Normal random walk example in more detail

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Suppose that $X_1, X_2, \ldots$ are i.i.d. $N(-\mu, 1)$ random variables, where $\mu > 0$. Let $S_0 = 0$ and $S_n = X_1 + X_2 + \ldots + X_n$ and let $T = \inf\{n : S_n \geq \ell\}$ for $\ell > 0$. We want to show that

$$\mathbb{E}\left[\exp(-pT)\right] \sim \exp(-(\mu + \sqrt{\mu^2 + 2p})\ell) \qquad \textbf{for } p \geq 0$$

for large ℓ.

We have that $(M_n)_{n \geq 0}$ is a martingale, where

$$M_n = \exp\left(\alpha(S_n + \mu n) - \frac{\alpha^2}{2} n\right).$$

Note that we do have $T < \infty$ with positive probability, since $\mathbb{P}\left[T < \infty\right] \geq \mathbb{P}\left[X_1 > \ell\right] > 0$. However, T is not a bounded stopping time since it can take arbitrarily large values. So we can't just apply the basic version of the Optional Stopping Theorem. Let's apply it with $T \wedge n = \min\{T, n\}$ instead:

$$1 = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_{T \wedge n}\right] = \mathbb{E}\left[\exp\left(\alpha S_{T \wedge n} + \left(\alpha\mu - \frac{\alpha^2}{2}\right)(T \wedge n)\right)\right].$$

If we set $\alpha = \mu + \sqrt{\mu^2 + 2p}$ then we have $\alpha > 0$ and $\alpha\mu - \alpha^2/2 = -p$.

It's useful to split the right-hand side up according to whether $T = \infty$ or $T < \infty$:

$$\mathbb{E}\left[\exp\left(\alpha S_{T \wedge n} - p(T \wedge n)\right) \mathbb{1}_{[T < \infty]} + \exp\left(\alpha S_{T \wedge n} - p(T \wedge n)\right) \mathbb{1}_{[T = \infty]}\right].$$

The law of large numbers tells us that $S_n / n \to -\mu$ as $n \to \infty$ and so, in particular, $S_n \to -\infty$ almost surely. Hence, on the event $\{T = \infty\}$, we have

$$\exp\left(\alpha S_{T \wedge n} - p(T \wedge n)\right) \to 0$$

almost surely, as $n \to \infty$. Since $S_n < \ell$ for all n on the event $\{T = \infty\}$ and $p > 0$, this whole quantity is also bounded by $e^{\alpha\ell}$ for every n. So, by the bounded convergence theorem,

$$\lim_{n \to \infty} \mathbb{E}\left[\exp\left(\alpha S_{T \wedge n} - p(T \wedge n)\right) \mathbb{1}_{[T = \infty]}\right] = 0.$$

On the other hand, on the event $\{T < \infty\}$, $S_{T \wedge n} \approx \ell$. Moreover, again on the event $\{T < \infty\}$, we have $S_{T \wedge n} \leq S_T$ for all $n \geq 0$. So (as long as you're willing to believe that $\mathbb{E}\left[\exp(\alpha S_T)\, \mathbb{1}_{[T < \infty]}\right] \sim e^{\alpha\ell} < \infty$, for large ℓ!), the dominated convergence theorem tells us that

$$\lim_{n \to \infty} \mathbb{E}\left[\exp\left(\alpha S_{T \wedge n} - p(T \wedge n)\right) \mathbb{1}_{[T < \infty]}\right] \sim e^{\alpha\ell}\, \mathbb{E}\left[\exp(-pT)\, \mathbb{1}_{[T < \infty]}\right] = e^{\alpha\ell}\, \mathbb{E}\left[\exp(-pT)\right].$$

Putting everything together, we get

$$e^{\alpha\ell}\, \mathbb{E}\left[\exp(-pT)\right] \sim 1,$$

which rearranges to the expression we wanted if we substitute in $\alpha = \mu + \sqrt{\mu^2 + 2p}$.