APTS ASP Simple Exercises 1

1. Suppose that $p_{x,y}$ are transition probabilities for a discrete state-space Markov chain satisfying detailed balance. Show that if the system of probabilities given by π_x satisfy the detailed balance equations then they must also satisfy the equilibrium equations.

2. Show that unconstrained simple symmetric random walk has period 2. Show that simple symmetric random walk subject to double reflection "by prohibition" must be aperiodic.

3. Solve the equilibrium equations for simple symmetric random walk on $\{0, 1, \ldots, k\}$ subject to double reflection "by prohibition".

4. Suppose that $X_0, X_1, \ldots,$ is a simple symmetric random walk with double reflection "by prohibition" as above.

   - Use the Markov property to deduce that $X_0, X_1, \ldots, X_{n-1}$ is conditionally independent of $X_{n+1}, X_{n+2}, \ldots$ given X_n. Suppose the reversed chain has kernel $\overline{p}_{y,x}$.
   - Use the definition of conditional probability to compute

   $$\overline{p}_{y,x} = \mathbb{P}\left[X_{n-1} = x\,,\, X_n = y\right] / \, \mathbb{P}\left[X_n = y\right]\,,$$

   - then show that

   $$\mathbb{P}\left[X_{n-1} = x\,,\, X_n = y\right] / \, \mathbb{P}\left[X_n = y\right] = \mathbb{P}\left[X_{n-1} = x\right] p_{x,y} / \, \mathbb{P}\left[X_n = y\right]\,,$$

   - now substitute, using $\mathbb{P}\left[X_n = i\right] = \frac{1}{k+1}$ for all i so $\overline{p}_{y,x} = p_{x,y}$.
   - Use the symmetry of the kernel ($p_{x,y} = p_{y,x}$) to show that the backwards kernel $\overline{p}_{y,x}$ is the same as the forwards kernel $\overline{p}_{y,x} = p_{y,x}$.

5. Show that if $X_0, X_1, \ldots,$ is a simple *asymmetric* random walk with double reflection "by prohibition", running in statistical equilibrium, then it also has the same statistical behaviour as its reversed chain.

6. Show that detailed balance doesn't work for the 3-state chain with transition probabilities $\frac{1}{3}$ for $0 \to 1$, $1 \to 2$, $2 \to 0$ and $\frac{2}{3}$ for $2 \to 1$, $1 \to 0$, $0 \to 2$.

7. Use Burke's theorem for a feed-forward $\cdot/M/1$ queueing network (no loops) to show that in equilibrium each queue viewed in isolation is $M/M/1$. This uses the fact that independent thinnings and superpositions of Poisson processes are still Poisson ....

8. Work through the Random Chess example to compute the mean return time to a corner of the chessboard.

9. Verify for the Ising model that

   $$\mathbb{P}\left[\mathbf{S} = \mathbf{s} \middle| \mathbf{S} \in \{\mathbf{s}, \mathbf{s}^{(i)}\}\right] = \frac{\exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}{\exp\left(J \sum_{j:j\sim i} s_i s_j\right) + \exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}\,.$$

   Determine how this changes in the presence of an external field. Confirm that detailed balance holds for the heat-bath Markov chain.

10. Write down the transition probability kernel for the Metropolis-Hastings sampler. Verify that it has the desired probability distribution as an equilibrium distribution.