

Computer Practical 3

Markov Chains and Monte Carlo

1. A warm-up. In a simplified model of the game of Monopoly, we consider the motion of the piece around a loop of 40 spaces. We can model this as a Markov chain with a state space consisting of the integers $0, \dots, 39$ in which the transition kernel adds the result of two six-sided dice to the current state modulo 40 to obtain the new state.
 - (a) Implement a piece of R code which simulates this Markov chain.
 - (b) Run the code for a large number of iterations, say 100,000, and plot a histogram of the states visited.
 - (c) Based on the output of the chain, would you conjecture that there is an invariant distribution for this Markov chain? If so, what?
 - (d) Write the transition kernel down mathematically.
 - (e) Check whether the Markov kernel you have written down is invariant with respect to any distribution conjectured in part (c).
2. Gibbs Sampling: recall the Poisson changepoint model discussed in lectures, and on p28 of the supporting notes, and think about the following closely related model.

Observations y_1, \dots, y_n comprise a sequence of M iid $\mathbf{N}(\mu_1, 1)$ random variables followed by a second sequence of $n - M$ iid $\mathbf{N}(\mu_2, 1)$ random variables. M , μ_1 and μ_2 are unknown.

The prior distribution over M is a discrete uniform distribution on $\{1, \dots, n - 1\}$ (there is at least one observation of each component). The prior distribution over μ_i ($i = 1, 2$) is $\mathbf{N}(0, 10^2)$. The three parameters are treated as being a priori independent.

- (a) Write down the joint density of $y_1, \dots, y_n, \mu_1, \mu_2$ and M and obtain the posterior distribution of μ_1, μ_2 and M , up to proportionality, in as simple a form as you can.
- (b) Find the “full conditional” distributions of μ_1, μ_2 and M . (i.e. the conditional distributions of each of these variables given all other variables).
- (c) Implement a Gibbs sampler which makes use of these full conditional distributions in order to target the posterior distribution identified in part (b).
- (d) Simulate some data from the model for various parameter values and test your Gibbs sampler.
- (e) How might you extend this algorithm if instead of a changepoint model you had a mixture model in which every observation is drawn from a mixture, i.e.:

$$Y_1, \dots, Y_n \stackrel{\text{iid}}{\sim} p\mathbf{N}(\cdot; \mu_1, 1) + (1 - p)\mathbf{N}(\cdot; \mu_2, 1)$$

(so the likelihood is $\prod_{i=1}^n [p\mathbf{N}(y_i; \mu_1, 1) + (1 - p)\mathbf{N}(y_i; \mu_2, 1)]$ in which p, μ_1 and μ_2 are unknown (and M is no longer a parameter of the model)?)

Consider the following things:

- i. The prior distribution over p .
- ii. Any other variables you may need to introduce.
- iii. The resulting algorithm.

If you have time, implement the resulting algorithm and apply it to some simulated data.