Bayesian Statistics, De Finetti and The Gambler's Fallacy

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Contents

The Gambler's Fallacy

- A series of studies in Experimental Psychology that indicate systematic bias in human behavior in situations of uncertainty (Kahneman, Slovic e Tversky, 1982).

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- *The Gambler's Fallacy* - as a person observes a long sequence of ***heads*** in a coin flipping process, he believes ***tails*** becomes more likely on the next flip.

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- It is also known as *the law of maturity of chances*.

The Gambler's Fallacy

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- The Bayesian approach can be a very interesting tool for this problem, helping to understand which would be the coherent behavior given a set of assumptions (such as exchangeability, for example).

The Gambler's Fallacy

- Two studies in the Statistics literature that examine this matter:
  - O'NEILL, P. AND PUZA, B. (2005) In Defence of the Reverse Gambler's Belief. *The Mathematical Scientist*, **30**.
  - RODRIGUES, F.W. AND WECHSLER, S. (1993) A Discrete Bayes Explanation of a Failure-Rate Paradox. *IEEE Transactions on Reliability*, **42(1)**, pp 132-133.

Contents

O'Neill & Puza

- O'Neill & Puza consider a model in which one obsevers a subsequence of discrete random variables ($\mathrm{x}_n = (x_1, x_2, \ldots, x_n)$) which take values in 1,2,...,k (sides of a dice, for example)
- It is assumed that $\mathrm{x} \equiv (x_1, x_2, \ldots)$ is infinitely exchangeable.

O'Neill & Puza

How would be the Gambler's Fallacy in this context?

n_i: frequency of outcomes of the face i in a dice.
$p(i)$: predictive probability of the face i in a dice.

$$n_5 < n_1 < n_2 < n_6 < n_3 < n_4$$
$$p(5) > p(1) > p(2) > p(6) > p(3) > p(4)$$

O'Neill & Puza

Reverse Gambler's Belief

Let n_a and n_b be the frequencies observed of the characteristics a and b, respectively; Given the previous assumptions (x infinitively exchangeable and exchangeable prior distribution for $\mathbf{\Theta}$), for any a, b in $\{1, \ldots, k\}$, we have that:

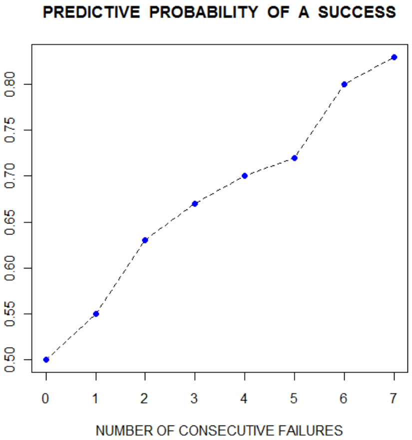
1. se $n_a \geq n_b$ then $p(a|\mathrm{x}_n) \geq p(b|\mathrm{x}_n)$; e
2. se $n_a > n_b$ e $\mathbf{\Theta}$ non-degenerate then $p(a|\mathrm{x}_n) > p(b|\mathrm{x}_n)$.

=> The opposite to the Gambler's Fallacy!

Rodrigues & Wechsler

- The model presented in Rodrigues & Wechsler describes a situation in which one observes a subsequence of 0-1 random variables (failure or success), and supposes the sequence $\mathrm{x} = (x_1, x_2, ...)$ is infinitely exchangeable.
- In this model, the law of maturity is defined as believing that, the bigger the sequence of consecutive failures one observes, the bigger the predictive probability of success will be.

Rodrigues & Wechsler



... success is getting “mature".

Rodrigues & Wechsler

- However, the main result given by Rodrigues & Wechsler is opposite to the law of maturity.
- Thus, the bigger the number of consecutively observed failures, the smaller will be the predictive probability of the next trial being a success. This behavior is named ***belief in reverse maturity***.

Considerations on the Studies

	O'Neill & Puza	*Rodrigues & Wechsler*
Variable Type	$1, 2, \ldots, k$	0 ou 1
Observations	exchangeable	exchangeable
Priori for Θ	exchangeable	any
Result	***reverse gambler's belief***	***belief in reverse maturity***

Considerations on the Studies

- Both studies show that given their assumptions and following a Bayesian approach, the rational behavior should oppose the one found in the *gambler's fallacy*.

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Considerations on the Studies

- Both studies show that given their assumptions and following a Bayesian approach, the rational behavior should oppose the one found in the *gambler's fallacy*.
- Even so, it is not possible to classify the gambler's behavior as irrational. This would depend on his judgment of reasonability of the assumptions presented.
- A new model was proposed based on the alteration of the infinite exchangeability assumption.

Contents

A New Model

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- The assumption of infinite exchangeability of the previous studies can be questionable when regarding average human perception.
- The main idea of this model is to take that what is observed (0 or 1 - failure or success) is part of a exchangeable finite population.

A New Model

Definitions and assumptions:

- Let $\mathbf{x}_N = (X_1, X_2, \ldots, X_N)$ be a vector of unknown values which can be only **0 or 1** ("population")
- Observations: x_1, x_2,. . . , x_n
- Predictive probability of interest: $p(x_{n+1} = x|\mathbf{x}_n)$
- let γ be the count of success in $\mathbf{x}_N$
- We assume $\mathbf{x}_N = (X_1, X_2, \ldots, X_N)$ exchangeable

A New

- Specification of Prior Distributions Studied for the New Model
  1. Binomial Distribution
  2. Binomial Mixture
  3. Distributions Tighter than the Binomial
  4. Distributions Second-order Tighter than the Binomial

The Binomial Distribution

Indifferent Belief

If the prior distribution for γ is a Binomial(N, π), then:

$$p(x_{n+1} = 1|\mathrm{x}_n) = \pi, \text{ for any } \mathrm{x}_n \in \{0,1\}^n \text{ and } n \leq N$$

The model is equivalent to an experiment of withdrawal with replacement of black and white balls from an urn in which each ball of the urn has probability π of being black and is independent of the remaining.

Distributions obtained by Binomial Mixture

Equivalence of Models

Let $Q(\pi)$ be a distribution function on $[0, 1]$; if the prior distribution chosen for γ can be expressed as

$$p(\gamma) = \int_0^1 \binom{N}{\gamma} \cdot \pi^{\gamma} \cdot (1-\pi)^{N-\gamma}\, dQ(\pi),$$

then:

$$p(x_1, x_2, \ldots, x_n) = \int_0^1 \pi^{\Sigma x_i} \cdot (1-\pi)^{n-\Sigma x_i}\, dQ(\pi),\ \forall\, n \leq N$$

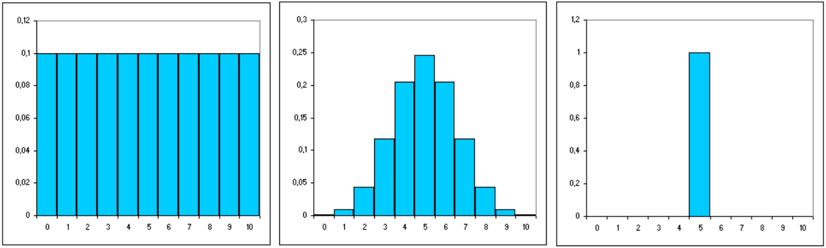
Distributions obtained by Binomial Mixture

- This way, for this class (Binomial Mixture) there is some equivalence of learning with the model in which there is the assumption of infinite exchangeability.

- Thus, we can use the results from previous studies to draw the following conclusions:
  1. If $Q(\pi)$ is not-degenerated $\Rightarrow$ ***belief in reverse maturity***
  2. If $Q(\pi)$ is symmetric on 1/2 $\Rightarrow$ ***reverse gambler's belief***

Distributions Tighter than the Binomial

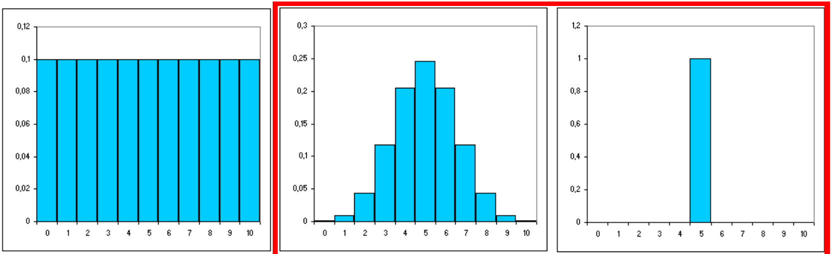
Maturity according to the prior distribution for γ



Reverse Maturity **Indifference** **Maturity**

Distributions Tighter than the Binomial

Maturity according to the prior distribution for γ



Reverse Maturity **Indifference** **Maturity**

Distributions Tighter than the Binomial

- This way, one might think there exists a class of priors which share the property of *maturity* "between" both cases.
- In order to examine this point, we define a class of distributions whose shape is *tighter* than the Binomial.

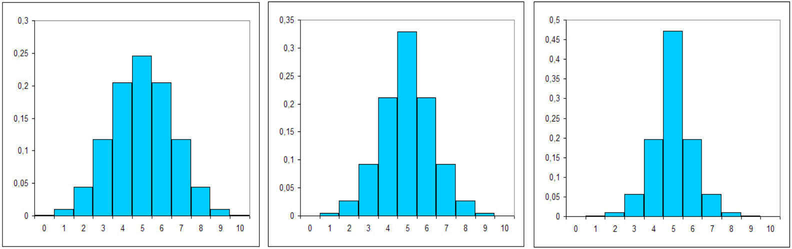
Distributions Tighter than the Binomial

A discrete random variable Y has its probability distribution *tighter than the Binomial(N, 1/2)* if it satisfies the following conditions:

1. Its support is the same as that of Binomial(N,1/2);
2. It is symmetric on $N/2$;
3. $$\frac{P(Y=y)}{P(Y=y-1)} > \frac{N-y+1}{y}, \quad y = 1, \ldots, \lfloor N/2 \rfloor$$

Distributions Tighter than the Binomial

Examples of Distributions Tighter than the Binomial$(10, 1/2)$



1 - Binomial, **2** - Tighter (I) and **3** - Tighter (II)

Distributions Tighter than the Binomial

Gambler's Belief

If one chooses a prior distribution tighter than the Binomial(N, 1/2) for γ, then:

$$\text{if } n_1 < n_0 \text{ then } p_{n+1}(1|\mathrm{x}_n) > p_{n+1}(0|\mathrm{x}_n)$$

In words:

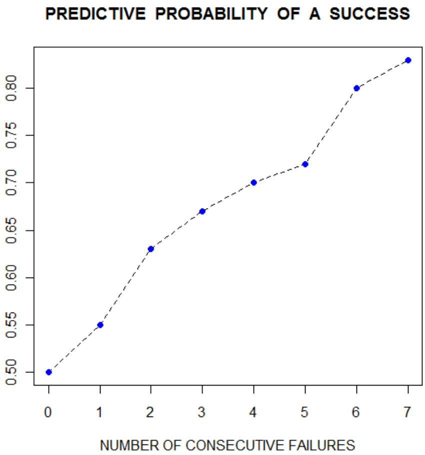
If the number observed of **failures** is less than of **successes**

=> p(failure) > p(success)

Distributions Second-order Tighter than the Binomial

- Additionally, we define a subclass within the distributions tighter than the Binomial so that ***belief in maturity*** is valid.
- ***Belief in maturity***: the bigger the number of failures consecutively observed, the more one believes in the occurrence of a success in the next trial. This leads to the idea that the success is getting “mature".

Distributions Second-order Tighter than the Binomial



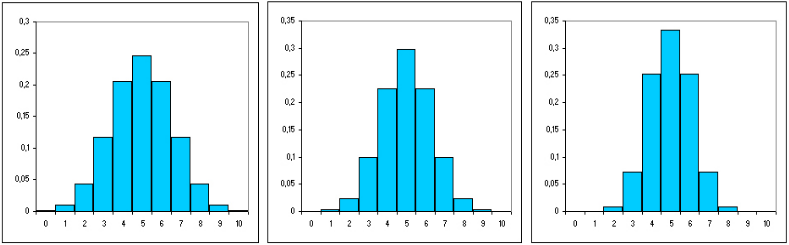
... success is getting “mature".

Distributions Second-order Tighter than the Binomial

A discrete random variable Y has its probability distribution *second-order tighter than the Binomial(N, 1/2)* if it satisfies the following conditions:

1. Its distribution is tighter than the Binomial(N,1/2);
2. $$\frac{P(Y = y+1)/P(Y = y)}{P(Y = y)/P(Y = y-1)} < \frac{(N-y)/(y+1)}{(N-y+1)/y} \; ,$$
   $y = 1, \ldots, \lfloor N/2 \rfloor - 1$

Graph of the Binomial$(10, 1/2)$ *and two examples which are second-order tighter than it.*



1 - Binomial, **2** - Second-order Tighter (I) and **3** - Second-order Tighter (II)

Distributions Second-order Tighter than the Binomial

Belief in Maturity

If the prior distribution chosen for γ ($p(\gamma)$) is second-order tighter than the Binomial(N, 1/2), then:

$$r(m) = p(x_m = 1 | x_1 = 0, \ldots, x_{m-1} = 0) \textit{ is } \textbf{\textit{increasing}} \textit{ in } m.$$

Contents

Review of the major points of study

- Specifications of Prior Distributions in **this work**

  1. Binomial Distribution
     $\Rightarrow$ *Indifference*
  2. Binomial Mixture
     $\Rightarrow$ *Equivalence of Models*
  3. Distributions Tighter than the Binomial
     $\Rightarrow$ ***Gambler's Belief***
  4. Distributions Second-order Tighter than the Binomial
     $\Rightarrow$ ***Belief in Maturity***

Final Considerations

- As was shown, previous studies proved that *reverse gambler's belief* and *reverse maturity* are a logical consequence of the assumption of infinite exchangeability.

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- As was shown, previous studies proved that *reverse gambler's belief* and *reverse maturity* are a logical consequence of the assumption of infinite exchangeability.
- In this work we presented a new model in which a finite exchangeable population is considered and, therefore, the main assumption of the previous studies is not verified.

Final Considerations

- The new model proposed is compatible not only with the behavior prescribed by the *gambler's fallacy* but also with the reverse.

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- Because of this, when a person believes in the adopted premises, it is not necessarily unreasonable to behave according to *maturity*.

Final Considerations

- The new model proposed is compatible not only with the behavior prescribed by the *gambler's fallacy* but also with the reverse.
- Because of this, when a person believes in the adopted premises, it is not necessarily unreasonable to behave according to *maturity*.
- These results may contribute on the judgment of how reasonable the assumption of infinite exchangeability is relative to typical human perception.

Thank you for the attention!

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