A Minimal Model of Railway Network Congestion

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We consider an interconnected network of stations which for simplicity we take to form a square grid with nearest neighbour connections. Each station $i$is characterised by a capacity C_i being the number of trains it can admit, platform and despatch per unit time. Arrivals in excess of capacity lengthen the queue of Q_i trains held (just upstream of) each station.

For simplicity we work with a discrete timestep and we assume this absorbs the transit time between adjacent stations.

We make one further simplifying assumption which could influence the results in more than detail, which is that the outgoing flux of trains from a given station at a given timestep is distributed uniformly over neighbour destinations (rather than allocated according to proor train routings). This greatly simplifies computing and interpreting the model.

Detailed Algorithm

At time t each station receives its inbound trains which added to its inbound queue, give a total load

$$q_j(t) = Q_j(t-1) + \sum_{i \text{ neighbouring } j} J_{ij}(t-1),$$

where $J_{ij}(t-1)$ is the number of trains despatched from i towards j at time $t-1$. Then station j despatches outbound a total number of trains

$$J_j(t) = \min\left(C_j, q_j(t)\right)$$

leaving a queue of trains held over

$$Q_j(t) = q_j(t) - J_j(t).$$

Finally for the results displayed here we allocate the outgoing trains uniformly over directions so

$$J_{ji}(t) = J_j(t)/z_j$$

where station j is directly connected to z_j other stations.

Note that the sum of the loads q_j is conserved, being the total number of trains. Thus the load stage is in practice a convenient starting point in simulation and we specify the initial condition in terms of station loads which can be disordered. The system can be made intrinsically disordered through disorder in the capacities.

Simulation Results with Uniform Capacity

We first consider the case where the capacity is uniform and all disorder comes from the initial loads. Figure 1 shows results from such a simulation where the mean load was exactly matched to the uniform capacity, exhibiting critical behaviour. The cumulative autocovariance is defined as $ca(r) = \int_0^r \langle q(r')q(R+r') \rangle \, d^2R$ and the measurements clearly exhibit a regime of power law behaviour consistent with fractal scaling $ca(r) \propto r^D$ and a fractal dimension $D \simeq 0.8$. Both the queues and the loads exhibit the same scaling out to a lengthscale which increases with time. The amplitudes approach each other at late times and deay increasingly slowly.

Figures 2 and 3 show the behaviour for mean load to capacity at ratio 1.05 and 0.95 respectively, which we interpret as super- and sub-critical resectively. They both show the fractal scaling extending less far in distance than the matched load case, which matches the interpretation that they are off-critical. They differ markedly in their later time dependence: the supercritical case exhibits a fixation of the fractal correlations both in terms of spatial pattern and in terms of amplitude of correlation, whereas in the subcritical case all of the correlations wash out.

Simulation Results with non-uniform capacity

If we supply fixed disorder through the station capacities then the behaviour is very different. Fig 4 shows results for a case of matched load to that of the avergage station capacity, and similar apply when there is mismatch and when the intial loads are also disordered. Offsite correlation is now negative (as indicated by the negative slope of cumulative correlation), and the coarsenning process selects a uniform dust. What seems to be happenning is that the lowest capacity sites have ever growing queus whihc can never be mitigated.

We have also investigated giving the station capacities time dependent white noise. As shown in fig 5, the bahaviour is much as in the static disorder case except that there is no coarse graining. Essentially queues are popping up but then disipating locally everywhere all of the time.

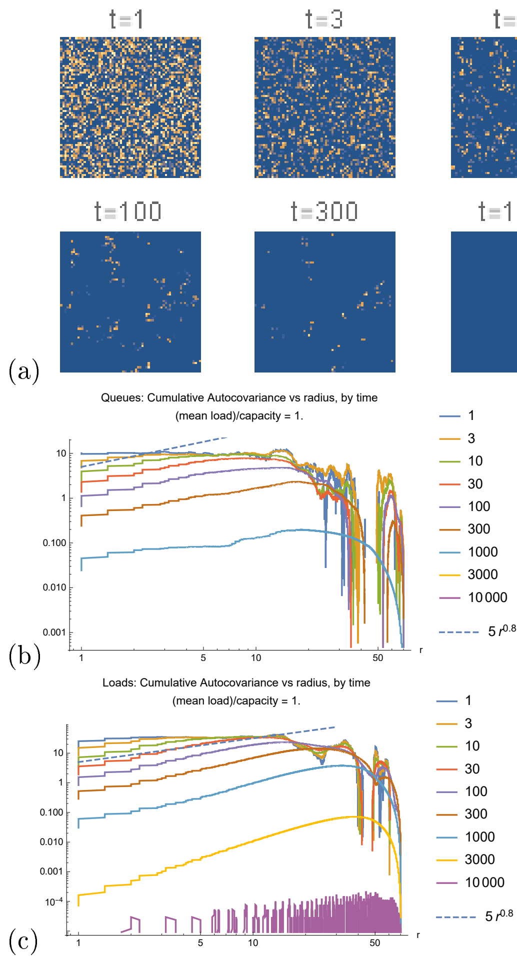


Figure 1: Results from a 100x100 simulation with disordered inital loads of mean matched to uniform station capacity. (a) Snapshots of the queue distribution for times t as labelled. Dark blue corresponds to zero queue. (b) Cumulative Spatial Autocovarince of the queues for times per the legend, consistent with fractal scaling r^D with $D \simeq 0.8$ but amplitude of the correlations decreasing with time. (c) Cumulative Spatial Autocovariance of the load distribution exhibits the same spatial scaling.

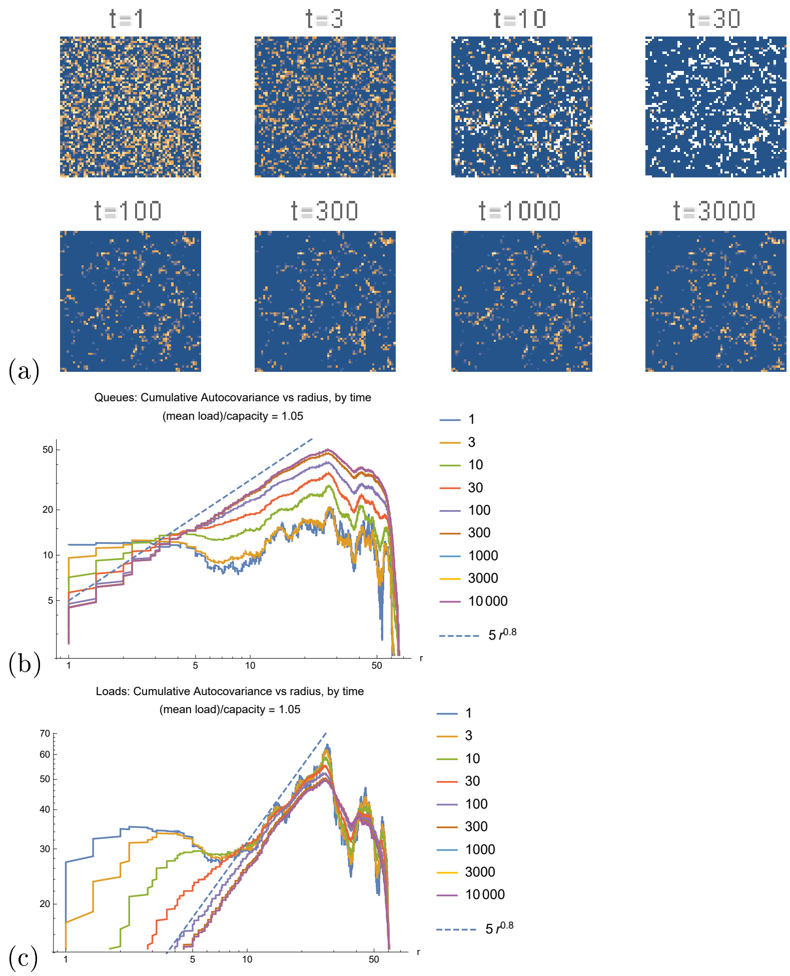


Figure 2: Results for a supercritcal simulation with mean load/capacity = 1.05, and other details as per figure 1. The autovoariance of the queues (b) shows that at long times the fractal correlations fixate rather than decaying in amplitude, but they do extend less far in space (a trend more noticeable at higher excess load). Note that at late times the loads (c) show the same correlation as the queues, in both amplitude and exponent.

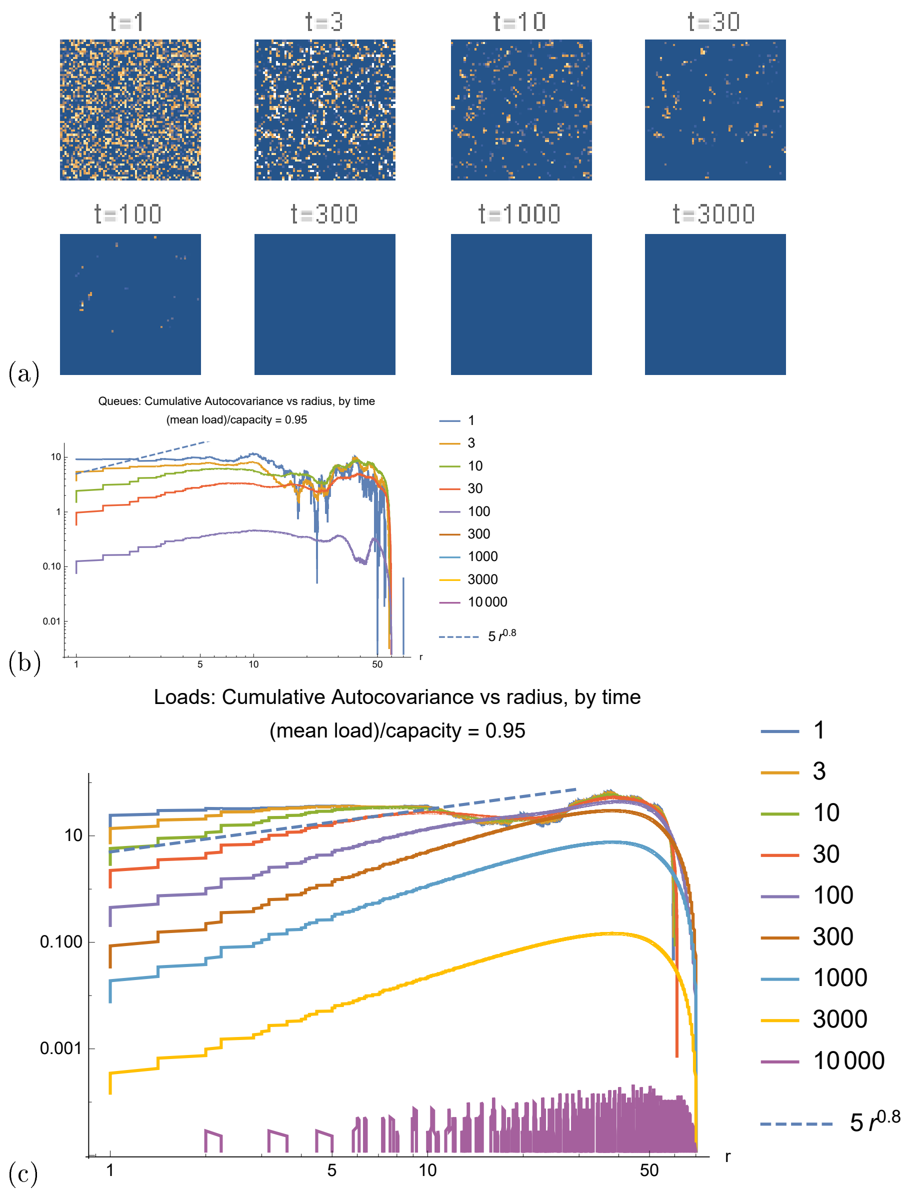


Figure 3: Results for a subcritical simulation (a) with mean load/capacity = 0.95, and other details as per figure 1. Strikingly the non-trivial correlations wash out at later times. The queue autocovariance (b) shows fractal region per higher loads limited to shorter distances, as expected for a subcritical problem. At larger distances the queues exhibit higher apparent correlation dimension whilst the loads (c) exhibit lower apparent dimension. with perhaps a lower dimension.

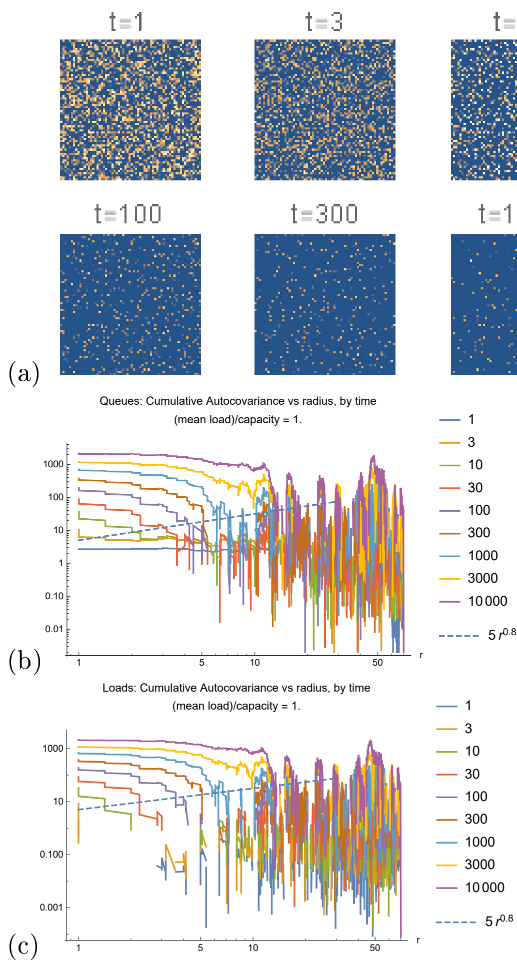


Figure 4: (a) Simulation with frozen disorder in the station capacities and uniform initial load. This simulation has mean capacity matched to load, but moderately mismatched results are similar. The declining cumulative covariance plots for queues (a) and loads (b) indicate only weak negative correlation at distance. There is coarsening, but the residual structure is concentrated on a uniform dust rather than a fractal.

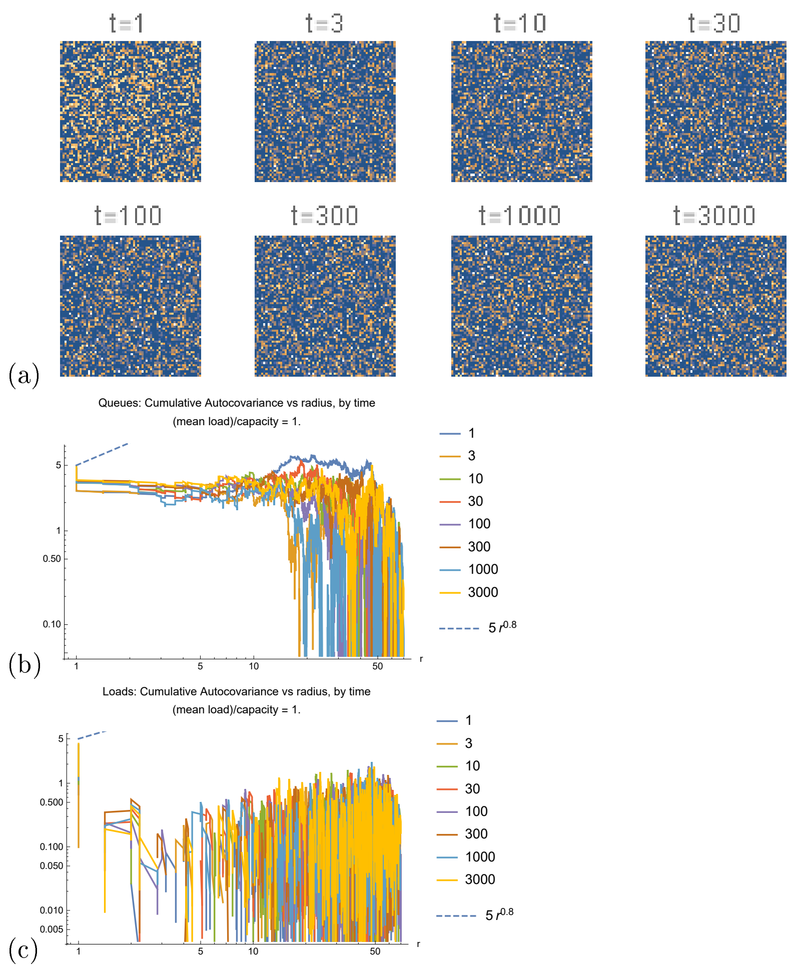


Figure 5: (a) Simulation as per figure 4 but with volatile disorder in the station capacities. Again there is weak negative correlation at distance, but now without any coarsenning.

Model with fixed route trains

In this section we refine the model such that, after any queueing, each train moves always in the same direction, meaning that the train keeps a fixed route which is a stright loop round the periodic boundary condition. Each station now has four queues, one for each of directions $\pm\hat{x}$, $\pm\hat{y}$. The total load is then given by $q_j = \sum_{\text{directions } d} q_{jd}$ and the total outflux then as before by $J_j(t) = \min(C_j, q_j(t))$, which is then repartitioned in proportion to the loads so that $J_{jd} = J_j q_{jd}/q_j$. This is not quite FIFO, but shares the feature that if one direction dominates the total load, then it also dominates the total outflow.

Figrure 6 shows that whilst the spatial correlation of the queues is significantly directed by the route bias of the trains, when the correlation is averaged over directions the same power law behaviour emerges as in the model with equal outfluxes. Again the sub- and super-critical trends are as before. It should be boted that this simulation strictly conserves the number of trains travelling in each direction along a given track, and this freezes in quite significant fluctuations from the intial condition, of order 10% for $L = 100$ as simulated here. Figure 7 shows the resutls of mitigating this by constraining the number of trains on each track direction to have equal total, so that the frozen fluctuations are zero.

Theoretical Interpretation

We focus on the case where the disorder lies only in the initial loads, in which case the key challenge is to understand how an apparently fractal spatial distribution of queues emerges around matched mean loading, which eventually fixates for supercritical loading and washes out in the subcritical case. We begin by focussing on the critical case for which mean load is exactly matched to (uniform and fixed) station capacity. Local uncorrelated fluctuations in the inital loads lead to the local average load in regions of radius r having deviation from the global mean $\delta\rho(r) \simeq \pm r$. In a characteristic time $\tau(r)$ regions of small radius r with negative fluctuation can even out and draw down any outstanding queues within them, whereas their equivalents with positive fluctuation have fluxes limited by station capacity: these fluxes balance leaving no overall net eflux from sites with a queue. We can then make the same argument for slightly larger regions on correspondingly longer timescale. The result is then that queues of order unity persist totalling r trains in the originally positive fluctuation regions of radius r, so at this level we explain a fractal distribution of queues with fractal dimension $D = 1$, which is conceivably in correspondance with the observations of $D \simeq 0.8$.

In the supercritical case, there will come a stage where the excess train density held in queues, $\rho_{queue} = Ar(t)^{D-2}$ matches the global excess load $\delta\rho$, at time given by $t = \tau(r)$: the queues then fixate because all stations operate at capacity. In the critical case there is a diminishing density deficit which by time $\tau(L)$ is spread uniformaly over the sites without queues. All queues then

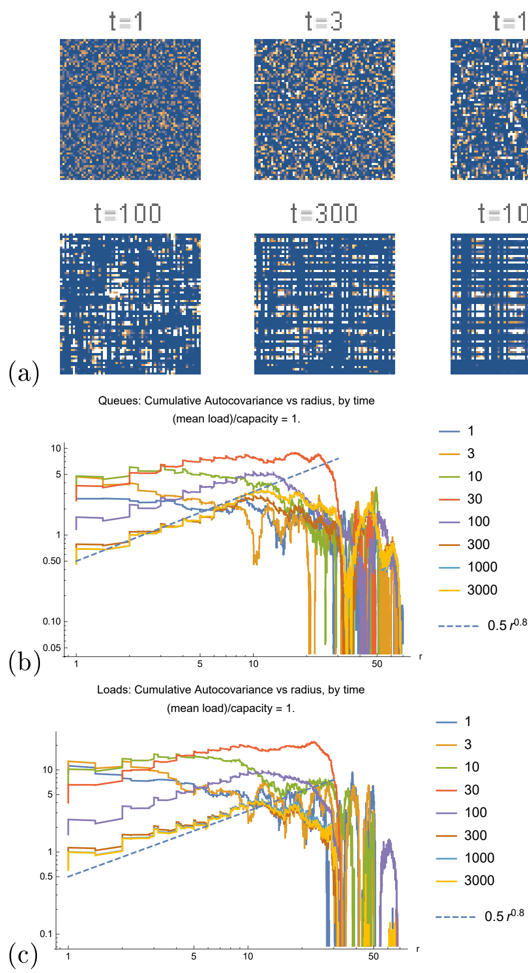
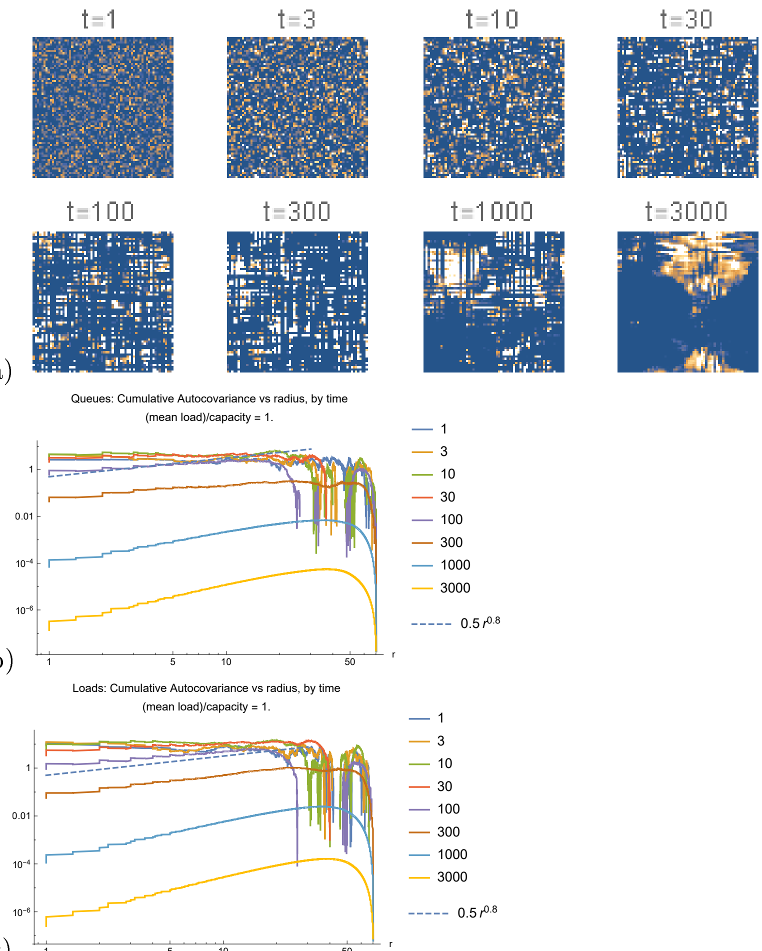


Figure 6: Simulation as per figure 1 but with trains continuing along fixed lines rather than being rerouted uniformly at each station. The linear routing clearly biases the alignment of queues seen in the queue denisty plot (a) but averaged over orientations the correlations of queues (b) and loads (c) both show the same radial dependence as before.



(a) (b) (c)

Figure 7: Simulation as per figure 6 but with equal number of trains trains travelling along each track direction. As per figure 1 there is is no longer fixation but the same fractal dimension can be seen.

erode at a system size dependent rate AL^{D-2} leading to the amplitude $A(t)$ of the fractal correlations decaying with decay time $\tau_A \sim L^{2-D}$. In the subcritcal regime exponential decay of the amplitudes sets in earlier and faster when the net underloading dominates, leading to $\tau \sim |\delta\rho|$.

All of these arguments are in good qualitative agreement with observation. To test them more sharply we introduce further characterisation of the queue evolution: the weight average queue size $Q_w(t) = \sum_{ij} q_{ij}^2 / \sum_{ij} q_{ij}$ and the apparent correlation length $\xi(t) = \int r C_{cum}(r) dr / \int C_{cum}(r) dr$.