + m^2\right]\phi = 0$$

$$(\partial^\mu \partial_\mu - e^2 A^\mu A_\mu - ie\partial^\mu A_\mu - ieA^\mu \partial_\mu + m^2)\phi = 0$$

$$\therefore (\partial^\mu \partial_\mu + m^2)\phi = -V\phi = ie(\partial^\mu A_\mu + A^\mu \partial_\mu) + e^2 A^\mu A_\mu$$

$$\therefore V = -ie(\partial^\mu A_\mu + A^\mu \partial_\mu) + e^2 A^\mu A_\mu$$

2

Katie Graham

RQM 5: From Invariant Amplitudes to Invariant Quantities

1) Starting from transition rate

$$dW = \frac{|M_{fi}|^2}{V^{n_i}}(2\pi)^4\delta^4(p_f - p_i)\prod_{f=1}^{n_f}\frac{d^3\vec{p}_f}{2E_f(2\pi)^3}$$

show that cross section for scattering process $a+b \rightarrow c+d$ can be written as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2(p_a+p_b)^2}\frac{|\vec{p}_c|}{|\vec{p}_a|}|M_{fi}|^2.$$ [10]

$$dW = \int\frac{|M_{fi}|^2}{V^{n_i}}(2\pi)^4\delta^4(P_C + P_D - P_A - P_B)\frac{d^3P_C}{(2\pi)^3 2E_C}\frac{d^3P_D}{(2\pi)^3 2E_D}$$

$$= \int\frac{|M_{fi}|^2}{V^{n_i}}\frac{1}{4\pi^2}\frac{d^3P_C}{2E_C}\frac{1}{2E_D}\delta(E_A + E_B - E_C - E_D)$$

In CM frame, $\sqrt{s} = E = E_A + E_B$, $|\vec{P_C}| = |\vec{P_D}|$, $d^3P_C = P_C^2 dP_C d\Omega$,

$$E = E_C + E_D = \sqrt{m_C^2 + P_C^2} + \sqrt{m_D^2 + P_C^2}$$

$$\therefore dW = \frac{|M_{fi}|^2}{V^{n_i}}\frac{1}{4\pi^2}\frac{P_C^2 dP_C d\Omega}{4E_C E_D}\delta(E - E_C - E_D)$$

$$\delta(f(x)) = \left|\frac{df}{dx}\right|_{x_0}^{-1}\delta(x - x_0) \rightarrow \frac{dE}{dP_C} = |\vec{P}_C|\frac{E_C + E_D}{E_C E_D}$$

$$\therefore dW = \frac{|M_{fi}|^2}{V^{n_i}}\frac{1}{4\pi^2}\frac{|\vec{P_C}|}{4(E_C+E_D)}\cancel{dE\,\delta(E - E_C - E_D)}\,d\Omega$$

$$d\sigma = \frac{dW}{4|\vec{P_A}|\sqrt{s}} = \frac{|M_{fi}|^2}{V^{n_i}}\frac{1}{64\pi^2}\frac{|\vec{P_C}|}{|\vec{P_A}|}\frac{1}{(E_C+E_D)^2}d\Omega$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2(P_a+P_b)^2}\frac{|\vec{P_c}|}{|\vec{P_a}|}|M_{fi}|^2$$

10