# 1 Introduction: Radians

Before dealing with the inverse trigonometric functions we will re-visit the definition of an angle in terms of radians. Consider the unit circle as we have done before. The circumference of the unit circle is $2\pi$, while the area is $\pi$. We can identify an angle by the length of the arc is spans with the positive $x$-axis. For instance, an angle of $90°$ will span an arc which is equal to a quarter of the circumference of the circle. We can thus specify the angle either by $90°$ or $2\pi/4 = \pi/2$. The latter is an angle in pseudo-units of radians, rad. It's a pseudo-unit as it is only introduced to make it clear we are talking about an angle, but we do not require it to be there, while, in contrast, for angles in degrees we require the $°$. Angles in radians are unitless and this is why they are so useful. The relationship between angles in degrees and radians is linear and, specifying an angle in degress by a greek letter, $\alpha$, and an angle in radians by a latin letter, $x$, we can write,

$$\alpha = \frac{180°}{\pi}x; \quad x = \frac{\pi}{180°}\alpha.$$

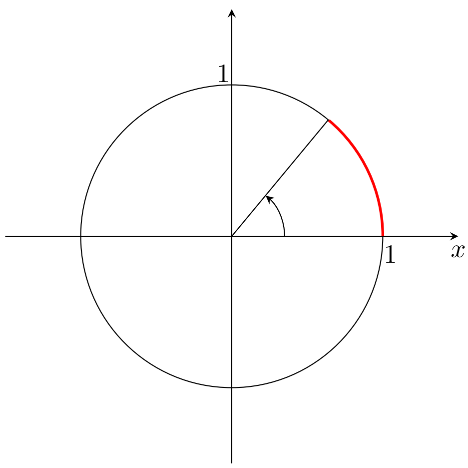


Figure 1: Visualisation of an angle as the length of an arc (red). The angle is $50°$ and the arc has a length of $x = 5\pi/18$.

These relationships simply take a fixed point that is particularly easy to remember ($180° = \pi$). Interestingly, for a unit circle, the area of the sector spanned by the angle has a numerical value equal to half of the arc length. So if the arc has a length of $\pi$ the are spanned by it has a magnitude of $\pi/2$. This way, we can define the so called circular angle either by the length of the arc or by twice the are of the sector. Bear that in mind when we come to talk about hyperbolic functions.

# 2 Inverting Trigonometric Functions

When using trigonometric functions we have seen that we assign the coordinates of some intercept between a unit circle or one of its tangents with a side of the angle under consideration to this angle. The reverse operation will give us the value of an angle by taking these coordinates as argument. For instance, the sine of an angle of $45°$ is $1/\sqrt{2}$. We could equally ask, what is the angle that gives us a sine of $1/\sqrt{2}$? Basically, we want to invert the trigonometric functions. While this can be simple for a multitude of functions, it is not that straight-forward for trigonometric functions. To understand that, we need to remind ourselves what a function actually is. A function is defined as an instruction, or map, that gives for a given value of $x$ (its argument) one and only one value on $y$. Note the specifier 'one and only one'. That is, if we know $x$ we know $y$. If we want to invert a function we require this for the new function as well. However, now our new $x$ is our old $y$ and our new $y$ is our old $x$. So if we want this one-$x$-to-one-$y$ in our new function, we not only require a one-$x$-to-one-$y$ but also a one-$y$-to-one-$x$ mapping in out initial function.

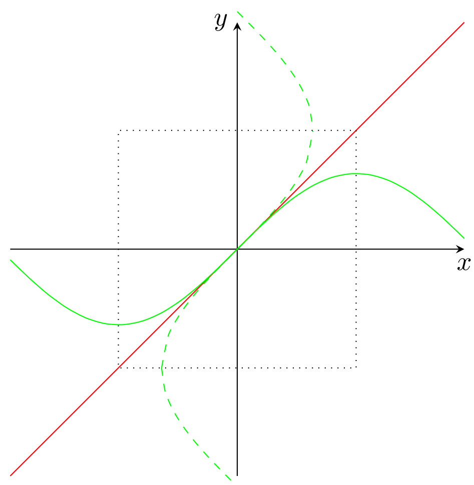


Figure 2: Graphical inversion of the sine function (green) on the $y = x$ line (red) into the arc-sine function (green). The dotted square shows the principal range of the sine function and its inverse.

While for trigonometric functions we see a clear $x \to y$ mapping, the reverse does not hold true. For a given value of $y$ we have infinitely many $x$ values, due to the periodicity of the functions. In order to invert the functions we need to define the so called *principal values*, i.e. the main range of interest for a given function. This does not mean that all other $x$ values that give a certain $y$ value become invalid, it is merely a trick to invert the function. We can make sure to capture all possible solutions to the inverse functions by adding multiples of the period to the answer. An illustration is below.

# 3 Arcus Functions

There are many ways to denote the inverse trigonometric functions, the most common (and the one used in lectures) is by a $^{-1}$ next to the function, e.g. $\sin^{-1}(x)$. There are some issues with this notation and I urge you **strongly** not to use it. This is because it is ambiguous, at best. The convention for powers of trigonometric functions is to write it between function and argument, something that is problematic in itself, e.g. $\cos^2(x)$ instead of $(\cos(x))^2$. In line with this, $\sin^{-1}(x) = 1/\sin(x) = \csc(x)$ and not the inverse. Another way of denoting the inverse which is in no way ambiguous is by using the prefix 'arc-', e.g. $\arcsin(x)$ instead of $\sin^{-1}(x)$. The prefix originates from the fact that the result of the operation of an inverse trigonometric function is an angle, which can be expressed (as shown above) as the length of an *arc*.

The most straight-forward definition of the arcus functions is as the inverse operation of the trigonometric functions.

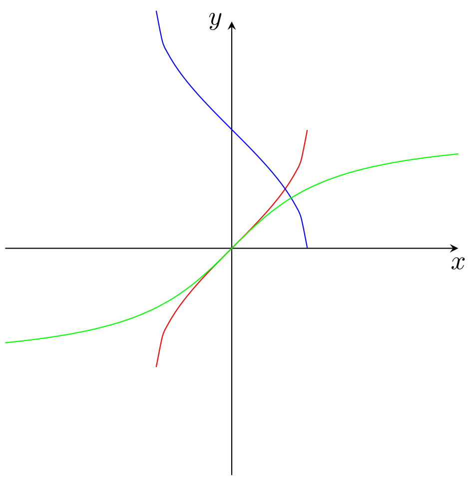


Figure 3: Principal branches of arcsin (red) and arccos (blue) as well as the graph of arctan (green)

# 4 Analytics

The arcus functions are most useful in integration as their derivatives can be expressed analytically by fairly simple functions. Let's find the differential for $\arcsin(x)$. We can say that,

$$y = \arcsin(x),$$

for $-\pi/2 \leq x \leq \pi/2$ (as this is the principal range). Then we can invert as,

$$\sin(y) = x,$$

and we can differentiate both sides wrt. $x$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(y) = \frac{\mathrm{d}}{\mathrm{d}x}x.$$

The RHS evaluates trivally as 1, while for the LHS we need to use the chain rule as $y = y(x)$. Then,

$$\cos(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

We can now write $\cos(y) = \sqrt{1-\sin^2(y)}$ and hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-\sin^2(y)}},$$

but $\sin(y) = x$ and therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Other derivatives are:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}; \quad \frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}.$$

# 5 Integrals with arc-Functions

The following two types of integral will appear in many exercises and in the exam, so it's worthwhile understanding how to solve them:

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\mathrm{d}x.$$

First, we take a factor of $a$ out of the denominator:

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\mathrm{d}x = \frac{1}{a}\int \frac{\mathrm{d}x}{\sqrt{1-\frac{b^2x^2}{a^2}}}.$$

Now substitute $bx/a = \sin(u)$ (or $bx/a = \cos(u)$ - both are possible and correct). Then,

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\mathrm{d}x = \frac{1}{a}\frac{a}{b}\int \frac{\cos(u)\mathrm{d}u}{\sqrt{1-\sin^2(u)}} = \frac{1}{b}\int \mathrm{d}u = \frac{1}{b}u + C = \frac{1}{b}\arcsin\left(\frac{bx}{a}\right) + C.$$

Now consider

$$\int \frac{\mathrm{d}x}{a^2+b^2x^2}.$$

Again, first remove a factor of $a^2$ in the denominator,

$$\int \frac{\mathrm{d}x}{a^2+b^2x^2} = \frac{1}{a^2}\int \frac{\mathrm{d}x}{1+\frac{b^2x^2}{a^2}}.$$

Now substitute $bx/a = \tan(u)$ to get,

$$\int \frac{\mathrm{d}x}{a^2+b^2x^2} = \frac{1}{a^2}\frac{a}{b}\int \frac{1}{\cos^2(u)}\frac{\mathrm{d}u}{1+\tan^2(u)} = \frac{1}{ab}\int \frac{\mathrm{d}u}{\cos^2(u)+\sin^2(u)} = \frac{1}{ab}\int \mathrm{d}u = \frac{u}{ab}+C = \frac{1}{ab}\arctan\left(\frac{bx}{a}\right)+C.$$