APTS ASP Simple Exercises 2

1. Use the martingale property to deduce that

$$\mathbb{E}\left[X_{n+k}|\mathcal{F}_n\right] = X_n\,, \quad k = 0, 1, 2, \ldots\,.$$

2. Recall Thackeray's martingale: let $Y_1, Y_2, \ldots$ be a sequence of independent random variables, with $\mathbb{P}\left[Y_1 = 1\right] = \mathbb{P}\left[Y_1 = -1\right] = 1/2$. Define the Markov chain M by

$$M_0 = 0; \qquad M_n = \begin{cases} 1 - 2^n & \text{if } Y_1 = Y_2 = \cdots = Y_n = -1, \\ 1 & \text{otherwise.} \end{cases}$$

   (a) Compute $\mathbb{E}\left[M_n\right]$ from first principles.

   (b) What should be the value of $\mathbb{E}\left[\widetilde{M}_n\right]$ if $\widetilde{M}$ is computed as for M but stopping play if M hits level $1 - 2^N$?

3. Consider a branching process Y, where $Y_0 = 1$ and Y_{n+1} is the sum $Z_{n+1,1} + \ldots + Z_{n+1,Y_n}$ of Y_n independent copies of a non-negative integer-valued family-size r.v. Z.

   (a) Suppose $\mathbb{E}\left[Z\right] = \mu < \infty$. Show that $X_n = Y_n/\mu^n$ is a martingale.

   (b) Show that Y is itself a supermartingale if $\mu < 1$ and a submartingale if $\mu > 1$.

   (c) Let $H_n = Y_0 + \ldots + Y_n$ be the total of all populations up to time n. Show that $s^{H_n}/(G(s)^{H_{n-1}})$ is a martingale.

   (d) How should these three expressions be altered if $Y_0 = k \geq 1$?

4. Consider asymmetric simple random walk, stopped when it first returns to 0. Show that this is a supermartingale if jumps have non-positive expectation, a submartingale if jumps have non-negative expectation (and therefore a martingale if jumps have zero expectation).

5. Consider Thackeray's martingale based on asymmetric random walk. Show that this is a supermartingale or submartingale depending on whether jumps have negative or positive expectation.

6. Show, using the conditional form of Jensen's inequality, that if X is a martingale then $|X|$ is a submartingale.

7. Suppose that a coin, with probability of heads equal to p, is repeatedly tossed: each Head earns £1 and each Tail loses £1. Let X_n denote your fortune at time n, with $X_0 = 0$. Show that

$$\left(\frac{1-p}{p}\right)^{X_n} \qquad \text{defines a martingale.}$$

8. A shuffled pack of cards contains b black and r red cards. The pack is placed face down, and cards are turned over one at a time. Let B_n denote the number of black cards left *just before* the n^{th} card is turned over. Let

$$Y_n = \frac{B_n}{r + b - (n-1)}\,.$$

   (So Y_n equals the proportion of black cards left just before the n^{th} card is revealed.) Show that Y is a martingale.

9. Suppose N_1, N_2, … are independent identically distributed normal random variables of mean 0 and variance σ^2, and put $S_n = N_1 + \ldots + N_n$.

   (a) Show that S is a martingale.

   (b) Show that $Y_n = \exp\left(S_n - \frac{n}{2}\sigma^2\right)$ is a martingale.

   (c) How should these expressions be altered if $\mathbb{E}\left[N_i\right] = \mu \neq 0$?

10. Let X be a discrete-time Markov chain on a countable state-space S with transition probabilities $p_{x,y}$. Let $f : S \to \mathbb{R}$ be a bounded function. Let $\mathcal{F}_n$ contain all the information about $X_0, X_1, \ldots, X_n$. Show that

$$M_n = f(X_n) - f(X_0) - \sum_{i=0}^{n-1} \sum_{y \in S} (f(y) - f(X_i)) p_{X_i,y}$$

defines a martingale. (Hint: first note that $\mathbb{E}\left[f(X_{i+1}) - f(X_i) | X_i\right] = \sum_{y \in S} (f(y) - f(X_i)) p_{X_i,y}$. Using this and the Markov property of X, check that $\mathbb{E}\left[M_{n+1} - M_n | \mathcal{F}_n\right] = 0$.)

11. Let Y be a discrete-time birth-death process absorbed at zero:

$$p_{k,k+1} = \frac{\lambda}{\lambda + \mu}, \quad p_{k,k-1} = \frac{\mu}{\lambda + \mu}, \qquad \text{for } k > 0, \text{ with } 0 < \lambda < \mu.$$

   (a) Show that Y is a supermartingale.

   (b) Let $T = \inf\{n : Y_n = 0\}$ (so $T < \infty$ a.s.), and define

$$X_n = Y_{n \wedge T} + \left(\frac{\mu - \lambda}{\mu + \lambda}\right)(n \wedge T).$$

   Show that X is a non-negative supermartingale, converging to

$$Z = \left(\frac{\mu - \lambda}{\mu + \lambda}\right) T.$$

   (c) Deduce that

$$\mathbb{E}\left[T | Y_0 = y\right] \leq \left(\frac{\mu + \lambda}{\mu - \lambda}\right) y.$$

12. Let $L(\theta; X_1, X_2, \ldots, X_n)$ be the likelihood of parameter θ given a sample of independent and identically distributed random variables, $X_1, X_2, \ldots, X_n$.

   (a) Check that if the "true" value of θ is θ_0 then the likelihood ratio

$$M_n = \frac{L(\theta_1; X_1, X_2, \ldots, X_n)}{L(\theta_0; X_1, X_2, \ldots, X_n)}$$

   defines a martingale with $\mathbb{E}\left[M_n\right] = 1$ for all $n \geq 1$.

   (b) Using the strong law of large numbers and Jensen's inequality, show that

$$\frac{1}{n} \log M_n \to -c \text{ as } n \to \infty.$$

   (c) Show that the result is still true even if the random variables are neither independent nor identically distributed.