APTS ASP Simple Exercises 3

1. Suppose that N is a Poisson process of rate λ. Working with the result
$$\mathbb{P}\left[N_t = k\right] = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad \text{for } k = 0, 1, 2, \ldots,$$
show that
   (i) $\frac{\mathrm{d}}{\mathrm{d}\, t} \mathbb{P}\left[N_t > 0\right] \to \lambda$ as $t \to 0$,
   (ii) $\frac{\mathrm{d}}{\mathrm{d}\, t} \mathbb{P}\left[N_t > 1\right] \to 0$ as $t \to 0$,
   (iii) $\frac{\mathrm{d}}{\mathrm{d}\, t} \mathbb{P}\left[N_t = 0\right] \to -\lambda$ as $t \to 0$.

2. In the context of question 1, show that
$$\frac{\mathrm{d}}{\mathrm{d}\, t} \mathbb{P}\left[N_t > k\right] = \lambda \times \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad \text{for } k = 0, 1, 2, \ldots.$$
Hence, deduce that the time to the k^{th} incident has a Gamma distribution, and write down the Gamma distribution parameters.

3. Suppose that X is a continuous-time Markov chain on a countable state-space S. Suppose that S is irreducible and recurrent. Let $i \in S$ be fixed and set $X_0 = i$. Show carefully that the number N_t of returns to state i by time t forms a renewal process.

4. Suppose that N is a Poisson process of rate λ and let
$$X_t = N_t - \lambda t$$
and
$$Y_t = X_t^2 - \lambda t.$$
Show that $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are both martingales.

5. Show that the hazard rate associated with the Exponential(λ) density is λ and deduce that a Poisson process of rate λ has constant hazard rate λ.

6. Suppose now that X is merely a nonnegative random variable, and h is an integrable function on $[0, t]$. Show that
$$\mathbb{E}\left[\int_0^{\min\{t,X\}} h(u)\,\mathrm{d}\, u\right] = \int_0^t \mathbb{P}\left[X > u\right] h(u)\,\mathrm{d}\, u\,.$$

7. With X as in question 5, suppose that
$$\mathbb{P}\left[X > t\right] = \exp\left(-\int_0^t h(s)\,\mathrm{d}\, s\right).$$
Show that
$$\mathbb{I}_{[X \leq t]} - \int_0^{\min\{t,X\}} h(u)\,\mathrm{d}\, u$$
determines a martingale.

8. Develop the result of question 6 to show that if $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with common density f and associated hazard rate h and if $N_t = \#\{i : X_i \leq t\}$ then

$$N_t - \int_0^t h(s)(n - N_s)\,\mathrm{d}\,s$$

determines a martingale.

9. Suppose that X_1, X_2, ... are independent mean-zero unit-variance random variables, such that for some constant C we have $\mathbb{E}\left[|X_i|^3\right] < C$ for all i. Show that the sequence X_1, X_2, ... satisfies the Lindeberg condition (so that $(X_1 + \ldots + X_n)/s_n \xrightarrow{d} N(0,1)$).
(Hint: $\mathbb{E}\left[X_i^2; X_i^2 > \varepsilon^2 n\right] \leq \frac{1}{\varepsilon\sqrt{n}}\,\mathbb{E}\left[|X_i|^3\right]$.)

10. Show that in general the condition

$$r_n^3 = \sum_{i=1}^{n} \mathbb{E}\left[|X_i - \mu_i|^3\right] < \infty \text{ and } r_n/s_n \to 0$$

(a special case of the so-called Lyapunov condition) implies the Lindeberg condition.

11. Using the fact that a Cauchy distributed random variable X has characteristic function $\mathbb{E}\left[e^{itX}\right] = e^{-|t|}$, show that if $X_1, X_2, \ldots, X_n$ are i.i.d. Cauchy then $\frac{1}{n}\sum_{i=1}^{n} X_i$ is also Cauchy.