APTS ASP Simple Exercises 4

1. Recall that the total variation distance between two probability distributions μ and ν on $\mathcal{X}$ is given by
$$\operatorname{dist}_{\mathrm{TV}}(\mu,\nu) \;=\; \sup_{A\subseteq\mathcal{X}} \{\mu(A)-\nu(A)\}\,.$$
Show that this is equivalent to the distance
$$\sup_{A\subseteq\mathcal{X}} |\mu(A)-\nu(A)|\,.$$

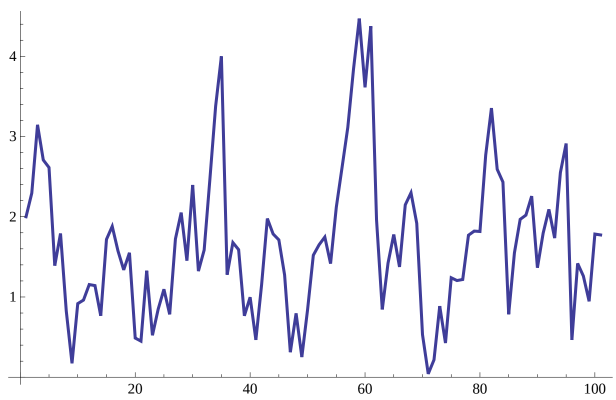
2. Show that if $\mathcal{X}$ is discrete, then
$$\operatorname{dist}_{\mathrm{TV}}(\mu,\nu) = \tfrac{1}{2}\sum_{y\in\mathrm{X}} |\mu(y)-\nu(y)|\,.$$
(Here we *do* need to use the absolute value on the RHS!)
Hint: consider $A=\{y:\mu(y)>\nu(y)\}$.

3. Suppose now that μ and ν are density functions on $\mathbb{R}$. Show that
$$\operatorname{dist}_{\mathrm{TV}}(\mu,\nu) = 1-\int_{-\infty}^{\infty}\min\{\mu(y),\nu(y)\}dy\,.$$
Hint: remember that $|\mu-\nu| = \mu+\nu-2\min\{\mu,\nu\}$.

4. Let X be a random walk on $\mathbb{R}$, with increments given by the standard normal distribution. Recall that any bounded set is small of lag 1. Does there exist $k\geq 1$ such that *the whole state space* is small of lag k?

5. Consider a Markov chain X with continuous transition density kernel. Show that it possesses *many* small sets of lag 1.

6. Consider a Vervaat perpetuity X, where
$$X_0 = 0; \qquad X_{n+1} = U_{n+1}(X_n+1)\,,$$
and where $U_1, U_2, \ldots$ are independent Uniform$(0,1)$ (simulated below).



Find a small set for this chain.

7. Recall the regeneration idea from lecture 7: suppose that C is a small set (with lag 1) for a ϕ-recurrent chain X, i.e. for $x \in C$,

$$\mathbb{P}\left[X_1 \in A | X_0 = x\right] \geq \alpha\nu(A).$$

Suppose that $X_n \in C$. Then with probability α let $X_{n+1} \sim \nu$, and otherwise let it have transition distribution $\frac{p(x,\cdot)-\alpha\nu(\cdot)}{1-\alpha}$.

   (a) Check that the latter expression really gives a probability distribution.

   (b) Check that X_{n+1} constructed in this manner obeys the correct transition distribution from X_n.

8. Define a reflected random walk as follows: $X_{n+1} = \max\{X_n + Z_{n+1}, 0\}$, for $Z_1, Z_2, \ldots$ i.i.d. with continuous density $f(z)$,

$$\mathbb{E}\left[Z_1\right] < 0 \quad \text{and} \quad \mathbb{P}\left[Z_1 > 0\right] > 0\,.$$

Show that the Foster-Lyapunov criterion for positive recurrence holds, using $\Lambda(x) = x$.

9. Define a reflected simple asymmetric random walk as follows: $X_{n+1} = \max\{X_n + Z_{n+1}, 0\}$, for $Z_1, Z_2, \ldots$ i.i.d. with

$$\mathbb{P}\left[Z_1 = -1\right] = 1 - \mathbb{P}\left[Z_1 = +1\right] > \frac{1}{2}\,.$$

Show that X is counting-measure-irreducible on the non-negative integers. (Check that this is the same as ordinary irreducibility for discrete state-space Markov chains.)