CONDITIONING AN ADDITIVE FUNCTIONAL OF A MARKOV CHAIN TO STAY NON-NEGATIVE II: HITTING A HIGH LEVEL [1]

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Abstract

Let $(X_t)_{t\geq 0}$ be a continuous-time irreducible Markov chain on a finite statespace E, let $v : E \to \mathbb{R}\backslash\{0\}$ and let $(\varphi_t)_{t\geq 0}$ be defined by $\varphi_t = \int_0^t v(X_s)ds$. We consider the cases where the process $(\varphi_t)_{t\geq 0}$ is oscillating and where $(\varphi_t)_{t\geq 0}$ has a negative drift. In each of the cases we condition the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that $(\varphi_t)_{t\geq 0}$ hits level y before hitting zero and prove weak convergence of the conditioned process as $y \to \infty$. In addition, we show the relation between conditioning the process $(\varphi_t)_{t\geq 0}$ with a negative drift to oscillate and conditioning it to stay non-negative until large time, and the relation between conditioning $(\varphi_t)_{t\geq 0}$ with a negative drift to drift to drift to $+\infty$ and conditioning it to hit large levels before hitting zero.

1 Introduction

Let $(X_t)_{t\geq 0}$ be a continuous-time irreducible Markov chain on a finite statespace E, let v be a map $v : E \to \mathbb{R}\backslash\{0\}$, let $(\varphi_t)_{t\geq 0}$ be an additive functional defined by $\varphi_t = \varphi + \int_0^t v(X_s)ds$ and let H_y, $y \in \mathbb{R}$, be the first hitting time of level y by the process $(\varphi_t)_{t\geq 0}$. In the previous paper Jacka, Najdanovic, Warren (2005) we discussed the problem of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative, that is the event $\{H_0 = +\infty\}$. In the oscillating case and in the case of the negative drift of the process $(\varphi_t)_{t\geq 0}$, when the event $\{H_0 = +\infty\}$ is of zero probability, the process $(X_t, \varphi_t)_{t\geq 0}$ can instead be conditioned on some approximation of the event $\{H_0 = +\infty\}$. In Jacka et al. (2005) we considered the approximation by the events $\{H_0 > T\}$, $T > 0$, and proved weak convergence as $T \to \infty$ of the process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on this approximation.

In this paper we look at another approximation of the event $\{H_0 = +\infty\}$ which is the approximation by the events $\{H_0 > H_y\}$, $y \in \mathbb{R}$. Again, we are interested in weak convergence as $y \to \infty$ of the process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on this approximation.

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Our motivation comes from a work by Bertoin and Doney. In Bertoin, Doney (1994) the authors considered a real-valued random walk $\{S_n, n \geq 0\}$ that does not drift to $+\infty$ and conditioned it to stay non-negative. They discussed two interpretations of this conditioning, one was conditioning S to exceed level n before hitting zero, and another was conditioning S to stay non-negative up to time n. As it will be seen, results for our process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on the event $\{H_0 = +\infty\}$ appear to be analogues of the results for a random walk.

Furthermore, similarly to the results obtained in Bertoin, Doney (1994) for a real-valued random walk $\{S_n, n \geq 0\}$ that does not drift to $+\infty$, we show that in the negative drift case

(i) taking the limit as $y \to \infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < +\infty\}$ and then further conditioning on the event $\{H_0 = +\infty\}$ yields the same result as the limit as $y \to \infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_0 > H_y\}$;

(ii) conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ oscillates and then further conditioning on $\{H_0 = +\infty\}$ yields the same result as the limit as $T \to \infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_0 > T\}$.

The organisation of the paper is as follows: in Section 2 we state the main theorems in the oscillating and in the negative drift case; in Section 3 we calculate the Green's function and the two-sided exit probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$ that are needed for the proofs in subsequent sections; in Section 4 we prove the main theorem in the oscillating case; in Section 5 we prove the main theorem in the negative drift case. Finally, Sections 6 and 7 deal with the negative drift case of the process $(\varphi_t)_{t\geq 0}$ and commuting diagrams in conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$ and $\{H_0 > T\}$, respectively, listed in (i) and (ii) above.

All the notation in the present paper is taken from Jacka et al. (2005).

2 Main theorems

First we recall some notation from Jacka et al. (2005).

Let the process (X_t, φ_t) be as defined in Introduction. Suppose that both $E^+ = v^{-1}(0, \infty)$ and $E^- = v^{-1}(-\infty, 0)$ are non-empty. Let, for any $y \in \mathbb{R}$, E_y^+ and E_y^- be the halfspaces defined by $E_y^+ = (E \times (y, +\infty)) \bigcup (E^+ \times \{y\})$ and $E_y^- = (E \times (-\infty, y)) \bigcup (E^- \times \{y\})$. Let H_y, $y \in \mathbb{R}$, be the first crossing time of the level y by the process $(\varphi_t)_{t\geq 0}$ defined by

$$H_y = \begin{cases} \inf\{t > 0 : \varphi_t < y\} & \text{if } (X_t, \varphi_t)_{t\geq 0} \text{ starts in } E_y^+ \\ \inf\{t > 0 : \varphi_t > y\} & \text{if } (X_t, \varphi_t)_{t\geq 0} \text{ starts in } E_y^-. \end{cases}$$

Let $P_{(e,\varphi)}$ denote the law of the process $(X_t, \varphi_t)_{t\geq 0}$ starting at (e, φ) and let $E_{(e,\varphi)}$ denote the expectation operator associated with $P_{(e,\varphi)}$. Let Q denote the conservative irreducible Q-matrix of the process $(X_t)_{t\geq 0}$ and let V be the diagonal matrix $diag(v(e))$.

Let $V^{-1}Q\Gamma = \Gamma G$ be the unique Wiener-Hopf factorisation of the matrix $V^{-1}Q$ (see Lemma 3.4 in Jacka et al. (2005)). Let J, J_1 and J_2 be the matrices

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad J_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and let a matrix Γ_2 be given by $\Gamma_2 = J\Gamma J$. For fixed $y > 0$, let $P^{[y]}_{(e,\varphi)}$ denote the law of the process $(X_t, \varphi_t)_{t\geq 0}$, starting at $(e,\varphi) \in E_0^+$, conditioned on the event $\{H_y < H_0\}$, and let $P^{[y]}_{(e,\varphi)}|_{\mathcal{F}_t}$, $t \geq 0$, be the restriction of $P^{[y]}_{(e,\varphi)}$ to $\mathcal{F}_t$. We are interested in weak convergence of $(P^{[y]}_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ as $y \to +\infty$.

Theorem 2.1 *Suppose that the process $(\varphi_t)_{t\geq 0}$ oscillate. Then, for fixed $(e,\varphi) \in E_0^+$ and $t \geq 0$, the measures $(P^{[y]}_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the probability measure $P^{h_r}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$ which is defined by*

$$P^{h_r}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_r(X_t,\varphi_t) I\{t < H_0\}\Big)}{h_r(e,\varphi)}, \qquad t \geq 0,\ A \in \mathcal{F}_t,$$

where h_r is a positive harmonic function for the process $(X_t, \varphi_t)_{t\geq 0}$ given by

$$h_r(e,y) = e^{-yV^{-1}Q} J_1 \Gamma_2 r(e), \qquad (e,y) \in E \times \mathbb{R},$$

and $V^{-1}Qr = 1$.

By comparing Theorem 2.1 and Theorem 2.1 in Jacka et al. (2005) we see that the measures $(P^{[y]}_{(e,\varphi)})_{y\geq 0}$ and $(P^{T}_{(e,\varphi)})_{T\geq 0}$ converge weakly to the same limit. Therefore, in the oscillating case conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$, $y > 0$, and conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_0 > T\}$, $T > 0$, yield the same result.

Let f_{max} be the eigenvector of the matrix $V^{-1}Q$ associated with its eigenvalue with the maximal non-positive real part. The weak limit as $y \to +\infty$ of the sequence $(P^{[y]}_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ in the negative drift case is given in the following theorem:

Theorem 2.2 *Suppose that the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$. Then, for fixed $(e,\varphi) \in E_0^+$ and $t \geq 0$, the measures $(P^{[y]}_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the probability measure $P^{h_{f_{max}}}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$ which is given by*

$$P^{h_{f_{max}}}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_{f_{max}}(X_t,\varphi_t) I\{t < H_0\}\Big)}{h_{f_{max}}(e,\varphi)}, \qquad t \geq 0,\ A \in \mathcal{F}_t,$$

where the function $h_{f_{max}}$ is positive and harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and is given by

$$h_{f_{max}}(e,y) = e^{-yV^{-1}Q} J_1 \Gamma_2 f_{max}(e), \qquad (e,y) \in E \times \mathbb{R}.$$

Before we prove Theorems 2.1 and 2.2, we recall some more notation from Jacka et al. (2005) that will be in use in the sequel.

The matrices G^+ and G^- are the components of the matrix G and the matrices Π^+ and Π^- are the components of the matrix Γ determined by the Wiener-Hopf factorisation of the matrix $V^{-1}Q$, that is

$$G = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix}.$$

In other words, the matrix G^+ is the Q-matrix of the process $(X_{H_y})_{y\geq 0}$, $(X_0, \varphi_0) \in E^+ \times \{0\}$, the matrix G^- is the Q-matrix of the process $(X_{H_{-y}})_{y\geq 0}$, $(X_0, \varphi_0) \in E^- \times \{0\}$, and the matrices Π^- and Π^+ determine the probability distribution of the process $(X_t)_{t\geq 0}$ at the time when $(\varphi_t)_{t\geq 0}$ hits zero, that is the probability distribution of X_{H_0} (see Lemma 3.4 in Jacka et al. (2005)).

A matrix $F(y)$, $y \in \mathbb{R}$, is defined by

$$F(y) \quad = \quad \begin{cases} J_1 \, e^{yG} = e^{yG} \, J_1, & y > 0 \\ J_2 \, e^{yG} = e^{yG} \, J_2, & y < 0. \end{cases}$$

For any vector g on E, let g^+ and g^- denote its restrictions to E^+ and E^- respectively. We write the column vector g as $g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix}$ and the row vector μ as $\mu = \begin{pmatrix} \mu^+ & \mu^- \end{pmatrix}$.

A vector g is associated with an eigenvalue λ of the matrix $V^{-1}Q$ if there exists $k \in \mathbb{N}$ such that $(V^{-1}Q - \lambda I)^k g = 0$.

$\mathcal{B}$ is a basis in the space of all vectors on E such that there are exactly $n = |E^+|$ vectors $\{f_1, f_2, \ldots, f_n\}$ in $\mathcal{B}$ such that each vector f_j, $j = 1, \ldots, n$ is associated with an eigenvalue α_j of $V^{-1}Q$ for which $Re(\alpha_j) \leq 0$, and that there are exactly $m = |E^-|$ vectors $\{g_1, g_2, \ldots, g_m\}$ in $\mathcal{B}$ such that each vector g_k, $k = 1, \ldots, m$, is associated with an eigenvalue β_k of $V^{-1}Q$ with $Re(\beta_k) \geq 0$. The vectors $\{f_1^+, f_2^+, \ldots, f_n^+\}$ form a basis $\mathcal{N}^+$ in the space of all vectors on E^+. and the vectors $\{g_1^-, g_2^-, \ldots, g_m^-\}$ form a basis $\mathcal{P}^-$ in the space of all vectors on E^-.

The matrix $V^{-1}Q$ cannot have strictly imaginary eigenvalues. All eigenvalues of $V^{-1}Q$ with negative (respectively positive) real part coincide with the eigenvalues of G^+ (respectively $-G^-$). G^+ and G^- are irreducible Q-matrices and

$$\alpha_{max} \equiv \max_{1\leq j\leq n} Re(\alpha_j) \leq 0 \quad \text{and} \quad -\beta_{min} \equiv \max_{1\leq k\leq m} Re(-\beta_k) = -\min_{1\leq k\leq m} Re(\beta_k) \leq 0$$

are simple eigenvalues of G^+ and G^-, respectively. f_{max} and g_{min} are the eigenvectors of the matrix $V^{-1}Q$ associated with its eigenvalues α_{max} and β_{min}, respectively, and therefore f_{max}^+ and g_{min}^- are the Perron-Frobenius eigenvectors of the matrices G^+ and G^-, respectively.

If the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$, then $\alpha_{max} < 0$ and $\beta_{min} = 0$. If the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$, then $\alpha_{max} = 0$ and $\beta_{min} > 0$. If the process $(\varphi_t)_{t\geq 0}$ oscillates then $\alpha_{max} = \beta_{min} = 0$ and there exists a vector r such that $V^{-1}Qr = 1$.

3 The Green's function and the hitting probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$

The Green's function of the process $(X_t, \varphi_t)_{t\geq 0}$, denoted by $G((e,\varphi),(f,y))$, for any $(e,\varphi),(f,y) \in E\times\mathbb{R}$, is defined as

$$G((e,\varphi),(f,y)) = E_{(e,\varphi)}\Big(\sum_{0\leq s<\infty} I(X_s = f, \varphi_s = y)\Big),$$

noting that the process $(X_t, \varphi_t)_{t\geq 0}$ hits any fixed state at discrete times. For simplicity of notation, let $G(\varphi, y)$ denote the matrix $(G((\cdot,\varphi),(\cdot,y)))_{E\times E}$.

Theorem 3.1 *In the drift cases,*

$$G(0,0) = \Gamma_2^{-1} = \begin{pmatrix} (I-\Pi^-\Pi^+)^{-1} & \Pi^-(I-\Pi^+\Pi^-)^{-1} \\ \Pi^+(I-\Pi^-\Pi^+)^{-1} & (I-\Pi^+\Pi^-)^{-1} \end{pmatrix}.$$

In the oscillating case, $G(0,0) = +\infty$.

Proof: By the definition of $G(0,0)$ and the matrices Π^+, Π^- and Γ_2,

$$G(0,0) = \sum_{n=1}^{\infty} \begin{pmatrix} 0 & \Pi^- \\ \Pi^+ & 0 \end{pmatrix}^n = \sum_{n=1}^{\infty} (I-\Gamma_2)^n.$$

Suppose that the process $(\varphi_t)_{t\geq 0}$ drifts either to $+\infty$ of $-\infty$. Then by (3.8) and Lemma 3.5 (iv) in Jacka et al. (2005) exactly one of the matrices Π^+ and Π^- is strictly substochastic. In addition, the matrix $\Pi^-\Pi^+$ is positive and thus primitive. Therefore, the Perron-Frobenius eigenvalue λ of $\Pi^-\Pi^+$ satisfies $0<\lambda<1$ which, by the Perron-Frobenius theorem for primitive matrices (see Seneta (1981)), implies that

$$\lim_{n\to\infty} \frac{(\Pi^-\Pi^+)^n}{(1+\lambda)^n} = const. \neq 0.$$

Therefore, $(\Pi^-\Pi^+)^n \to 0$ elementwise as $n\to+\infty$, and similarly $(\Pi^+\Pi^-)^n \to 0$ elementwise as $n\to+\infty$. Hence, $(I-\Gamma_2)^n \to 0$, $n\to+\infty$. Since

$$I - (I-\Gamma_2)^{n+1} = \Gamma_2 \sum_{k=0}^{n} (I-\Gamma_2)^k,$$

and, by Lemma 3.5 (ii) in Jacka et al. (2005), Γ_2^{-1} exists, by letting $n\to+\infty$ we obtain

$$G(0,0) = \sum_{n=0}^{\infty} (I-\Gamma_2)^n = \Gamma_2^{-1}. \tag{3.1}$$

Suppose now that the process $(\varphi_t)_{t\geq 0}$ oscillates. Then again by (3.8) and Lemma 3.5 (iv) in Jacka et al. (2005), the matries Π^+ and Π^- are stochastic. Thus, $(I - \Gamma_2)1 = 1$ and

$$G(0,0)1 = \sum_{n=0}^{\infty} (I - \Gamma_2)^n 1 = \sum_{n=0}^{\infty} 1 = +\infty. \tag{3.2}$$

Since the matrix Q is irreducible, it follows that $G(0,0) = +\infty$. □

Theorem 3.2 *In the drift cases, the Green's function $G((e,\varphi),(f,y))$ of the process $(X_t, \varphi_t)_{t\geq 0}$ is given by the $E \times E$ matrix $G(\varphi, y)$, where*

$$G(\varphi, y) = \begin{cases} \Gamma\ F(y-\varphi)\ \Gamma_2^{-1}, & \varphi \neq y \\ \Gamma_2^{-1}, & \varphi = y. \end{cases}$$

Proof: By Theorem 3.1, $G(y,y) = G(0,0) = \Gamma_2^{-1}$, and by Lemma 3.5 (vii) in Jacka et al. (2005),

$$P_{(e,\varphi-y)}(X_{H_0} = e', H_0 < +\infty) = \Gamma\ F(y-\varphi)(e,e'), \qquad \varphi \neq y.$$

The theorem now follows from

$$G((e,\varphi),(f,y)) = \sum_{e' \in E} P_{(e,\varphi-y)}(X_{H_0} = e', H_0 < +\infty)\ G((e',0),(f,0)).$$

□

The Green's function $G_0((e,\varphi),(f,y))$, $(e,\varphi),(f,y) \in E \times \mathbb{R}$, of the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero (in matrix notation $G_0(\varphi,y)$) is defined by

$$G_0((e,\varphi),(f,y)) = E_{(e,\varphi)}\Big(\sum_{0\leq s<H_0} I(X_s = f, \varphi_s = y) \Big).$$

It follows that $G_0(\varphi, y) = 0$ if $\varphi y < 0$, that $G_0(\varphi, 0) = 0$ if $\varphi \neq 0$, and that $G_0(0,0) = I$. To calculate $G_0(\varphi, y)$ for $|\varphi| \leq |y|$, $\varphi y \geq 0$, $y \neq 0$, we use the following lemma:

Lemma 3.1 *Let $(f,y) \in E^+ \times (0,+\infty)$ be fixed and let the process $(X_t,\varphi_t)_{t\geq 0}$ start at $(e,\varphi) \in E \times (0,y)$. Let $(e,\varphi) \mapsto h((e,\varphi),(f,y))$ be a bounded function on $E \times (0,y)$ such that the process $(h((X_{t\wedge H_0 \wedge H_y}, \varphi_{t\wedge H_0\wedge H_y}),(f,y)))_{t\geq 0}$ is a uniformly integrable martingale and that*

$$h((e,0),(f,y)) = 0, \qquad e \in E^- \tag{3.3}$$

$$h((e,y),(f,y)) = G_0((e,y),(f,y)). \tag{3.4}$$

Then

$$h((e,\varphi),(f,y)) = G_0((e,\varphi),(f,y)), \quad (e,\varphi) \in E \times (0,y).$$

Proof: The proof of the lemma is based on the fact that a uniformly integrable martingale in a region which is zero on the boundary of that region is zero everywhere. Therefore we omit the proof the lemma. $\square$

Let A_y, B_y, C_y and D_y be components of the matrix $e^{-yV^{-1}Q}$ such that, for any $y \in \mathbb{R}$,

$$e^{-yV^{-1}Q} = \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix}. \tag{3.5}$$

Theorem 3.3 *The Green's function $G_0((e,\varphi),(f,y))$, $|\varphi| \le |y|$, $\varphi y \ge 0$, $y \ne 0$, $e, f \in E$, is given by the $E \times E$ matrix $G_0(\varphi, y)$ with the components*

$$G_0(\varphi,y) = \begin{cases} \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1} & A_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \\ C_\varphi(A_y - \Pi^- C_y)^{-1} & C_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \end{pmatrix}, & 0 \le \varphi < y \\ \begin{pmatrix} B_\varphi(D_y - \Pi^+ B_y)^{-1}\Pi^+ & B_\varphi(D_y - \Pi^+ B_y)^{-1} \\ D_\varphi(D_y - \Pi^+ B_y)^{-1}\Pi^+ & D_\varphi(D_y - \Pi^+ B_y)^{-1} \end{pmatrix}, & y < \varphi \le 0, \\ \begin{pmatrix} (I - \Pi^- C_y A_y^{-1})^{-1} & \Pi^-(I - C_y A_y^{-1}\Pi^-)^{-1} \\ C_y A_y^{-1}(I - \Pi^- C_y A_y^{-1})^{-1} & (I - C_y A_y^{-1}\Pi^-)^{-1} \end{pmatrix}, & \varphi = y > 0 \\ \begin{pmatrix} (I - B_y D_y^{-1}\Pi^+)^{-1} & ByD_y^{-1}(I - \Pi^+ B_y D_y^{-1})^{-1} \\ \Pi^+(I - B_y D_y^{-1}\Pi^+)^{-1} & (I - \Pi^+ B_y D_y^{-1})^{-1} \end{pmatrix}, & \varphi = y < 0, \end{cases}$$

In the drift cases, $G_0(\varphi, y)$ written in matrix notation is given by

$$G_0(\varphi,y) = \begin{cases} \Gamma\ e^{-\varphi G}\ \Gamma_2\ F(y)\ \Gamma_2^{-1}, & 0 \le \varphi < y \quad \text{or} \quad y < \varphi \le 0 \\ \Gamma\ F(-\varphi)\ \Gamma_2\ e^{yG}\ \Gamma_2^{-1}, & 0 < y < \varphi \quad \text{or} \quad \varphi < y < 0 \\ \Big(I - \Gamma F(-y)\Gamma F(y)\Big)\Gamma_2^{-1}, & \varphi = y \ne 0. \end{cases}$$

In addition, the Green's function $G_0(\varphi, y)$ is positive for all $\varphi, y \in \mathbb{R}$ except for $y = 0$ and for $\varphi y < 0$.

Proof: We prove the theorem for $y > 0$. The case $y < 0$ can be proved in the same way.

Let $y > 0$. First we calculate the Green's function $G_0(y, y)$. Let Y_y denote a matrix on $E^- \times E^+$ with entries

$$Y_y(e, e') = P_{(e,y)}(X_{H_y} = e', H_y < H_0).$$

Then

$$G_0(y,y) = \begin{pmatrix} I & \Pi^- \\ Y_y & I \end{pmatrix} \begin{pmatrix} \sum_{n=0}^\infty (\Pi^- Y_y)^n & 0 \\ 0 & \sum_{n=0}^\infty (Y_y \Pi^-)^n \end{pmatrix}.$$

By Lemma 3.5 (vi) in Jacka et al. (2005), the matrix Y_y is positive and $0 < Y_y 1^+ < 1^-$. Hence, $\Pi^- Y_y$ is positive and therefore irreducible and its Perron-Frobenius eigenvalue λ satisfies $0 < \lambda < 1$. Thus,

$$\lim_{n\to\infty} \frac{(\Pi^- Y_y)^n}{(1+\lambda)^n} = const. \ne 0,$$

which implies that $(\Pi^- Y_y)^n \to 0$ elementwise as $n \to +\infty$. Similarly, $(Y_y \Pi^-)^n \to 0$ elementwise as $n \to +\infty$.

Furthermore, the essentially non-negative matrices $(\Pi^- Y_y - I)$ and $(Y_y \Pi^- - I)$ are invertible because their Perron-Frobenius eigenvalues are negative and, by the same argument, the matrices $(I - \Pi^- Y_y)^{-1}$ and $(I - Y_y \Pi^-)^{-1}$ are positive. Since

$$\begin{array}{rcl}\sum_{k=0}^{n}(\Pi^- Y_y)^k & = & (I - \Pi^- Y_y)^{-1}\,(I - (\Pi^- Y_y)^{n+1}) \\ \sum_{k=0}^{n}(Y_y \Pi^-)^k & = & (I - Y_y \Pi^-)^{-1}\,(I - (Y_y \Pi^-)^{n+1}).\end{array}$$

by letting $n \to \infty$ we finally obtain

$$G_0(y,y) = \begin{pmatrix} (I - \Pi^- Y_y)^{-1} & \Pi^-(I - \Pi^- Y_y)^{-1} \\ Y_y(I - Y_y \Pi^-)^{-1} & (I - Y_y \Pi^-)^{-1} \end{pmatrix} = \begin{pmatrix} I & -\Pi^- \\ -Y_y^{-1} & I \end{pmatrix}^{-1}. \tag{3.6}$$

By Lemma 3.5 (i) and (vi) in Jacka et al. (2005), the matrices Π^- and Y_y are positive. Since the matrices $(I - \Pi^- Y_y)^{-1}$ and $(I - Y_y \Pi^-)^{-1}$ are also positive, it follows that $G_0(y,y)$, $y > 0$ is positive.

Now we calculate the Green's function $G_0(\varphi, y)$ for $0 \le \varphi < y$. Let $(f, y) \in E^+ \times (0, +\infty)$ be fixed and let the process $(X_t, \varphi_t)_{t \ge 0}$ start in $E \times (0, y)$. Let

$$h((e,\varphi),(f,y)) = e^{-\varphi V^{-1} Q} g_{f,y}(e), \tag{3.7}$$

for some vector $g_{f,y}$ on E. Since by (3.6) in Jacka et.al (2005) $\mathcal{G}h = 0$, the process $(h((X_t, \varphi_t),(f,y)))_{t \ge 0}$ is a local martingale, and because the function h is bounded on every finite interval, it is a martingale. In addition, $(h((X_{t \wedge H_0 \wedge H_y}, \varphi_{t \wedge H_0 \wedge H_y}),(f,y)))_{t \ge 0}$ is a bounded martingale and therefore a uniformly integrable martingale.

We want the function h to satisfy the boundary conditions in Lemma 3.1. Let $h_y(\varphi)$ be an $E \times E^+$ matrix with entries

$$h_y(\varphi)(e,f) = h((e,\varphi),(f,y)).$$

Then, from (3.7) and the boundary condition (3.3),

$$h_y(\varphi) = \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix}\begin{pmatrix} M_y \\ 0 \end{pmatrix} = \begin{pmatrix} A_\varphi M_y \\ C_\varphi M_y \end{pmatrix}, \quad 0 \le \varphi < y,$$

for some $E^+ \times E^+$ matrix M_y. From the boundary condition (3.4),

$$A_y M_y = (I - \Pi^- Y_y)^{-1} \qquad \text{and} \qquad C_y M_y = Y_y (I - \Pi^- Y_y)^{-1}, \tag{3.8}$$

which implies that $M_y = (A_y - \Pi^- C_y)^{-1}$ and $Y_y = C_y A_y^{-1}$. Hence,

$$h_y(\varphi) = \begin{pmatrix} A_\varphi (A_y - \Pi^- C_y)^{-1} \\ C_\varphi (A_y - \Pi^- C_y)^{-1} \end{pmatrix}, \qquad 0 \le \varphi < y,$$

and the function $h((e,\varphi),(f,y))$ satisfies the boundary conditions (3.3) and (3.4) in Lemma 3.1. Therefore, for $0 \le \varphi < y$, $G_0(\varphi, y) = h_y(\varphi)$ on $E \times E^+$, and because $G_0(\varphi, y) = h_y(\varphi)\Pi^-$ on $E \times E^-$,

$$G_0(\varphi, y) = \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1} & A_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \\ C_\varphi(A_y - \Pi^- C_y)^{-1} & C_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \end{pmatrix}, \qquad 0 \le \varphi < y.$$

Finally, since $G_0(y,y)$, $y > 0$, is positive, by irreducibility $G_0(\varphi, y)$ for $0 \le \varphi < y$ is also positive. □

Lemma 3.2 *For $y \neq 0$ and any $(e,f) \in E \times E$*

$$\begin{aligned} &P_{(e,\varphi)}(X_{H_y} = f, H_y < H_0) = G_0(\varphi, y)(G_0(y,y))^{-1}(e,f), \qquad 0 < |\varphi| < |y|, \\ &P_{(e,y)}(X_{H_y} = f, H_y < H_0) = \Big(I - (G_0(y,y))^{-1}\Big)(e,f). \end{aligned}$$

Proof: By Theorem 3.3, the matrix $G_0(y,y)$ is invertible. Therefore, the equalities

$$\begin{aligned} G_0((e,\varphi),(f,y)) &= \textstyle\sum_{e' \in E} P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0)\; G_0((e',y),(f,y)), \;\; \varphi \neq y \neq 0, \\ G_0((e,y),(f,y)) &= I(e,f) + \textstyle\sum_{e' \in E} P_{(e,y)}(X_{H_y} = e', H_y < H_0) G_0((e',y),(f,y)), y \neq 0, \end{aligned}$$

prove the lemma. □

4 The oscillating case: Proof of Theorem 2.1

Let $t \ge 0$ be fixed and let $A \in \mathcal{F}_t$. We start by looking at the limit of $P^{[y]}_{(e,\varphi)}(A)$ as $y \to +\infty$. For $(e,\varphi) \in E_0^+$ and $y > \varphi$, by Lemma 3.5 (vi) in Jacka et al. (2005), $P_{(e,\varphi)}(H_y < H_0) > 0$ for all $y > 0$. Hence, by the Markov property, for any $(e,\varphi) \in E_0^+$ and any $A \in \mathcal{F}_t$,

$$\begin{aligned} P^{[y]}_{(e,\varphi)}(A) &= P_{(e,\varphi)}(A \mid H_y < H_0) \\ &= \frac{1}{P_{(e,\varphi)}(H_y < H_0)}\, E_{(e,\varphi)}\Big(I(A)(I\{t < H_0 \wedge H_y\} P_{(X_t,\varphi_t)}(H_y < H_0) \\ &\qquad + I\{H_y \le t < H_0\} + I\{H_y < H_0 \le t\})\Big). \end{aligned} \tag{4.9}$$

Lemma 4.1 *Let r be a vector such that $V^{-1}Qr = 1$. Then,*

$$\begin{aligned} &(i) \qquad h_r(e,\varphi) \equiv -e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e) > 0, \quad (e,\varphi) \in E_0^+, \\ &(ii) \quad \lim_{y \to +\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{e^{-\varphi' V^{-1}Q} J_1 \Gamma_2 r(e')}{e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e)}, \quad (e,\varphi),(e',\varphi') \in E_0^+. \end{aligned}$$

Proof: (i) For any $y \in \mathbb{R}$, let the matrices A_y and C_y be the components of the matrix $e^{-yV^{-1}Q}$ as given in (3.5), that is

$$e^{-yV^{-1}Q} = \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix}.$$

Then, for any $\varphi \in \mathbb{R}$.

$$h_r(\cdot, \varphi) = -e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r = -\begin{pmatrix} A_\varphi(r^+ - \Pi^- r^-) \\ C_\varphi(r^+ - \Pi^- r^-) \end{pmatrix}.$$

The outline of the proof is the following: first we show that the vector $A_\varphi(r^+ - \Pi^- r^-)$ has a constant sign by showing that it is a Perron-Frobenius vector of some positive matrix. Then, because $C_\varphi(r^+ - \Pi^- r^-) = C_\varphi A_\varphi^{-1}\ A_\varphi(r^+ - \Pi^- r^-)$ and because by Lemma 3.2, Theorem 3.3 and by Lemma 3.5 (vi) in Jacka et al. (2005) the matrix $C_\varphi A_\varphi^{-1}$ is positive, we conclude that the vector $C_\varphi(r^+ - \Pi^- r^-)$ has the same constant sign and that the function h_r has a constant sign. Finally, by Lemma 4.1 (ii) in Jacka et al. (2005), we conclude that h_r is always positive.

Therefore, all we have to prove is that the vector $A_\varphi(r^+ - \Pi^- r^-)$ has a constant sign for any $\varphi \in \mathbb{R}$. Let r be fixed vector such that $V^{-1}Qr = 1$. Then

$$e^{yV^{-1}Q} r = r + y1 \quad \Leftrightarrow \quad \begin{array}{r} A_{-y} r^+ + B_{-y} r^- = r^+ + y1^+ \\ C_{-y} r^+ + D_{-y} r^- = r^- + y1^-. \end{array}$$

By (3.8), the matrix A_φ is invertible. Thus, because $1^+ = \Pi^- 1^-$, $(A_{-y} - \Pi^- C_{-y}) = (A_y - \Pi^- C_y)^{-1}$ and $(B_{-y} - \Pi^- D_{-y}) = -(A_{-y} - \Pi^- C_{-y})\Pi^-$,

$$\left(A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}\right)\ A_\varphi(r^+ - \Pi^- r^-) = A_\varphi(r^+ - \Pi^- r^-).$$

By Theorem 3.3 the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1}$ is positive for any $\varphi \neq y$. By Lemma 3.2, Theorem 3.3 and by Lemma 3.5 (vi) in Jacka et al. (2005), the matrix A_φ^{-1} is also positive. Hence, the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$, $\varphi \neq y$, is positive and it has the Perron-Frobenius eigenvector which has a constant sign.

Suppose that $A_\varphi(r^+ - \Pi^- r^-) = 0$. Then, because A_φ is invertible, $(r^+ - \Pi^- r^-) = 0$. If $r^+ = \Pi^- r^-$ then r is a linear combination of the vectors g_k, $k = 1, \ldots, m$ in the basis $\mathcal{B}$, but that is not possible because r is also in the basis $\mathcal{B}$ and therefore independent from g_k, $k = 1, \ldots, m$. Hence, the vector $A_\varphi(r^+ - \Pi^- r^-) \neq 0$ and by the last equation it is the eigenvector of the matrix $A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1}$ which corresponds to its eigenvalue 1.

We want to show that 1 is the Perron-Frobenius eigenvalue of the matrix $A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1}$. It follows from

$$\left(A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}\right)\ A_\varphi(I - \Pi^- \Pi^+) = A_\varphi(I - \Pi^- \Pi^+)\ e^{yG^+} \tag{4.10}$$

that if α is a non-zero eigenvalue of the matrix G^+ with some algebraic multiplicity, then $e^{\alpha y}$ is an eigenvalue of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1}A_\varphi^{-1}$ with the same algebraic multiplicity. Since all $n-1$ non-zero eigenvalues of G^+ have negative real parts, all eigenvalues $e^{\alpha_j y}$, $\alpha_j \neq 0$, $j = 1, \ldots, n$, of $A_\varphi(A_y - \Pi^- C_y)^{-1}A_\varphi^{-1}$ have real parts strictly less than 1. Thus, 1 is the Perron-Frobenius eigenvalue of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1}A_\varphi^{-1}$ and the vector $A_\varphi(r^+ - \Pi^- r^-)$ is its Perron-Frobenius eigenvector, and therefore has a constant sign.

(ii) The statement follows directly from the equality

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \lim_{y\to+\infty} \frac{G_0(\varphi', y)1(e')}{G_0(\varphi, y)1(e)},$$

where $G_0(\varphi, y)$ is the Green's function for the killed process defined and determined in Section 3, and from the representation of $G_0(\varphi, y)$ given by

$$G_0(\varphi, y)1 = \sum_{j,\alpha_j\neq 0} a_j\ e^{-\varphi V^{-1}Q} J_1\Gamma_2\ e^{yV^{-1}Q} f_j +\ c\ e^{-\varphi V^{-1}Q} J_1\Gamma_2 r,$$

for some constants a_j, $j = 1, \ldots, n$ and $c \neq 0$, where vectors f_j, $j = 1, \ldots, n$, form a part of the basis $\mathcal{B}$ in the space of all vectors on E and are associated with the eigenvalues α_j, $j = 1, \ldots, n$, of the matrix G^+. Since $Re(\alpha_j) < 0$ for all $\alpha_j \neq 0$, $j = 1, \ldots, n$, it can be shown that for every j, $j = 1, \ldots, n$, such that $\alpha_j \neq 0$, $e^{yV^{-1}Q} f_j \to 0$ as $y \to +\infty$, which proves the statement. For the details of the proof see Najdanovic (2003). □

Proof of Theorem 2.1: By Lemmas 4.1 (ii) and 4.3 in Jacka et al. (2005), the function $h_r(e, \varphi)$ is positive and harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$. Therefore, the measure $P^{h_r}_{(e,\varphi)}$ is well-defined.

For fixed $(e, \varphi) \in E_0^+$, $t \in [0, +\infty)$ and any $y \geq 0$, let Z_y be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$\begin{aligned} Z_y \;&=\; \frac{1}{P_{(e,\varphi)}(H_y < H_0)} \Big(I\{t < H_0 \wedge H_y\} P_{(X_t,\varphi_t)}(H_y < H_0) \\ &\qquad + I\{H_y \leq t < H_0\} + I\{H_y < H_0 \leq t\} \Big). \end{aligned}$$

By Lemma 4.1 (ii) and by Lemmas 4.1 (ii) , 4.2 (i) and 4.3 in Jacka et al. (2005) the random variables Z_y converge to $\frac{h_r(X_t,\varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}$ in $L^1(\Omega, \mathcal{F}, P_{(e,\varphi)})$ as $y \to +\infty$. Therefore, by (4.9), for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\lim_{y\to+\infty} P^{[y]}_{(e,\varphi)}(A) = \lim_{y\to+\infty} E_{(e,\varphi)}\Big(I(A) Z_y\Big) = P^{h_r}_{(e,\varphi)}(A),$$

which, by Lemma 4.2 (ii) in Jacka et al. (2005), implies that the measures $(P^{[y]}_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to $P^{h_r}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$. □

5 The negative drift case: Proof of Theorem 2.2

Again, as in the oscillating case, we start with the limit of $P^{[y]}_{(e,\varphi)}(A)$ as $y \to +\infty$ by looking at $\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)}$. First we prove an auxiliary lemma.

Lemma 5.1 *For any vector g on E* $\lim_{y\to+\infty} F(y)g = 0$.

In addition, for any non-negative vector g on E $\lim_{y\to+\infty} e^{-\alpha_{max} y}F(y)g = c\ J_1 f_{max}$ for some positive constant $c \in \mathbb{R}$.

Proof: Let

$$g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \text{ and } g^+ = \sum_{j=1}^{n} a_j f_j^+,$$

for some coefficients a_j, $j = 1, \ldots, n$, where vectors f_j^+, $j = 1, \ldots, n$, form the basis in the space of all vectors on E^+ and are associated with the eigenvalues α_j, $j = 1, \ldots, n$, of the matrix G^+. Then, the first equality in the lemma follows from

$$F(y)g = \begin{pmatrix} e^{yG^+} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} e^{yG^+} g^+ \\ 0 \end{pmatrix} = \sum_{j=1}^{n} a_j \begin{pmatrix} e^{yG^+} f_j^+ \\ 0 \end{pmatrix}, \qquad y > 0, \quad (5.11)$$

since, for $Re(\alpha_j) < 0$, $j = 1, \ldots, n$, $e^{yG^+} f_j^+ \to 0$ as $y \to +\infty$.

Moreover, by Lemma 3.5 (iii) in Jacka et al. (2005), the matrix G^+ is an irreducible Q-matrix with the Perron-Frobenius eigenvalue α_{max} and Perron-Frobenius eigenvector f^+_{max}. Thus, for any non-negative vector g on E^+, by Lemma 3.6 (ii) in Jacka et al. (2005),

$$\lim_{y\to+\infty} e^{-\alpha_{max} y}\ e^{yG^+} g(e) = c\ f^+_{max}(e), \quad (5.12)$$

for some positive constant $c \in \mathbb{R}$. Therefore, from (5.11) and (5.12)

$$\lim_{y\to+\infty} e^{-\alpha_{max} y}\ F(y)g = \lim_{y\to+\infty} \begin{pmatrix} e^{-\alpha_{max} y}\ e^{yG^+} g^+ \\ 0 \end{pmatrix} = c \begin{pmatrix} f^+_{max} \\ 0 \end{pmatrix} = c\ J_1 f_{max}.$$

□

Now we find the limit $\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)}$.

Lemma 5.2

$$(i) \qquad h_{f_{max}}(e,\varphi) \equiv e^{-\varphi V^{-1}Q} J_1 \Gamma_2 f_{max}(e) > 0, \quad (e,\varphi) \in E_0^+,$$

$$(ii) \quad \lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{e^{-\varphi' V^{-1}Q} J_1 \Gamma_2 f_{max}(e')}{e^{-\varphi V^{-1}Q} J_1 \Gamma_2 f_{max}(e)}, \quad (e,\varphi), (e',\varphi') \in E_0^+.$$

Proof: (i) The function $h_{f_{max}}$ can be rewritten as

$$h_{f_{max}}(\cdot,\varphi) = e^{-\varphi V^{-1}Q} J_1 \Gamma_2 f_{\max} = \begin{pmatrix} A_\varphi(I - \Pi^-\Pi^+)f^+_{max} \\ C_\varphi(I - \Pi^-\Pi^+)f^+_{max} \end{pmatrix}$$

where A_φ and C_φ are given by (3.5).

First we show that the vector $A_\varphi(I - \Pi^-\Pi^+)f^+_{max}$ is positive. By (3.8) the matrix A_φ is invertible and, by (3.8) and Lemma 3.5 (ii) and (iv) in Jacka et al. (2005), the matrix $(I - \Pi^-\Pi^+)$ is invertible. Therefore,

$$A_\varphi(A_{-y} - \Pi^- C_{-y})A^{-1}_\varphi = A_\varphi(I - \Pi^-\Pi^+)e^{yG^+}(I - \Pi^-\Pi^+)^{-1}A^{-1}_\varphi.$$

By Theorem 3.3 the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1}$, $\varphi \neq y$, is positive and by Lemma 3.2, Theorem 3.3 and by Lemma 3.5 (vi) in Jacka et al.(2005), the matrix A^{-1}_φ is also positive. Hence, the matrix $A_\varphi(A_{-y} - \Pi^- C_{-y})A^{-1}_\varphi$, $\varphi \neq y$ is positive and is similar to e^{yG^+}. Thus, $A_\varphi(A_{-y} - \Pi^- C_{-y})A^{-1}_\varphi$ and e^{yG^+} have the same Perron-Frobenius eigenvalue and because the Perron-Frobenius eigenvector of e^{yG^+} is f^+_{max}, it follows that $A_\varphi(I - \Pi^-\Pi^+)f^+_{max}$ is the Perron-Frobenius eigenvector of $A_\varphi(A_{-y} - \Pi^- C_{-y})A^{-1}_\varphi$ and therefore positive. In addition,

$$C_\varphi(I - \Pi^-\Pi^+)f^+_{max} = C_\varphi A^{-1}_\varphi \; A_\varphi(I - \Pi^-\Pi^+)f^+_{max},$$

and by Lemma 3.2, Theorem 3.3 and by Lemma 3.5 (vi) in Jacka et al. (2005), the matrix $C_\varphi A^{-1}_\varphi$ is positive. Therefore, the function $h_{f_{max}}$ is positive.

(ii) By Lemmas 3.2, 5.1 and Theorem 3.3,

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} == \lim_{y\to+\infty} \frac{e^{-\varphi' V^{-1}Q}\Gamma\Gamma_2 F(y)1(e')}{e^{-\varphi V^{-1}Q}\Gamma\Gamma_2 F(y)1(e)}.$$

Since the vector 1 is non-negative and because $\Gamma\Gamma_2 J_1 f_{max} = J_1\Gamma_2 f_{max}$, the statement in the lemma follows from Lemma 5.1. □

The function $h_{f_{max}}$ has the property that the process $\{h_{f_{max}}(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. We prove this in the following lemma.

Lemma 5.3 *The function $h_{f_{max}}(e,\varphi)$ is harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$ and the process $\{h_{f_{max}}(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$.*

Proof: The function $h_{f_{max}}(e,\varphi)$ is continuously differentiable in φ and therefore by (3.6) in Jacka et al. (2005) it is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t,\varphi_t)_{t\geq 0}$ and $\mathcal{G}h_{f_{max}} = 0$. Thus, the function $h_{f_{max}}(e,\varphi)$ is harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$ and the process $(h_{f_{max}}(X_t,\varphi_t))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$. It follows that the process $(h_{f_{max}}(X_{t\wedge H_0},\varphi_{t\wedge H_0}) = h_{f_{max}}(X_t,\varphi_t)I\{t < H_0\})_{t\geq 0}$ is also a local martingale under $P_{(e,\varphi)}$ and, because it is bounded on every finite interval, that it is a martingale. □

Proof of Theorem 2.2: The proof is exactly the same as the proof of Theorem 2.1 with the function $h_{f_{max}}$ substituting for h_r (and we therefore appeal to Lemma 5.2 rather than Lemma 4.1 for the desired properties of $h_{f_{max}}$). □

6 The negative drift case: conditioning $(\varphi_t)_{t\geq 0}$ to drift to $+\infty$

The process $(X_t, \varphi_t)_{t\geq 0}$ can also be conditioned first on the event that $(\varphi_t)_{t\geq 0}$ hits large levels y regardless of crossing zero (that is taking the limit as $y \to \infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < +\infty\}$), and then the resulting process can be conditioned on the event that $(\varphi_t)_{t\geq 0}$ stays non-negative. In this section we show that these two conditionings performed in the stated order yield the same result as the limit as $y \to +\infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$.

Let $(e, \varphi) \in E_0^+$ and $y > \varphi$. Then, by Lemma 3.5 (vii) in Jacka et al. (2005), the event $\{H_y < +\infty\}$ is of positive probability and the process $(X_t, \varphi_t)_{t\geq 0}$ can be conditioned on $\{H_y < +\infty\}$ in the standard way.

For fixed $t \geq 0$ and any $A \in \mathcal{F}_t$,

$$P_{(e,\varphi)}(A \mid H_y < +\infty) = \frac{E_{(e,\varphi)}\Big(I(A)P_{(X_t,\varphi_t)}(H_y < +\infty)I\{t < H_y\} + I(A)I\{H_y < t\}\Big)}{P_{(e,\varphi)}(H_y < +\infty)}. \tag{6.13}$$

Lemma 6.1 *For any* $(e, \varphi), (e', \varphi') \in E_0^+$,

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < +\infty)}{P_{(e,\varphi)}(H_y < +\infty)} = \frac{e^{-\alpha_{max}\varphi'} f_{max}(e')}{e^{-\alpha_{max}\varphi} f_{max}(e)}.$$

Proof: By Lemma 3.7 in Jacka et al. (2005), for $0 \leq \varphi < y$,

$$P_{(e,\varphi)}(H_y < +\infty) = P_{(e,\varphi-y)}(H_0 < +\infty) = \Gamma F(y - \varphi)1.$$

The vector 1 is non-negative. Hence, by Lemma 5.1 and because $\Gamma J_1 f_{max} = f_{max}$,

$$\begin{aligned}\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < +\infty)}{P_{(e,\varphi)}(H_y < +\infty)} &= \lim_{y\to+\infty} \frac{e^{-\alpha_{max}\varphi'}\Gamma e^{-\alpha_{max}(y-\varphi')}F(y-\varphi)1(e')}{e^{-\alpha_{max}\varphi}\Gamma e^{-\alpha_{max}(y-\varphi)}F(y-\varphi)1(e)} \\ &= \frac{e^{-\alpha_{max}\varphi'} f_{max}(e')}{e^{-\alpha_{max}\varphi} f_{max}(e)}.\end{aligned}$$

□

Let $h_{max}(e, \varphi)$ be a function on $E \times \mathbb{R}$ defined by

$$h_{max}(e, \varphi) = e^{-\alpha_{max}\varphi} f_{max}(e).$$

Lemma 6.2 *The function $h_{max}(e, \varphi)$ is harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and the process $(h_{max}(X_t, \varphi_t))_{t\geq 0}$ is a martingale under $P_{(e,\varphi)}$.*

Proof: The function $h_{max}(e,\varphi)$ is continuously differentiable in φ and therefore by (3.6) in Jacka et al. (2005) it is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t,\varphi_t)_{t\geq 0}$ and $\mathcal{G}h_{max}=0$. It follows that the function $h_{max}(e,\varphi)$ is harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$ and that the process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$. Since the function $h_{max}(e,\varphi)$ is bounded on every finite interval, the process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a martingale under $P_{(e,\varphi)}$. □

By Lemmas 6.1 and 6.2 we prove

Theorem 6.1 *For fixed $(e,\varphi)\in E_0^+$, let $P^{h_{max}}_{(e,\varphi)}$ be a measure defined by*

$$P^{h_{max}}_{(e,\varphi)}(A)=\frac{E_{(e,\varphi)}\Big(I(A)\ h_{max}(X_t,\varphi_t)\Big)}{h_{max}(e,\varphi)},\qquad t\geq 0, A\in\mathcal{F}_t.$$

Then, $P^{h_{max}}_{(e,\varphi)}$ is a probability measure and, for fixed $t\geq 0$,

$$\lim_{y\to+\infty}P_{(e,\varphi)}(A\mid H_y<+\infty)=P^{h_{max}}_{(e,\varphi)}(A),\quad A\in\mathcal{F}_t.$$

Proof: By the definition, the function h_{max} is positive. By Lemma 6.2, it is harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$ and the process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a martingale under $P_{(e,\varphi)}$. Hence, $P^{h_{max}}_{(e,\varphi)}$ is a probability measure.

For fixed $(e,\varphi)\in E_0^+$ and $t\geq 0$ and any $y\geq 0$, let Z_y be a random variable defined on the probability space $(\Omega,\mathcal{F},P_{(e,\varphi)})$ by

$$Z_y=\frac{P_{(X_t,\varphi_t)}(H_y<+\infty)I\{t<H_y\}+I\{H_y<t\}}{P_{(e,\varphi)}(H_y<+\infty)}.$$

By Lemma 6.1 and by Lemmas 4.2 (i) and 4.3 in Jacka et al. (2005) the random variables Z_y converge to $\frac{h_{max}(X_t,\varphi_t)}{h_{max}(e,\varphi)}$ in $L^1(\Omega,\mathcal{F},P_{(e,\varphi)})$ as $y\to+\infty$. Therefore, by (6.13), for fixed $t\geq 0$ and $A\in\mathcal{F}_t$,

$$\lim_{y\to+\infty}P_{(e,\varphi)}(A\mid H_y<+\infty)=\lim_{y\to+\infty}E_{(e,\varphi)}\Big(I(A)\ Z_y\Big)=P^{h_{max}}_{(e,\varphi)}(A).$$

□

We now want to condition the process $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ on the event $\{H_0=+\infty\}$. By Theorem 7.1, $(X_t)_{t\geq 0}$ is Markov under $P^{h_{max}}_{(e,\varphi)}$ with the irreducible conservative Q-matrix $Q^{h_{max}}$ given by

$$Q^{h_{max}}(e,e')=\frac{f_{max}(e')}{f_{max}(e)}(Q-\alpha_{max}V)(e,e'),\qquad e,e'\in E,$$

and, by the same theorem, the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$ under $P^{h_{max}}_{(e,\varphi)}$. We find the Wiener-Hopf factorization of the matrix $V^{-1}Q^{h_{max}}$.

Lemma 6.3 *The unique Wiener-Hopf factorization of the matrix* $V^{-1}Q^{h_{max}}$ *is given by* $V^{-1}Q^{h_{max}}\,\Gamma^{h_{max}} = \Gamma^{h_{max}}\,G^{h_{max}}$*, where, for any* $(e,e') \in E \times E$*,*

$$G^{h_{max}}(e,e') = \frac{f_{max}(e')}{f_{max}(e)}\,(G - \alpha_{max} I)(e,e') \quad \textit{and} \quad \Gamma^{h_{max}}(e,e') = \frac{f_{max}(e')}{f_{max}(e)}\,\Gamma(e,e').$$

In addition, if

$$G^{h_{max}} = \begin{pmatrix} G^{h_{max},+} & 0 \\ 0 & -G^{h_{max},-} \end{pmatrix} \quad \textit{and} \quad \Gamma^{h_{max}} = \begin{pmatrix} I & \Pi^{h_{max},-} \\ \Pi^{h_{max},+} & I \end{pmatrix},$$

then $G^{h_{max},+}$ *is a conservative Q-matrix and* $\Pi^{h_{max},+}$ *is stochastic, and* $G^{h_{max},-}$ *is not a conservative Q-matrix and* $\Pi^{h_{max},-}$ *is strictly substochastic.*

Proof: By the definition the matrices $G^{h_{max},+}$ and $G^{h_{max},-}$ are essentially non-negative. In addition, for any $e \in E^+$, $G^{h_{max},+}1(e) = 0$. Hence, $G^{h_{max},+}$ is a conservative Q-matrix. By Lemma 5.2 (i),

$$h^-_{f_{max}} = (\Pi^+ e^{-\varphi G^+} - e^{\varphi G^-}\Pi^+) f^+_{max} = e^{-\alpha_{max}\varphi}(I - e^{\varphi(G^- + \alpha_{max} I)}) f^-_{max} > 0.$$

Since

$$\lim_{\varphi \to 0} \frac{(I - e^{\varphi(G^- + \alpha_{max} I)})\; f^-_{max}}{\varphi} = -(G^- + \alpha_{max} I) f^-_{max},$$

and $(I - e^{\varphi(G^- + \alpha_{max} I)}) f^-_{max} > 0$, it follows that $(G^- + \alpha_{max} I) f^-_{max} \leq 0$. Thus, $G^{h_{max},-}1^- \leq 0$ and so $G^{h_{max},-}$ is a Q-matrix. Moreover, if $(G^- + \alpha_{max} I) f^-_{max} = 0$ then $h_{f_{max}}(e,\varphi) = 0$ for $e \in E^-$ which is a contradiction to Lemma 5.2. Therefore, the matrix $G^{h_{max},-}$ is not conservative.

The matrices $G^{h_{max}}$ and $\Gamma^{h_{max}}$ satisfy the equality $V^{-1}Q^{h_{max}}\,\Gamma^{h_{max}} = \Gamma^{h_{max}}\,G^{h_{max}}$, which, by Lemma 3.4 in Jacka et al. (2005), gives the unique Wiener-Hopf factorization of the matrix $V^{-1}Q^{h_{max}}$. Furthermore, by Lemma 3.5 (iv) in Jacka et al. (2005), $\Pi^{h_{max},+}$ is a stochastic and $\Pi^{h_{max},-}$ is a strictly substochastic matrix. □

Finally, we prove the main result in this section

Theorem 6.2 *Let* $P^{h_{f_{max}}}_{(e,\varphi)}$ *be as defined in Theorem 2.2. Then, for any* $(e,\varphi) \in E^+_0$ *and any* $t \geq 0$*,*

$$P^{h_{max}}_{(e,\varphi)}(A \mid H_0 = \infty) = P^{h_{f_{max}}}_{(e,\varphi)}(A), \qquad A \in \mathcal{F}_t.$$

Proof: By Theorem 7.1 the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ drifts to $+\infty$. Since in the positive drift case the event $\{H_0 = +\infty\}$ is of positive probability, for any $t \geq 0$ and any $A \in \mathcal{F}_t$,

$$P^{h_{\max}}_{(e,\varphi)}(A \mid H_0 = \infty) = \frac{E^{h_{\max}}_{(e,\varphi)}\Big(I(A)\; P^{h_{max}}_{(X_t,\varphi_t)}(H_0 = +\infty)\; I\{t < H_0\}\Big)}{P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty)}, \tag{6.14}$$

where $E^{h_{\max}}_{(e,\varphi)}$ denotes the expectation operator associated with the measure $P^{h_{max}}_{(e,\varphi)}$.

By Lemma 3.7 in Jacka et al. (2005) and by Lemma 6.3, for $\varphi > 0$,

$$\begin{aligned} P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty) &= 1 - \frac{e^{\alpha_{max}\varphi}}{f_{max}(e)} \sum_{e' \in E} \Gamma e^{-\varphi G}(e,e')\ J_2 1(e')\ f_{max}(e') \\ &= \frac{1}{h_{max}(e,\varphi)} \Big(e^{-\alpha_{max}\varphi}\ f_{max} - \Gamma F(-\varphi) f_{max} \Big)(e) \\ &= \frac{h_{f_{max}}(e,\varphi)}{h_{max}(e,\varphi)}, \end{aligned} \qquad (6.15)$$

where $h_{f_{max}}$ is as defined in Lemma 5.2. Similarly, for $e \in E^+$,

$$P^{h_{max}}_{(e,0)}(H_0 = +\infty) = \frac{f^+_{max} - \Pi^- f^-_{max})(e)}{f^+_{max}(e)} = \frac{h_{f_{max}}(e,0)}{h_{max}(e,0)}.$$

Therefore, the statement in the theorem follows from Theorem 6.1, (6.14) and (6.15). □

We summarize the results from this section: in the negative drift case, making the h-transform of the process $(X_t, \varphi_t)_{t \geq 0}$ by the function $h_{max}(e,\varphi) = e^{-\alpha_{max}\varphi} f_{max}(e)$ yields the probability measure $P^{h_{max}}_{(e,\varphi)}$ such that $(X_t)_{t\geq 0}$ is Markov under $P^{h_{max}}_{(e,\varphi)}$ and that $(\varphi_t)_{t\geq 0}$ has a positive drift under $P^{h_{max}}_{(e,\varphi)}$. The process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ is also the limiting process as $y \to +\infty$ in conditioning $(X_t, \varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}$ on $\{H_y < +\infty\}$. Further conditioning $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ on $\{H_0 = +\infty\}$ yields the same result as the limit as $y \to +\infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$. In other words, the diagram in Figure 1 commutes.

7 The negative drift case: conditioning $(\varphi_t)_{t\geq 0}$ to oscillate

In this section we condition the process $(\varphi_t)_{t\geq 0}$ with a negative drift to oscillate, and then condition the resulting oscillating process to stay non-negative.

Let $P^h_{(e,\varphi)}$ denote the h-transform of the measure $P_{(e,\varphi)}$ by a positive superharmonic function h for the process $(X_t, \varphi_t)_{t\geq 0}$. We want to find a function h such that $P^h_{(e,\varphi)}$ is honest; the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$ and the process $(\varphi_t)_{t\geq 0}$ oscillates under $P^h_{(e,\varphi)}$. These desired properties of the function h necessarily imply that it has to be harmonic.

First we find a form of a positive and harmonic function for the process $(X_t, \varphi_t)_{t\geq 0}$ such that the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$.

Lemma 7.1 *Suppose that a function h is positive and harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and that the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$. Then h is of the form*

$$h(e,\varphi) = e^{-\lambda\varphi} g(e), \qquad (e,\varphi) \in E \times \mathbb{R},$$

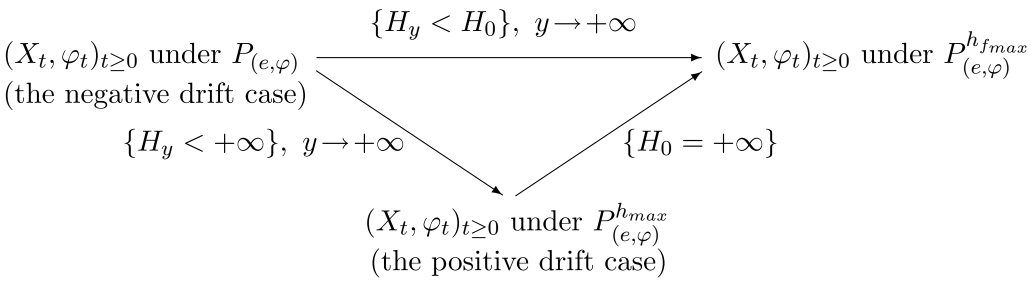

Figure 1: Conditioning of the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_y < H_0\}$, $y \geq 0$, in the negative drift case.

for some $\lambda \in \mathbb{R}$ *and some vector* g *on* E.

Proof: By the definition of $P^h_{(e,\varphi)}$, for any $(e,\varphi) \in E \times \mathbb{R}$ and $t \geq 0$,

$$P^h_{(e,\varphi)}(X_s = e, 0 \leq s \leq t) = \frac{h(e, \varphi + v(e)t)}{h(e,\varphi)}\ P_{(e,\varphi)}(X_s = e, 0 \leq s \leq t).$$

Since the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$, the probability $P^h_{(e,\varphi)}(X_s = e, 0 \leq s \leq t)$ does not depend on φ. Thus, the right-hand side of the last equation does not depend on φ. Since $P_{(e,\varphi)}(X_s = e, 0 \leq s \leq t)$ also does not depend on φ because $(X_t)_{t\geq 0}$ is Markov under $P_{(e,\varphi)}$, it follows that the ratio $\frac{h(e,\varphi+v(e)t)}{h(e,\varphi)}$ does not depend on φ. This implies that h satisfies

$$h(e, \varphi + y) = \frac{h(e,\varphi)\ h(e,y)}{h(e,0)}, \qquad e \in E,\ \varphi, y \in \mathbb{R}. \tag{7.16}$$

Let $e \in E$ be fixed. Since the function h is positive, we define a function $k_e(\varphi)$ by

$$k_e(\varphi) = \log\Big(\frac{h(e,\varphi)}{h(e,0)}\Big), \qquad \varphi \in (0, +\infty).$$

Then, by (7.16), the function k_e is additive. In addition, it is measurable because the function h is measurable as a harmonic function. Therefore, it is linear (see Aczel (1966)). It follows that the function h is exponential, that is

$$h(e,\varphi) = h(e,0)\ e^{\lambda(e)\varphi}, \qquad (e,\varphi) \in E_0^+$$

for some function $\lambda(e)$ on E.

Hence, the function h is continuously differentiable in φ which implies by (3.6) in Jacka et al. (2005) that the Q-matrix of the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is given by

$$\begin{aligned} Q^h(e,e') &= \frac{h(e',\varphi)}{h(e,\varphi)}Q + \frac{\frac{\partial h}{\partial \varphi}(e,\varphi)}{h(e,\varphi)}\, V(e,e') \\ &= \frac{h(e',0)}{h(e,0)}\, e^{(\lambda(e)-\lambda(e'))\varphi}\, Q + \lambda(e)\, V(e,e), \quad e,e' \in E. \end{aligned}$$

But, because $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$, Q^h does not depend on φ. This implies that $\lambda(e) = -\lambda = const.$

Finally, putting $g(e) = h(e,0)$, $e \in E$, proves the theorem. □

The following theorem characterizes all positive harmonic functions for the process $(X_t, \varphi_t)_{t\geq 0}$ with the properties stated at the beginning of the section.

Theorem 7.1 *There exist exactly two positive harmonic functions h for the process $(X_t, \varphi_t)_{t\geq 0}$ such that the measure $P^h_{(e,\varphi)}$ is honest and that the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$. They are given by*

$$h_{max}(e,\varphi) = e^{-\alpha_{max}\varphi} f_{max}(e) \qquad and \qquad h_{min}(e,\varphi) = e^{-\beta_{min}\varphi} g_{min}(e).$$

Moreover,
(i) if the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$ then $h_{max} = 1$ and the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{min}}_{(e,\varphi)}$ drifts to $-\infty$;
(ii) if the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$ then $h_{min} = 1$ and the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ drifts to $+\infty$;
(iii) if the process $(\varphi_t)_{t\geq 0}$ oscillates then $h_{max} = h_{min} = 1$.

Proof: We give a sketch of the proof. For the details see Najdanovic (2003).

Let a function h be positive and harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and let the process $(X_t)_{t\geq 0}$ be Markov under $P^h_{(e,\varphi)}$. Then by Lemma 7.1 the function h is of the form

$$h(e,\varphi) = e^{-\lambda\varphi} g(e), \qquad (e,\varphi) \in E \times \mathbb{R},$$

for some $\lambda \in \mathbb{R}$ and some vector g on E.

Since the function h is harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ it satisfies the equation $\mathcal{G}h = 0$ where $\mathcal{G}$ is the generator of the process $(X_t, \varphi_t)_{t\geq 0}$ given by (3.6) in Jacka et al. (2005). Hence, $\mathcal{G}h = (Q + V\frac{d}{d\varphi})h = 0$ and $h(e,\varphi) = e^{-\lambda\varphi} g(e)$ imply that $V^{-1}Qg = \lambda g$, that is λ is an eigenvalue and g its associated eigenvector of the matrix $V^{-1}Q$. In addition, by Lemma 3.6 (i) in Jacka et al. (2005) the only positive eigenvectors of the matrix $V^{-1}Q$ are f_{max} and g_{min}. Hence, $h(e,\varphi) = e^{-\alpha_{max}\varphi} f_{max}(e)$ or $h(e,\varphi) = e^{-\beta_{min}\varphi} g_{min}(e)$.

The equality $\mathcal{G}h = 0$ implies that the process $(h(X_t, \varphi_t))_{t\geq 0}$ is a local martingale. Since the function $h(e, \varphi) = e^{-\lambda\varphi} g(e)$ is bounded on every finite interval, the process $(h(X_t, \varphi_t))_{t\geq 0}$ is a martingale. It follows that the measure $P^h_{(e,\varphi)}$ is honest.

Let Q^h be the Q-matrix of the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$. It can be shown that the eigenvalues of the matrix $V^{-1}Q^{h_{min}}$ coincide with the eigenvalues of the matrix $V^{-1}(Q - \beta_{min} I)$, and that the eigenvalues of the matrix $V^{-1}Q^{h_{max}}$ coincide with the eigenvalues of the matrix $V^{-1}(Q - \alpha_{max} I)$. These together with (3.8) in Jacka et al. (2005) prove statements (i)-(iii). □

By Theorem 7.1 (ii) there does not exist a positive function h harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ such that $P^h_{(e,\varphi)}$ is honest, that the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$ and that the process $(\varphi_t)_{t\geq 0}$ oscillates under $P^h_{(e,\varphi)}$ (we recall that initially the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$ under $P_{(e,\varphi)}$). However, we can look for a positive space-time harmonic function h for the process $(X_t, \varphi_t)_{t\geq 0}$ that has the desired properties.

Lemma 7.2 *Suppose that a function h is positive and space-time harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and that the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$. Then h is of the form*

$$h(e, \varphi, t) = e^{-\alpha t} e^{-\beta\varphi} g(e), \qquad (e, \varphi) \in E \times \mathbb{R},$$

for some $\alpha, \beta \in \mathbb{R}$ and some vector g on E.

Proof: By the definition of $P^h_{(e,\varphi)}$, for any $(e, \varphi) \in E \times \mathbb{R}$ and $t \geq 0$, and any $s \geq 0$ and $y \in \mathbb{R}$,

$$P^h_{(e,\varphi,t)}(X_{t+s} = e, \varphi_{t+s} \in \varphi + y) = \frac{h(e, \varphi + y, t + s)}{h(e, \varphi, t)} \, P_{(e,\varphi,t)}(X_{t+s} = e, \varphi_{t+s} \in \varphi + y). \tag{7.17}$$

Since the process $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$, we have

$$P^h_{(e,\varphi,t)}(X_{t+s} = e, \varphi_{t+s} \in \varphi + y) = P^h_{(e,0,0)}(X_s = e, \varphi_s \in y).$$

And similarly

$$P_{(e,\varphi,t)}(X_{t+s} = e, \varphi_{t+s} \in \varphi + y) = P_{(e,0,0)}(X_s = e, \varphi_s \in y).$$

Therefore, it follows from (7.17) that the ratio $\frac{h(e,\varphi+y,t+s)}{h(e,\varphi,t)}$ does not depend on φ and t. This implies that h satisfies

$$h(e, \varphi + y, t + s) = \frac{h(e, \varphi, t) \; h(e, y, s)}{h(e, 0, 0)}, \qquad e \in E, \;\; \varphi, y \in \mathbb{R}, t, s \geq 0. \tag{7.18}$$

Since the function h is positive, we define a function $k(e, \varphi, t)$ by

$$k(e, \varphi, t) = \log\Big(\frac{h(e, \varphi, t)}{h(e, 0, 0)}\Big), \qquad (e, \varphi, t) \in E_0^+ \times [0, +\infty).$$

Then, by (7.18),

$$k(e, \varphi + y, t + s) = k(e, \varphi, t) + k(e, y, s), \qquad e \in E, \ \varphi, y \in \mathbb{R}, \ t, s \geq 0.$$

Let $t = s = 0$. Then

$$k(e, \varphi + y, 0) = k(e, \varphi, 0) + k(e, y, 0), \qquad e \in E, \ \varphi, y \in \mathbb{R}.$$

Hence, $k(e, \varphi, 0)$ is additive in φ and is measurable because the function h is measurable as a harmonic function. It follows (see Aczel (1966)) that $k(e, \varphi, 0)$ is linear in φ, that is

$$k(e, \varphi, 0) = \beta(e) \ \varphi$$

for some function β on E. Similarly, for $\varphi = y = 0$, we have

$$k(e, 0, t + s) = k(e, 0, t) + k(e, 0, s),$$

which implies that

$$k(e, 0, t) = \alpha(e) \ t$$

for some function α on E. Putting the pieces together, we obtain

$$k(e, \varphi, t) = \alpha(e) \ t + \beta(e) \ \varphi, \qquad (e, \varphi, t) \in E_0^+ \times [0, +\infty).$$

Then it follows from the definition of the function $k(e, \varphi, t)$ that

$$h(e, \varphi, t) = h(e, 0, 0) \ e^{\alpha(e)t} \ e^{\beta(e)\varphi}, \qquad (e, \varphi, t) \in E_0^+ \times [0, +\infty)$$

for some functions α and β on E.

Hence, the function h is continuously differentiable in φ and t which implies by (3.7) in Jacka et al. (2005) that the Q-matrix of the process $(X_t)_{t \geq 0}$ under $P^h_{(e,\varphi)}$ is given by

$$\begin{aligned} Q^h(e, e') &= \frac{h(e', \varphi, t)}{h(e, \varphi, t)} Q + \frac{\frac{\partial h}{\partial \varphi}(e, \varphi, t)}{h(e, \varphi, t)} \ V(e, e') + \frac{\frac{\partial h}{\partial t}(e, \varphi, t)}{h(e, \varphi, t)} \ I(e, e') \\ &= \frac{h(e', 0, 0)}{h(e, 0, 0)} \ e^{(\alpha(e') - \alpha(e))t} \ e^{(\beta(e') - \beta(e))\varphi} \ Q + \beta(e) \ V(e, e) \\ &\quad + \alpha(e) \ I(e, e), \quad e, e' \in E. \end{aligned}$$

But, because $(X_t)_{t \geq 0}$ is Markov under $P^h_{(e,\varphi)}$, Q^h does not depend on φ and t. This implies that $\alpha(e) = -\alpha = const.$ and $\beta(e) = -\beta = const.$.

Finally, putting $g(e) = h(e, 0, 0)$, $e \in E$, proves the theorem. □

Theorem 7.2 *All positive space-time harmonic functions h for the process $(X_t, \varphi_t)_{t\geq 0}$ continuously differentiable in φ and t such that $P^h_{(e,\varphi)}$ is honest and that $(X_t)_{t\geq 0}$ is Markov under $P^h_{(e,\varphi)}$ are of the form*

$$h(e, \varphi, t) = e^{-\alpha t} e^{-\beta\varphi} g(e), \quad (e, \varphi, t) \in E \times \mathbb{R} \times [0, +\infty),$$

where, for fixed $\beta \in \mathbb{R}$, α is the Perron-Frobenius eigenvalue and g is the right Perron-Frobenius eigenvector of the matrix $(Q - \beta V)$.

Moreover, there exists unique $\beta_0 \in \mathbb{R}$ such that

$$\begin{array}{lll} (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ drifts to } +\infty & \text{iff} & \beta < \beta_0 \\ (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ oscillates} & \text{iff} & \beta = \beta_0 \\ (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ drifts to } -\infty & \text{iff} & \beta > \beta_0, \end{array}$$

and β_0 is determined by the equation $\alpha'(\beta_0) = 0$, where $\alpha(\beta)$ is the Perron-Frobenius eigenvalue of $(Q - \beta V)$.

Proof: We again give a sketch of the proof. For the details see Najdanovic (2003).

Let a function h be positive and space-time harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and let the process $(X_t)_{t\geq 0}$ be Markov under $P^h_{(e,\varphi)}$. Then by Lemma 7.2 the function h is of the form

$$h(e, \varphi, t) = e^{-\alpha t} e^{-\beta\varphi} g(e), \qquad (e, \varphi, t) \in E \times \mathbb{R} \times [0, +\infty),$$

for some $\alpha, \beta \in \mathbb{R}$ and some vector g on E.

Since the function h is harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ it satisfies the equation $\mathcal{A}h = 0$ where $\mathcal{A}$ is the generator of the process $(X_t, \varphi_t)_{t\geq 0}$ given by (3.7) in Jacka et al. (2005). Hence, $\mathcal{A}h = (Q + V\frac{d}{d\varphi} + \frac{d}{dt})h = 0$ and $h(e, \varphi, t) = e^{-\alpha t} e^{-\beta\varphi} g(e)$ imply that $(Q - \beta V)g = \alpha g$, that is α is an eigenvalue and g its associated eigenvector of the matrix $(Q - \beta V)$. In addition, By Lemma 3.1 in Jacka et al. (2005) the matrix $(Q - \beta V)$ is irreducible and essentially non-negative. By the Perron-Frobenius theorem the only positive eigenvector of an irreducible and essentially non-negative matrix is its Perron-Frobenius eigenvector. Thus, α and g are Perron-Frobenius eigenvalue and eigenvector, respectively, of the matrix $(Q - \beta V)$.

The equation $\mathcal{A}h = 0$ implies that the process $h(X_t, \varphi_t, t)_{t\geq 0}$ is a local martingale. Since the function $h(e, \varphi, t) = e^{-\alpha t} e^{-\beta\varphi} g(e)$ is bounded on every finite interval, the process $h(X_t, \varphi_t, t)_{t\geq 0}$ is a martingale. It follows that the measure $P^h_{(e,\varphi)}$ is honest.

Let, for fix $\beta \in \mathbb{R}$, $h(e, \varphi, t) = e^{-\alpha(\beta)t} e^{-\beta\varphi} g(\beta)(e)$, where $\alpha(\beta)$ and $g(\beta)$ are Perron-Frobenius eigenvalue and right eigenvector, respectively, of the matrix $(Q - \beta V)$. Let μ_β denote the invariant measure of the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$, and let $g^{left}(\beta)$ denote the left eigenvector of the matrix $(Q - \beta V)$. Then it can be shown that $\mu_\beta V 1 = g^{left}(\beta) V g(\beta)$. Since $g^{left}(\beta)(e) g(\beta)(e) > 0$ for every $e \in E$, Lemma 3.9 and (3.8) in Jacka et al. (2005) imply the statement in the second part of the theorem. □

By Theorem 7.2, there exists exactly one positive space-time harmonic function h for the process $(X_t, \varphi_t)_{t\geq 0}$ with the desired properties and it is given by

$$h_0(e, \varphi, t) = e^{-\alpha_0 t} e^{-\beta_0 \varphi} g_0(e), \quad (e, \varphi, t) \in E \times \mathbb{R} \times [0, +\infty).$$

For fixed $(e, \varphi) \in E_0^+$, let a measure $P^{h_0}_{(e,\varphi)}$ be defined by

$$P^{h_0}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_0(X_t, \varphi_t, t)\Big)}{h_0(e, \varphi, 0)}, \quad A \in \mathcal{F}_t,\ t \geq 0, \tag{7.19}$$

and let $E^{h_0}_{(e,\varphi)}$ denote the expectation operator associated with the measure $P^{h_0}_{(e,\varphi)}$. Then, the process $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is Markov with the Q-matrix Q^0 given by

$$Q^0(e, e') = \frac{g_0(e')}{g_0(e)}(Q - \alpha_0 I - \beta_0 V)(e, e'), \qquad e, e' \in E. \tag{7.20}$$

and, by Theorem 7.2, the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates.

The aim now is to condition $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ on the event that $(\varphi_t)_{t\geq 0}$ stays non-negative. The following theorem determines the law of this new conditioned process.

Theorem 7.3 *For fixed* $(e, \varphi) \in E_0^+$*, let a measure* $P^{h_0, h_r^0}_{(e,\varphi)}$ *be defined by*

$$P^{h_0, h_r^0}_{(e,\varphi)}(A) = \frac{E^{h_0}_{(e,\varphi)}\Big(I(A) h_r^0(X_t, \varphi_t) I\{t < H_0\}\Big)}{h_r^0(e, \varphi)}, \quad A \in \mathcal{F}_t,\ t \geq 0,$$

where the function h_r^0 *is given by* $h_r^0(e, y) = e^{-yV^{-1}Q^0} J_1 \Gamma_2 r^0(e)$, $(e, y) \in E \times \mathbb{R}$, *and* $V^{-1}Q^0 r^0 = 1$*. Then,* $P^{h_0, h_r^0}_{(e,\varphi)}$ *is a probability measure.*

In addition, for $t \geq 0$ *and* $A \in \mathcal{F}_t$,

$$P^{h_0, h_r^0}_{(e,\varphi)}(A) = \lim_{y\to\infty} P^{h_0}_{(e,\varphi)}(A \mid H_y < H_0) = \lim_{T\to\infty} P^{h_0}_{(e,\varphi)}(A \mid H_0 > T),$$

and

$$P^{h_0, h_r^0}_{(e,\varphi)}(A) = P^{h_{r^0}}_{(e,\varphi)}(A),$$

where $P^{h_{r^0}}_{(e,\varphi)}$ *is as defined in Theorem 2.2 in Jacka et al. (2005).*

Proof: By definition (7.19) of the measure $P^{h_0}_{(e,\varphi)}$, for $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\begin{aligned} P^{h_0, h_r^0}_{(e,\varphi)}(A) &= \frac{E_{(e,\varphi)}\Big(I(A)\ h_0(X_t, \varphi_t, t)\ h_r^0(X_t, \varphi_t)\ I\{t < H_0\}\Big)}{h_0(e, \varphi, 0)\ h_r^0(e, \varphi)} \\ &= \frac{E_{(e,\varphi)}\Big(I(A)\ h_{r^0}(X_t, \varphi_t, t)\ I\{t < H_0\}\Big)}{h_{r^0}(e, \varphi, t)}, \end{aligned}$$

where $h_{r^0}(e,\varphi,t) = h_0(e,\varphi,t)\, h_r^0(e,\varphi) = e^{-\alpha_0 t} e^{-\beta_0\varphi}\, G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e)$ is as defined in Theorem 2.2 in Jacka et al. (2005). By Lemma 5.1 (i) in Jacka et al. (2005), the function $h_{r^0}(e,\varphi,t)$ is positive and by Lemma 5.5 in Jacka et al. (2005), the function $h_{r^0}(e,\varphi,t)$ is space-time harmonic for the process $(X_t,\varphi_t,t)_{t\geq 0}$. Thus, $P^{h_0,h_r^0}_{(e,\varphi)}$ is a probability measure, and by the definition of the measure $P^{h_{r^0}}_{(e,\varphi)}$ in Theorem 2.2 in Jacka et al. (2005),

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = P^{h_{r^0}}_{(e,\varphi)}(A), \quad A \in \mathcal{F}_t, \quad t \geq 0.$$

In addition, by (3.8) and Lemma 3.11 in Jacka et al. (2005), the Q-matrix Q^0 of the process $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is conservative and irreducible and the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates. Thus, by Theorem 2.1 and by Theorem 2.1 in Jacka et al. (2005), $P^{h_0,h_r^0}_{(e,\varphi)}$ denotes the law of $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ conditioned on $\{H_0 = +\infty\}$, and for any $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \lim_{y\to\infty} P^{h_0}_{(e,\varphi)}(A|H_y < H_0) = \lim_{T\to\infty} P^{h_0}_{(e,\varphi)}(A|H_0 > T).$$

□

We summarize the results in this section: in the negative drift case, making the h-transform of the process $(X_t,\varphi_t,t)_{t\geq 0}$ with the function $h_0(e,\varphi) = e^{-\alpha_0\varphi} e^{-\beta_0\varphi} g_0(e)$ yields the probability measure $P^{h_0}_{(e,\varphi)}$ such that $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is Markov and that $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates. Then the law of $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ conditioned on the event $\{H_0 = +\infty\}$ is equal to $P^{h_0,h_r^0}_{(e,\varphi)} = P^{h_{r^0}}_{(e,\varphi)}$. On the other hand, by Theorem 2.2 in Jacka et al. (2005), under the condition that all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, $P^{h_{r^0}}_{(e,\varphi)}$ is the limiting law as $T \to +\infty$ of the process $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_{r^0}}_{(e,\varphi)}$ conditioned on $\{H_0 > T\}$. Hence, under the condition that all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, the diagram in Figure 2 commutes.

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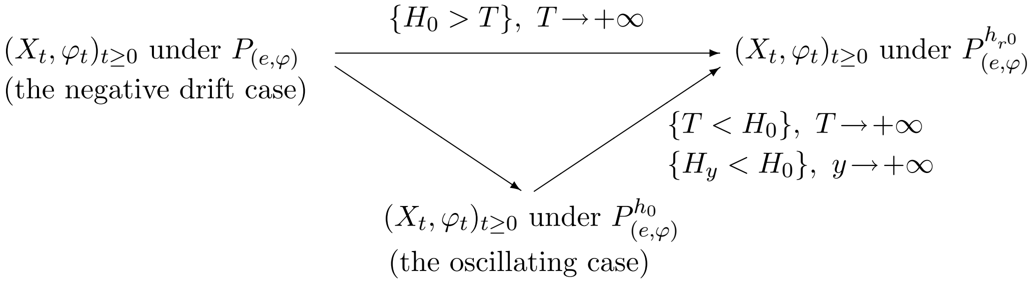

Figure 2: Conditioning of the process $(X_t, \varphi_t)_{t \geq 0}$ on the events $\{H_0 > T\}$, $T \geq 0$, in the negative drift case.

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