APTS Lab 1: solutions *

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1. See `Lab1.R`.

2. An asymptotic expansion of the expression above gives

$$\begin{aligned}
2\pi^{1/2}MISE(\hat{f}_h) &= \frac{1}{nh} + \Big(1 - \frac{1}{n}\Big)\frac{1}{(h^2+\sigma^2)^{1/2}} - \frac{2^{3/2}}{(h^2+2\sigma^2)^{1/2}} + \frac{1}{\sigma} \\
&= \frac{1}{nh} + \frac{1}{\sigma}\Big(1 + \frac{h^2}{\sigma^2}\Big)^{-1/2} - \frac{2}{\sigma}\Big(1 + \frac{h^2}{2\sigma^2}\Big)^{-1/2} + \frac{1}{\sigma} + o\Big(\frac{1}{nh}\Big) \\
&= \frac{1}{nh} + \frac{1}{\sigma}\Big(1 - \frac{h^2}{2\sigma^2} + \frac{3h^4}{8\sigma^4}\Big) - \frac{2}{\sigma}\Big(1 - \frac{h^2}{4\sigma^2} + \frac{3h^4}{32\sigma^4}\Big) + \frac{1}{\sigma} + o\Big(\frac{1}{nh} + h^4\Big) \\
&= \frac{1}{nh} + \frac{3h^4}{16\sigma^5} + o\Big(\frac{1}{nh} + h^4\Big).
\end{aligned}$$

Differentiating the leading terms of this expansion, we see that the asymptotically optimal bandwidth h_{AMISE} with respect to the MISE criterion is given by

$$h_{AMISE} = \left(\frac{4}{3n}\right)^{1/5}\sigma.$$

The general formula for the asymptotically optimal bandwidth for a second-order kernel is

$$h_{AMISE} = \left(\frac{R(K)}{R(f'')\mu_2(K)^2 n}\right)^{1/5}.$$

In this case, we have $\mu_2(K) = \mu_2(\phi) = 1$, and

$$R(K) = R(\phi) = \int_{-\infty}^{\infty} \phi(x)^2\,dx = \frac{1}{2\pi^{1/2}}.$$

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Finally, integrating by parts, we have

$$\begin{aligned}
R(f'') = R(\phi_\sigma'') = R\big(\sigma^{-3}\phi''(x/\sigma)\big) &= \frac{1}{\sigma^6}\int_{-\infty}^{\infty} \phi''\Big(\frac{x}{\sigma}\Big)^2\,dx \\
&= \frac{1}{\sigma^5}\int_{-\infty}^{\infty}(x^2-1)^2\phi(x)^2\,dx \\
&= \frac{1}{\sigma^5}\Big\{\int_{-\infty}^{\infty}\Big(\frac{3}{2}x^2-1\Big)\phi(x)^2\,dx + \phi_{2^{1/2}}(0)\Big\} \\
&= \frac{1}{\sigma^5}\int_{-\infty}^{\infty}\frac{3}{4}\phi(x)^2\,dx \\
&= \frac{3}{8\pi^{1/2}\sigma^5},
\end{aligned}$$

so $R(K)/R(f'') = 4\sigma^5/3$, which shows the desired result.

To verify the formula for $\text{MISE}(\hat{f}_h)$, first observe that, by using Fubini's theorem,

$$\begin{aligned}
\int_{-\infty}^{\infty}(K_h^2 * f)(x)\,dx &= \frac{1}{h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K\Big(\frac{x-y}{h}\Big)^2 f(y)\,dy\,dx \\
&= \frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(z)^2 f(x-hz)\,dz\,dx \\
&= \frac{1}{h}\int_{-\infty}^{\infty} K(z)^2\int_{-\infty}^{\infty} f(x-hz)\,dx\,dz \\
&= \frac{1}{h}\int_{-\infty}^{\infty} K(z)^2\,dz.
\end{aligned}$$

Thus, with $K_h(x) = \phi_h(x)$ and $f(x) = \phi_\sigma(x)$ and using the formula from class, the mean integrated squared error is given by

$$\begin{aligned}
\text{MISE}(\hat{f}_h) &= \frac{1}{n}\int_{-\infty}^{\infty}\{(K_h^2 * f)(x) - (K_h * f)^2(x)\}\,dx + \int_{-\infty}^{\infty}\{(K_h * f)(x) - f(x)\}^2\,dx \\
&= \frac{1}{nh}\int_{-\infty}^{\infty}\phi(x)^2\,dx + \Big(1-\frac{1}{n}\Big)\int_{-\infty}^{\infty}(\phi_h * \phi_\sigma)^2(x)\,dx \\
&\qquad - 2\int_{-\infty}^{\infty}(\phi_h * \phi_\sigma)(x)\phi_\sigma(x)\,dx + \int_{-\infty}^{\infty}\phi_\sigma(x)^2\,dx \\
&= \frac{1}{nh}\phi_{2^{1/2}}(0) + \Big(1-\frac{1}{n}\Big)\int_{-\infty}^{\infty}\phi_{(h^2+\sigma^2)^{1/2}}(x)^2\,dx - 2\int_{-\infty}^{\infty}\phi_{(h^2+\sigma^2)^{1/2}}(x)\phi_\sigma(x)\,dx \\
&\qquad + \phi_{2^{1/2}\sigma}(0) \\
&= \frac{1}{2\pi^{1/2}}\Big\{\frac{1}{nh} + \Big(1-\frac{1}{n}\Big)\frac{1}{(h^2+\sigma^2)^{1/2}} - \frac{2^{3/2}}{(h^2+2\sigma^2)^{1/2}} + \frac{1}{\sigma}\Big\}.
\end{aligned}$$