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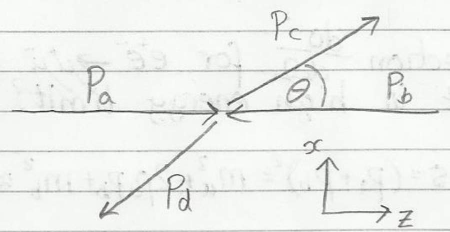
RQM 6: Cross-Section for Scattering of Spin-0 Particles

1) Starting from the result for $e^-\mu^- \to e^-\mu^-$ scattering

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\left(\frac{s-u}{t}\right)^2,$$

show that in the centre of mass system in the high energy limit where masses are negligible this has the form

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2. \qquad [3]$$



$$p_a = (p, 0, 0, p)$$
$$p_b = (p, 0, 0, -p)$$
$$p_c = (p, p\sin\theta, 0, p\cos\theta)$$
$$p_d = (p, -p\sin\theta, 0, -p\cos\theta)$$

$$s = (p_a+p_b)^2 = (p+p)^2 - (p-p)^2 = 4p^2$$

$$t = (p_a-p_c)^2 = (p-p)^2 - (0-p\sin\theta)^2 - (p-p\cos\theta)^2 = -2p^2(1-\cos\theta)$$

$$u = (p_a-p_d)^2 = (p-p)^2 - (0-p\sin\theta)^2 - (p+p\cos\theta)^2 = -2p^2(1+\cos\theta)$$

$$\frac{s-u}{t} = \frac{4p^2+2p^2(1+\cos\theta)}{-2p^2(1-\cos\theta)} = \frac{3+\cos\theta}{\cos\theta-1}$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2$$

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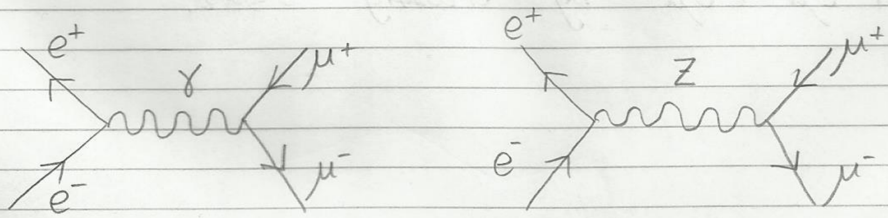
2) Draw all Feynmann diagrams at lowest order for $e^+e^- \to \mu^+\mu^-$. [2]



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3) By treating all particles as spinless particles derive invariant amplitude $-iM$ for $e^+e^- \to \mu^+\mu^-$. [2]

$$-iM = ie(p_a - p_b)^\mu \left(\frac{-ig_{\mu\nu}}{q^2}\right) ie(p_c - p_d)^\nu$$

$$= \frac{ie^2 (p_a - p_b)^\mu (p_c - p_d)_\mu}{(p_a + p_b)^2}$$

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4) Derive differential cross-section $\frac{d\sigma}{d\Omega}$ for $e^+e^- \to \mu^+\mu^-$ scattering. What can you say about it in high energy limit? [3]

In the high energy limit, $s = (p_a + p_b)^2 = m_a^2 + 2p_a p_b + m_b^2 \approx 2p_a \cdot p_b$,

$t \approx -2p_a \cdot p_c$ or $-2p_b \cdot p_d$, $u \approx -2p_a \cdot p_d$ or $-2p_b \cdot p_c$

$\therefore (p_a - p_b)^\mu (p_c - p_d)_\mu = p_a \cdot p_c - p_a \cdot p_d - p_b \cdot p_c + p_b \cdot p_d$

in high energy limit $\approx -\frac{1}{2}t + \frac{1}{2}u + \frac{1}{2}u - \frac{1}{2}t = u - t$

$\therefore -iM = ie^2 \left(\frac{u-t}{s}\right)$.

$$\therefore \frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}\left(\frac{u-t}{s}\right)^2$$

So in the high energy limit $e^+e^- \to \mu^+\mu^-$ can be obtained from $e^-\mu^- \to e^-\mu^-$ by 'crossing' $s \leftrightarrow t$.

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RQM 7: The Dirac Equation

1) Starting from properties of $\vec{\alpha}$ and β show that γ^μ matrices defined as $(\beta, \beta\vec{\alpha})$ satisfy the following relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$ [3]

$$\alpha_i^2 = 1, \quad \beta^2 = 1, \quad \alpha_i\alpha_j + \alpha_j\alpha_i = 0, \quad \alpha_i\beta + \beta\alpha_i = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \cancel{\gamma^\mu g^{\mu\nu}\gamma_\mu} + \cancel{g^{\mu\nu}_{\mu}\gamma_\mu\gamma^\mu}$$

I'm not getting this

$$\rightarrow g^{\mu\nu}\gamma_\nu\gamma^\nu + g^{\mu\nu}\gamma_\mu\gamma^\mu = g^{\mu\nu}(\gamma_\nu\gamma^\nu + \gamma_\mu\gamma^\mu)$$

$$= g^{\mu\nu}\left[\begin{pmatrix}\beta \\ \beta\alpha_i \\ \beta\alpha_j \\ \beta\alpha_k\end{pmatrix}(\beta, \beta\alpha_i, \beta\alpha_j, \beta\alpha_k) + \begin{pmatrix}\beta \\ \beta\alpha_l \\ \beta\alpha_m \\ \beta\alpha_n\end{pmatrix}(\beta, \beta\alpha_l, \beta\alpha_m, \beta\alpha_n)\right]$$

$$= g^{\mu\nu}\left[\begin{pmatrix}1 & \alpha_i & \alpha_j & \alpha_k \\ \alpha_i & 1 & \alpha_i\alpha_j & \alpha_i\alpha_k \\ \alpha_j & \alpha_i\alpha_j & 1 & \alpha_j\alpha_k \\ \alpha_k & \alpha_k\alpha_i & \alpha_k\alpha_j & 1\end{pmatrix} + \begin{pmatrix}1 & \alpha_l & \alpha_m & \alpha_n \\ \alpha_l & 1 & \alpha_l\alpha_m & \alpha_l\alpha_n \\ \alpha_m & \alpha_l\alpha_m & 1 & \alpha_m\alpha_n \\ \alpha_n & \alpha_m\alpha_l & \alpha_n\alpha_m & 1\end{pmatrix}\right]$$

$\neq 0$

$$= \begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{pmatrix}\begin{bmatrix}2 & \alpha_i+\alpha_l & \alpha_j+\alpha_m & \alpha_k+\alpha_n \\ \alpha_i+\alpha_l & 2 & \alpha_i\alpha_j+\alpha_l\alpha_m & \alpha_i\alpha_k+\alpha_l\alpha_n \\ \alpha_j+\alpha_m & \alpha_i\alpha_j+\alpha_l\alpha_m & 2 & \alpha_j\alpha_k+\alpha_m\alpha_n \\ \alpha_k+\alpha_n & \alpha_k\alpha_i+\alpha_m\alpha_l & \alpha_k\alpha_j+\alpha_n\alpha_m & 2\end{bmatrix}$$

$$= \begin{pmatrix}2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2\end{pmatrix} = 2g^{\mu\nu}$$

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2) Show that Dirac equation for conjugate spinor defined as $\bar{\Psi} = \Psi^{+}\gamma^{0}$ is

$$i\partial_\mu \bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0.$$ [3]

~~The Hermitian conjugate of the Dirac equation is~~

$$\cancel{\Psi^{+}(-i\gamma^{\mu}\partial_\mu - m) = 0}$$

Dirac equation: $(i\gamma^\mu\partial_\mu - m)\Psi = 0$

Taking the Hermitian conjugate, & using $\left[\gamma^0 \frac{\partial\Psi}{\partial t}\right]^{+} = \frac{\partial \Psi^{+}}{\partial t}\gamma^{0+}$, etc.,

$$-i\partial_\mu \Psi^{+}\gamma^{\mu+} - m\Psi^{+} = 0$$

$$-i\partial_\mu \Psi^{+}\gamma^0\gamma^\mu\gamma^0 - m\Psi^{+} = 0 \qquad (\gamma^{\mu+} = \gamma^0\gamma^\mu\gamma^0)$$

Multiplying on the right by γ^0,

$$-i\partial_\mu \Psi^{+}\gamma^0\gamma^\mu\gamma^0\gamma^0 - m\Psi^{+}\gamma^0 = 0.$$

Using $\gamma^0\gamma^0 = \mathbb{1}$, and $\bar{\Psi} = \Psi^{+}\gamma^0$,

$$+i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0$$

3

3) Derive current associated with Dirac equation. [4]

Multiplying Dirac by $\bar{\Psi}$ on the left & adjoint by Ψ on the right,

$$i\bar{\Psi}\gamma^\mu(\partial_\mu\Psi) - m\bar{\Psi}\Psi = 0 \quad \& \quad i(\partial_\mu\bar{\Psi})\gamma^\mu\Psi + m\bar{\Psi}\Psi = 0$$

Adding, $i\bar{\Psi}\gamma^\mu(\partial_\mu\Psi) + i(\partial_\mu\bar{\Psi})\gamma^\mu\Psi = 0$

$$= \partial_\mu(\bar{\Psi}\gamma^\mu\Psi) = 0$$

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$\therefore$ the conserved quantity is the current:

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi$$

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RQM 8: The Dirac Equation: Spin, Antiparticles

1) Consider operator $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

and show that its commutator with Hamiltonian $H_0 = \vec{\alpha}\cdot\vec{p} + \beta m$ is $-2i\vec{\alpha}\times\vec{p}$. [3]

$$H_0 = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\vec{p} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} m = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}$$

$$[\vec{\Sigma}, H_0] = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} - \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} \vec{\sigma}m & \vec{\sigma}\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\vec{\sigma}\cdot\vec{p} & -\vec{\sigma}m \end{pmatrix} - \begin{pmatrix} m\vec{\sigma} & \vec{\sigma}\cdot\vec{p}\cdot\vec{\sigma} \\ \vec{\sigma}\cdot\vec{p}\cdot\vec{\sigma} & -m\vec{\sigma} \end{pmatrix}$$

$\left([a_x, \sigma_y] = 2ia_z \text{ etc.}\right)$

$$= \begin{pmatrix} 0 & \vec{\sigma}(\vec{\sigma}\cdot\vec{p} - \vec{p}\cdot\vec{\sigma}) \\ \vec{\sigma}(\vec{\sigma}\cdot\vec{p} - \vec{p}\cdot\vec{\sigma}) & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \vec{\sigma}[\vec{p}, \vec{\sigma}] \\ \vec{\sigma}[\vec{p}, \vec{\sigma}] & 0 \end{pmatrix}$$

$$= -2i\vec{\alpha}\times\vec{p}$$

3

2) Find explicit expression for conjugated spinors $\bar{u}$ and $\bar{v}$ and show that

$$\bar{u}(p,r)u(p,s) = 2m\delta_{rs}$$
$$\bar{v}(p,r)v(p,s) = -2m\delta_{rs}$$
$$\bar{v}(p,r)u(p,s) = 0.$$

[4]

$$u(p,s) = (E+m)^{1/2}\begin{pmatrix} \phi^s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi^s \end{pmatrix} \quad \therefore \quad \bar{u}(p,s) = (E+m)^{1/2}\left(\phi^{s+}\cancel{\gamma^0}, \ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi^{s+}\cancel{\gamma^0}\right)\gamma^0 \ ?$$

?

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