Forecasting with Imprecise Probabilities [*IP*] – *some preliminary findings*

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Organization of this presentation

Part 1 **Background on de Finetti's twin criteria of *coherence*:**

***Coherence*$_1$: 2-sided previsions free from dominance through a *Book*.**

***Coherence*$_2$: Forecasts free from dominance under *Brier* (squared error) *score*.**

- **Two modest differences between these as coherence criteria.**

Part 2 ***IP* theory based on a scoring rule.**

(2.1) Coherence and elicitation of *IP* sets using a proper scoring rule

and the Bayes-choice rule of *E-admissibility*.

(2.2) An *IP* Brier-scoring rule allowing different IP-models of *coherence*

ε-contamination coherence

***Atomic Lower-Upper Probabilities* [ALUP] *coherence*.**

Part 1: Background on de Finetti's twin *coherence* criteria

De Finetti's theory [1] of *2-sided* previsions is the basis for *IP* generalizations. De Finetti's 2-person, zero-sum prevision-game has these ingredients:

a class of bounded variables $\mathcal{X} = \{X_i : i \in I\}$ (I is an index set) measurable with respect to some common space $\{\Omega, \mathcal{B}\}$.

One player, the *bookie*, posts a *fair*, or *2-sided* prevision $P[X_i]$ for each $X_i \in \mathcal{X}$.

The bookie's opponent, the *gambler*, may choose *finitely many* non-zero real numbers $\{\alpha_i\}$ where, when the state $\omega \in \Omega$ obtains,

the *bookie*'s payoff is $\Sigma_i\, \alpha_i(X_i(\omega) - P[X_i])$,

and the *gambler*'s payoff is the opposite, $-\Sigma_i\, \alpha_i(X_i(\omega) - P[X_i])$.

- **That is, the *bookie* is obliged either to buy (if $\alpha > 0$), or to sell (if $\alpha < 0$) $|\alpha|$-many units of X at the price, $P(X)$.**

Hence, the previsions are described as being *2-sided* or *fair* buy/sell prices.

The bookie's previsions are <u>*incoherent*$_1$</u> if the *gambler* has a strategy that insures a uniformly negative payoff for the *bookie*, i.e.,

if there exist a <u>finite set</u> $\{\alpha_i\}$ and $\varepsilon > 0$ such that, for each $\omega \in \Omega$,

$$\Sigma_i \alpha_i(X_i(\omega) - P[X_i]) < -\varepsilon.$$

Otherwise, the bookie's previsions are <u>*coherent*$_1$</u>.

- **De Finetti's *Fundamental Theorem of Previsions*:**

The *bookie*'s previsions $\{P(X): X \in \mathcal{X}\}$ are coherent$_1$ *iff*

There is a finitely additive probability P whose expected value for X, $E_P[X]$, is the *bookie*'s prevision:

$$\text{Coherence}_1 \quad \textit{if and only if} \quad E_P[X] = P[X].$$

This result extends to include *coherence*$_1$ for conditional expectations using the device of called-off previsions.

The *bookie'*s called-off prevision, $P_F[X]$, for X given event F has payoff to the bookie:

$$F\alpha(X(\omega) - P_F[X])$$

Then the payoff is 0 – the transaction is called-off – in case event F fails.

***Coherence*$_1$ then requires that**

$$\mathbf{E}_P[X \mid F] = P_F[X].$$

***NB*: This is a *static* account of conditional probability. There is no active learning, nor dynamic updating involved in this sense of *coherence*$_1$ for conditional probabilities.**

De Finetti [2] noted that *strategic* aspects of betting may affect <u>elicitation</u> of a *bookie*'s *fair* previsions.

- **When the *bookie* (believes he/she) knows the *gambler*'s betting odds, then <u>announcing</u> a prevision is a strategic play.**

***<u>Example</u>*$_1$: Suppose the *bookie*'s *fair* (2-sided) prevision for an event G is .50.**

The bookie is confident the *gambler*'s fair prevision for G is .75.

So the bookie <u>announces</u> $P[G] = .70$, anticipating that the *gambler* will find it profitable to buy units of G at the inflated price.

The *elictation* fails to identify the bookie's fair price for bets on/against G.

- **There are other strategic aspects in the prevision game!**

To mitigate such strategic aspects of the prevision-game, de Finetti used *probabilistic forecasting subject to Brier score* to formulate: *coherence*$_2$.

Focus on probabilistic forecasting of events. Each $X \in \mathcal{X}$ is an indicator function. The bookie's previsions serve as probabilistic forecasts subject to Brier score: squared-error loss. The penalty for the forecast $P[X]$ when the state $\omega \in \Omega$ obtains is given by two functions $\{g_1, g_0\}$

$$(X(\omega) - \mathrm{P}[X])^2 = \begin{cases} g_1(P[X]) = (1 - P[X])^2 & \text{if event } X \text{ obtains} \\ g_0(P[X]) = (0 - P[X])^2 & \text{if event } X^c \text{ obtains} \end{cases}$$

For the conditional (called-off) forecast $P_F[X]$, the score is

$$F(\omega)(X(\omega) - P[X])^2.$$

The score for a finite set of forecasts is the sum of the separate scores.

- **A forecast set $\{P[X]: X \in \mathcal{X}\}$ is <u>*coherent*</u>$_2$ if, for each finite subset of $\mathcal{X}$, no rival forecast set $\{P'[X]: X \in \mathcal{X}\}$ uniformly dominates (in Ω) the score from the corresponding *P*-forecasts.**

By a Euclidean projection, de Finetti showed that the two senses of coherence are equivalent.

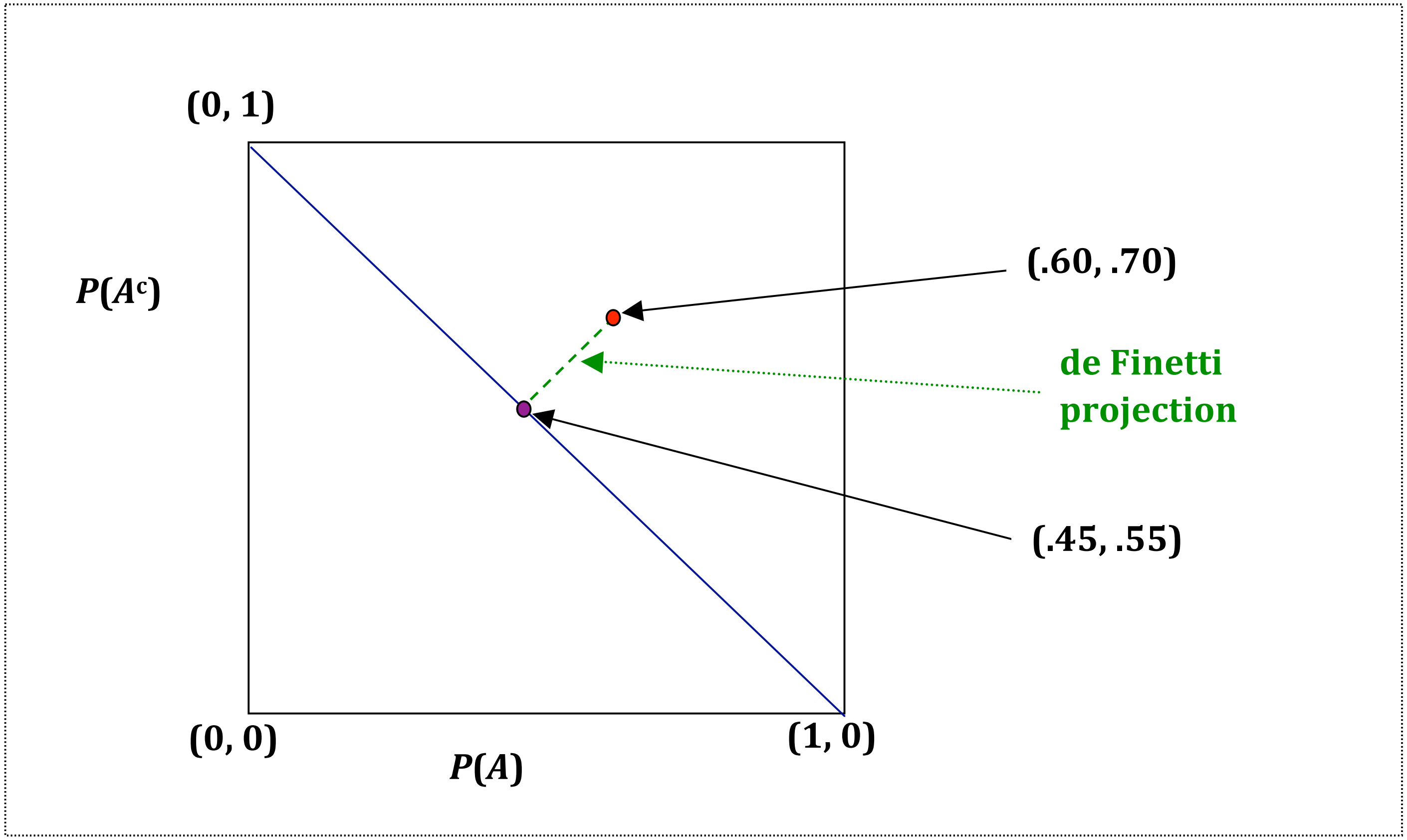
- **A set of previsions is *coherent*$_1$ in the prevision-game *if and only if* those same previsions are a *coherent*$_2$ set of forecasts under Brier score.**

***Coherence*$_2$ does *not* involve strategic forecasting as there is no opponent, in contrast with the 2-person previsions-game.**

Elicitation is straightforward with Brier score.

***Example*$_2$: De Finetti's projection for establishing that *coherence*$_1$ = *coherence*$_2$.**

The incoherent$_1$ previsions: $P(A) = .6$ and $P(A^c) = .7$



If the forecast previsions are not *coherent*$_1$, they lie outside the probability simplex. Project these *incoherent*$_1$ forecasts into the simplex. The resulting *coherent*$_1$ forecasts have dominating Brier score, showing that the initial forecasts are *incoherent*$_2$. No *coherent*$_1$ forecast set can be so dominated.

Two results relating to coherence with proper scoring rules.

Brier-score offers a *coherence*$_2$ criterion that is modestly more friendly to merely finitely additive probability than is *coherence*$_1$ from the prevision-game.

***Example*$_3$: Consider a countably infinite state space $\Omega = \{\omega_1, \omega_2, ...\}$ with its powerset serving as the σ-field of sets. Let $\mathcal{X} = \{W_i: i = 1, ...\}$ with W_i is the indicator for event $\{\omega_i\}$.**

Consider a (purely) finitely additive probability: $P(\omega_i) = 0, i = 1, ...$.

These previsions are *coherent*$_1$. But to secure *coherence*$_1$ only finitely many may be used.

On the contrary, as de Finetti noted, were countably many previsions used simultaneously, then for each $\omega \in \Omega$, choosing $\alpha_i = -1$, yields the constant payoff:

$$\Sigma_i -(W_i(\omega) - P[W_i]) = \Sigma_i -W_i(\omega) = -1$$ **which is a sure loss of -1 to the *bookie*.**

However, the infinite set of forecasts $\{P[Wi] = 0\}$ has a constant Brier score

$$-1 = \Sigma_i -(W_i(\omega) - P[W_i])^2$$ **which is *not* dominated by any set of forecasts, $\{P'[W_i]\}$, regardless whether or not these rival forecasts are *coherent*$_{1\text{-or-}2}$.**

This example generalizes to the following result.

Proposition[1]

- **Say that a set of variables $\{X_i: i = 1, \ldots, \}$ are *series-bounded,***

  if there exists a real number *b*, so that for each $\omega \in \Omega$, $\Sigma_i |X_i(\omega)| < b$

When series-boundedness obtains, it is *not* necessary to restrict Brier score to sums of *finitely* many unconditional forecasts in order to preserve the equivalence between the two coherence criteria.

***Note*: *Series-boundedness* is satisfied when forecasting events over a partition.**

However, the same result does *not* obtains when called-off forecasts are used. Then, because of non-conglomerablity of merely finitely additive probabilities, the equivalence between the two coherence criteria depends upon the clause that only finitely many Brier scores are summed up simultaneously.

Brier score is just one of an infinite class of (strictly) proper scoring rules, where a coherent$_1$-forecaster minimizes her/his expected score (uniquely) by forecasting her/his previsions.

- **A scoring rule for forecasting an event *A* given event *B*, the called-off score for the forecast P(*A*|*B*), is defined by two extended real-valued loss functions $\{g_0, g_1\}$, with arguments from [0,1], as follows.**

if *A* occurs, the loss is $Bg_1(\,P(A|B)\,)$

and if A^c occurs, the loss is $Bg_0(\,P(A|B)\,)$.

Again, even allowing different scoring rules for different forecasts, the combined score for a finite set of forecasts is the sum of the individual scores.

Savage [5], Schervish [6] characterize the (g_0, g_1) pairs for proper scoring rules.

Proposition$_2$ [7]:

- **When the scoring rule is proper, finite, and continuous, each incoherent$_1$ forecast set is dominated by some coherent$_1$ forecast set. (See also [4].)**
- **When the scoring rule is proper, finite, but not continuous, each incoherent$_1$ forecast set is dominated, but not necessarily by a coherent$_1$ forecast set. See Example$_4$, below.**

Here is an illustration where, in order to dominate an incoherent forecast, one must use a different *incoherent* forecast. No coherent forecast dominates. This example uses a strictly proper, discontinuous scoring rule based on Brier score.

***Example*$_4$:**

Consider a binary partition $\{A, A^c\}$ and a common scoring rule.

$$g_0(x) = \begin{cases} x^2 & \text{if } x \leq 1/2 \\ x^2 + 1/2 & \text{if } x > ½ \end{cases}$$

$$g_1(x) = \begin{cases} (1-x)^2 + 1/2 & \text{if } x \leq 1/2 \\ (1-x)^2 & \text{if } x > ½ \end{cases}$$

- **The incoherent forecast pair from *Example*$_2$,**

$$\mathbf{F_1} = \{P(A) = 0.6,\ P(A^c) = 0.7\}$$

has a score 1.15 if A obtains, and a score 0.95 if A^c obtains.

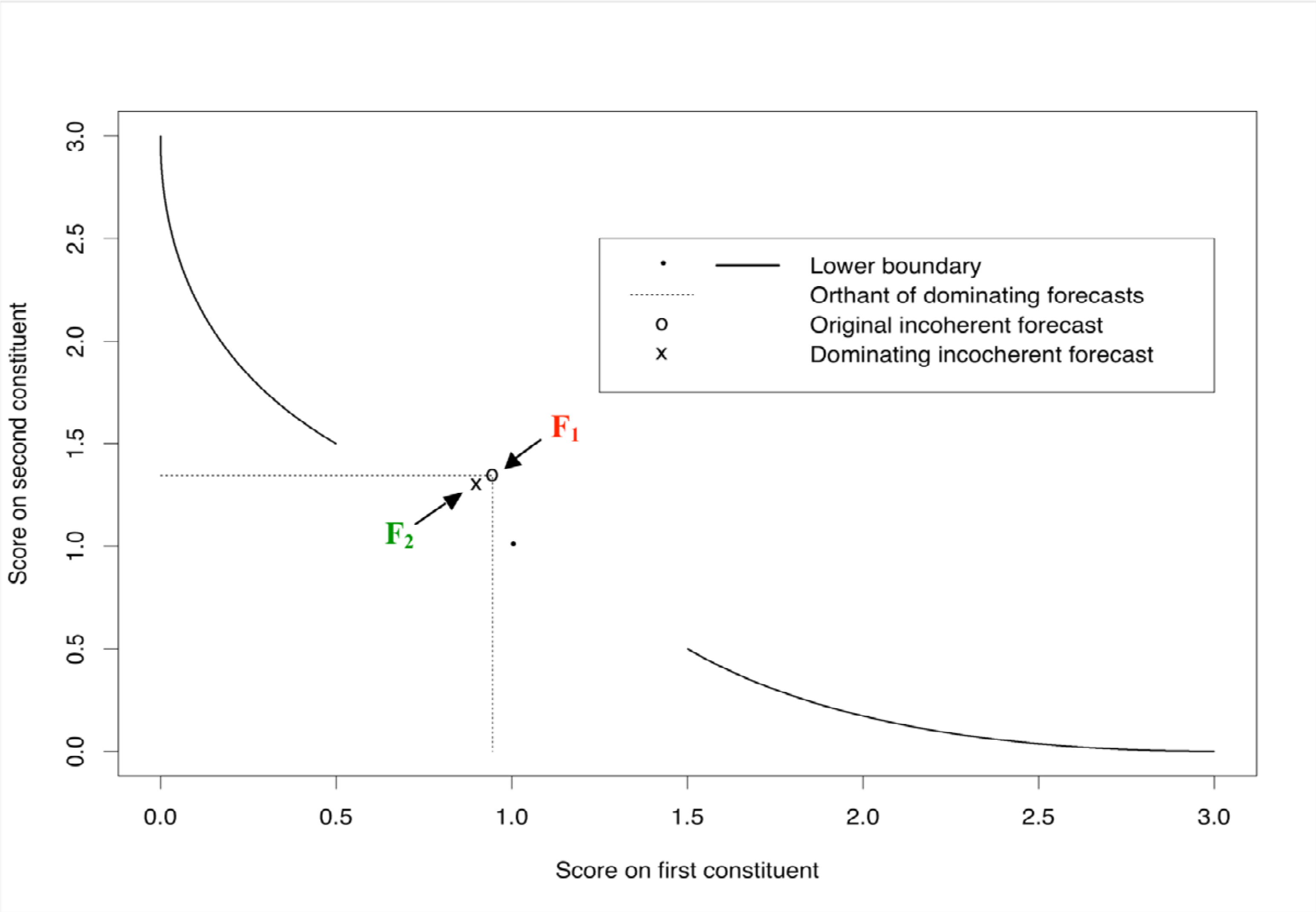
- **F_1 is *not* dominated by any coherent forecasts – see the graph on the next slide!**

- **But F_1 is dominated by the incoherent forecast pair**

$$\mathbf{F_2} = \{P'(A) = .55,\ P'(A^c) = .65\},$$

which has scores 1.125, and 0.925, respectively.

Example4: An incoherent forecast undominated by any coherent forecast



Part 2a – Coherence and elicitation of *IP* sets using a proper scoring rule based on the choice rule of *E-admissibility*

For general background on IP theory, please visit www.sipta.org

Let $\mathcal{P}$ be a set of probabilities over states.

Let *U* be a determinate cardinal utility over outcomes in decision problem.

Let $\mathcal{O}$ be a (closed) set of feasible options – a *choice set*.

***E-admissibility* (See Levi, [3]):**

An option $o \in \mathcal{O}$ is *E-admissible* provided that

***o* maximizes *P*-expected *U*-utililty for some $P \in \mathcal{P}$.**

See [9] for an axiomatization of this decision rule.

Heuristic Model – *Cooperative Bayesian Group Decision-making*.

Suppose that a *team* of Bayesian decision makers, with a common utility *U*, have different personal degrees of belief as represented by the set $\mathcal{P}$ of coherent probability distributions.

- **The *E-admissible* choices in a decision problem are precisely those that are *Bayes* for at least one of the team's members.**

***Proposition*$_3$:**

Given any set of (strictly) proper scoring rules, one for each forecast event, and given any set $\mathcal{P}$ of probabilities over states, the *E-admissible* forecasts are precisely those that elicit the set $\mathcal{P}$ of probabilities over these events.

Part 2b: a Brier-styled, *IP* scoring rule for varying *IP*-models.

To create an *IP*-scoring rule, parallel the familiar modification of de Finetti's game that permits the *bookie* to fix a *pair* of 1-sided previsions for each $X \in \mathcal{X}$:

The *bookie* announces one rate $\underline{P}(X)$ as a buying price,

and a possibly different selling price $\overline{P}(X)$.

The result is a generalized *Book* argument. (See [8] for some history.)

Then, a *bookie*'s 1-sided previsions are *coherent*$_1$ if and only if there is a maximal, non-empty (convex) set of finitely additive probabilities $\mathcal{P}$ where

$$\underline{P}(X) = \textit{infemum}_{P \in \mathcal{P}} \mathrm{E}_P[X]$$

and

$$\overline{P}(X) = \textit{supremum}_{P \in \mathcal{P}} \mathrm{E}_P[X]$$

Analogously:

use a *lower forecast* to assess a penalty score when the event *<u>fails</u>*,

use an *upper forecast* to assess a penalty score when the event *<u>obtains</u>*.

Let $\{E_i: i = 1, \ldots, m\}$ be m events over a finite partition $\Omega = \{\omega_j: j = 1, \ldots, n\}$.

***Forecaster* gives *lower* and *upper* probability forecasts $\{p_i, q_i\}$ for each event E_i .**

***<u>Scoring forecasts with a Brier-styled IP scoring rule</u>*: Fix a state $\omega \in \Omega$.**

- **If $\omega \in E_i$ the score for the forecast of E_i is $(1-q_i)^2 = g_1(q_i)$**
- **If $\omega \notin E_i$ the score for the forecast of E_i is $p_i^2 = g_0(p_i)$**

The score for the set of forecasts is the sum of the individual scores.

Dominance: **One forecast set (*strictly*) *dominates* another provided that, for each $\omega \in \Omega$,**

the score for the former set is (strictly) less than the score for the latter set.

But, the vacuous $\{0 = p_i, q_i = 1\}$ forecast dominates each rival $\{0 < p_i', q_i' < 1\}$.

Aside: This is analogous to a problem that is usually ignored within traditional *IP* theory.

With 1-sided previsions, it remains coherent to be strategic: announce a lower buying (and/or a higher selling) price than one is prepared to accept.

Therefore, *IP-coherence* needs to take into account *an index of imprecision* (the extent of indeterminacy) or degree of informativeness in a rival but dominating set of forecasts.

- **We illustrate one such index in the following examples.**

Let *M* be an *IP-model*, that is, a class of sets of probabilities satisfying a characteristic property that defines *M*.

***Illustrations of three IP-models*:**

- ***M* may be an ε*-contamination* class, which can be defined as an *IP-model* by specifying (*coherent*$_1$) lower probabilities for atomic events, and using the largest (closed) convex set of distributions satisfying these bounds.**
- ***M* may be an *Atomic Lower-Upper Probability* [ALUP] class. Give (*coherent*$_1$) lower and upper probabilities for atomic events, and use the largest (closed) convex set of distributions that satisfy these bounds.**
- ***M* may be an *Lower-Upper Probability* [LUP] class Give (*coherent*$_1$) lower and upper probabilities for events, and use the largest (closed) convex set of distributions that satisfy these bounds.**

***Definition*:**

Given an *IP*-scoring rule, *a* set F of *IP*-forecasts is *incoherent with respect to the IP-model* M provided that there is a <u>dominating set</u> of rival forecasts F' from the model M, and where the set F' is <u>at least as informative as</u> the set F.

In what follows we illustrate one index of *informativeness* associated with our Brier-styled *IP*-scoring rule.

***IP-forecasts over a finite partition for Brier-styled, ε-contamination coherence*:**

Let $F_1 = \{ \{p_i, q_i\}: i = 1, \ldots, n\}$ be forecasts for each $\omega_i \in \Omega = \{\omega_1, \ldots, \omega_n\}$.

Define F_1's *score set* as

$$S_1 = \{(q_1, p_2, \ldots, p_n), \{(p_1, q_2, \ldots, p_n), \ldots, \{(p_1, p_2, \ldots, q_n)\}.$$

- **The F_1 *IP-Brier-style*d score, evaluated at state ω_j, is the square of the Euclidean distance from the j^{th} point of S_1 to the j^{th} corner of the probability simplex on Ω.**

Clearly, the *IP*-score for a forecast set can be improved merely by moving a lower forecast closer to 0, or by moving an upper forecast closer to 1.

So, consider dominating forecast sets only when the dominator has a score set that is *at least as informative as* the dominated set.

Relative informativeness **of forecast sets is a utility concept.**

Here is one index which, when combined with our Brier-style *IP* score, allows a characterization of *ε-contamination IP*-coherence.

- **Say that forecast set F_1 is *at least as informative as* forecast set F_2 if the convex hull of score set S_1 is isomorphic to a subset of the convex hull of score set S_2.**

This index of *relative imprecision*, or *relative informativeness*, is a partial order.

***Proposition*$_4$: Assume $0 \le p_i \le q_i \le 1$, with forecasts *for atoms* in a finite partition.**

(*4.1*) A score set S lies entirely within the probability simplex on Ω

if and only if

the lower and upper forecasts F match an ε*-contamination* model.

(*4.2*) If all the elements of a score set S', associated with forecast set F', lie below or above the probability simplex on Ω, there is a dominating ε-contamination forecast model F that is at least as informative as F'.

(*4.3*) If only some (but not all) elements of score set S' lie in the simplex on Ω, this ε-contamination model F may have scores that only weakly dominate S'.

***Aside*: (*4.1*) is established by an elementary calculation;**

(*4.2*) by the *Brouwer Fixed-Point* Theorem: the de Finetti projections are continuous;

and (*4.3*) is a simple observation based on the strict propriety of Brier score.

Example$_5$: Illustrating ε-*contamination* coherence, *Prop*$_4$, with 5 forecast sets.

$\Omega = \{\omega_1, \omega_2, \omega_3\}$. Forecasts are for the atoms only!

Forecast sets F_j (j = 1, …,5) are of the form $\{ \{p_i, q_i\}$ for ω_i: $i = 1, 2, 3\}$.

Score sets are points with coordinates as described above.

$F_1 = \{ \{.55, .80\}, \{.55, .80\}, \{.55, .80\}\}$ $S_1 = \{(.80, .55, .55), (.55, .80, .55), (.55, .55, .80)\}$

$F_2 = \{ \{.25, .50\}, \{.25, .50\}, \{.25, .50\}\}$ $S_2 = \{(.50, .25, .25), (.25, .50, .25), (.25, .25, .50)\}$

$F_3 = \{ \{.20, .45\}, \{.20, .45\}, \{.20, .45\}\}$ $S_3 = \{(.45, .20, .20), (.20, .45, .20), (.20, .20, .45)\}$

$F_4 = \{ \{.10, .35\}, \{.10, .35\}, \{.10, .35\}\}$ $S_4 = \{(.35, .10, .10), (.10, .35, .10), (.10, .10, .35)\}$

$F_5 = \{ \{.05, .30\}, \{.05, .30\}, \{.05, .30\}\}$ $S_5 = \{(.30, .05, .05), (.05, .30, .05), (.05, .05, .30)\}$

Aside: See the *Appendix* for a more complicated illustration of Proposition$_4$: Example$_6$.

- **Forecast sets F_1 and F_5 are incoherent[1]. Their 1-sided previsions lead to sure losses as, respectively, the lower (upper) forecasts are too great (too small). There does not exist a probability distribution agreeing with either set of their lower and upper forecasts.**

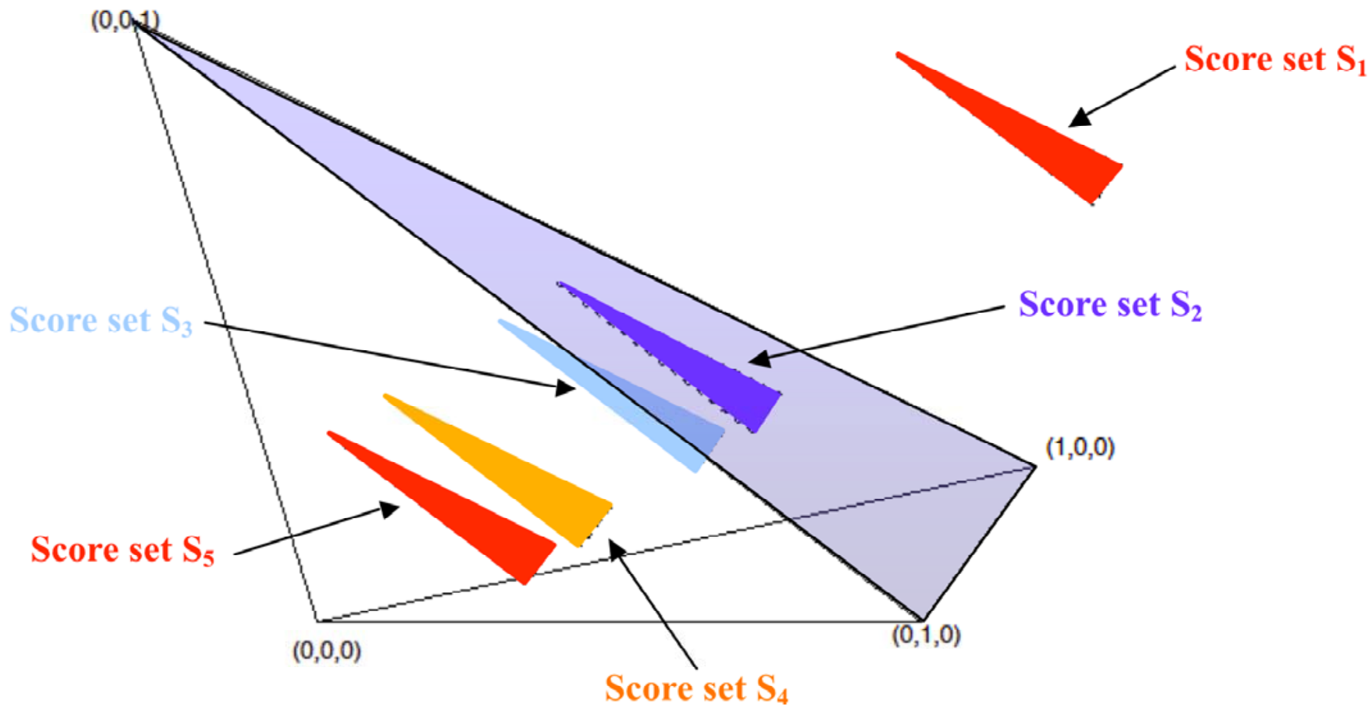
- **Forecast set F_2 corresponds to an ε-contamination model.**

  Score set S_2 lies in the probability simplex, as per Proposition4.

- **Forecast set F_3 has lower and upper forecasts agreeing with a closed convex set of probabilities. Those values agree with an ALUP model, but not with an ε-*contamination* model.**

- **Forecast set F_4 has lower and upper forecasts that do not agree with any closed convex set of probabilities. Its intervals are too wide. However, there exists a probability agreeing with these forecasts, e.g., (1/3, 1/3, 1/3)**

Example$_5$: Score set S_2 dominates the other four score sets and is as informative as each of them



Example$_5$ continued**: Illustrating ALUP-*****coherence***** with 3 forecast sets.**

$\Omega = \{\omega_1, \omega_2, \omega_3\}$. Forecasts are for atoms *<u>and their complements</u>*. That is, each forecast set is for 6 events. Forecasts F_j (j = 2, 3, 4) are given as 6 pairs: $\{p_i, q_i\}$ for ω_i, ω_i^c i = 1, 2, 3. Score sets are comprised by 3 points, corresponding to the 3 states. Each score set point has 6 coordinates, for forecasts of $(\omega_1, \omega_1^c, \omega_2, \omega_2^c, \omega_3, \omega_3^c)$.

ω_1	ω_1^c	ω_2	ω_2^c	ω_3	ω_3^c

F_2 = { {.25, .50}, {.50, .75}, {.25, .50}, {.50, .75}, {.25, .50}, {.50, .75} }

S_2 = {(.50, .50, .25, .75, .25, .75), (.25, .75, .50, .50, .25, .75), (.25, .75, .25, .75, .50, .50)}

F_3 = { {.20, .45}, {.55, .80}, {.20, .45}, {.55, .80}, {.20, .45}, {.55, .80} }

S_3 = {(.45, .55, .20, .80, .20, .80), (.20, .80, .45, .55, .20, .80), (.20, ,80, .20, .80 .45, .55)}

F_4 = { {.10, .35}, {.65, .90}, {.10, .35}, {.65, .90}, {.10, .35}, {.65, .90} }

S_4 = {(.35, .65, .10, .90, .10, .90), (.10, .90 .35, .65, .10, .90), (.10, .90, .10, .90, .35, .65)}

- **Forecast sets F_2 and F_3 are ALUP-*coherent*.**

  There do not exist "more informative" forecast sets from the ALUP-model that dominate either of these sets of forecasts.

  ***Aside*: Their score sets lie in the probability simplex for these 6 events.**

- **Forecast set F_4 is ALUP-incoherent. A de Finetti projection produces a more informative rival ALUP forecast with dominating *IP Brier-style* score.**

  In fact, a more informative ε*-contamination* model dominates.

  The respective *IP Brier-style* scores for F_4 and for F_2 are independent of ω:

  For F_4 a constant penalty of 0.885.

  For F_2 a constant penalty of 0.750.

Summary

When coherence$_1$ of previsions is not enough, and elicitation also is in focus, then

- **the use of a (strictly) proper scoring rule produces an equivalent criterion:**

  ***coherence$_2$* – avoid dominated forecasts**

- **and the scoring rule mitigates many of the strategic aspects in elicitation associated with previsions and betting.**

There are at least two ways to use proper scoring rules to underwrite IP theory.

- **A set of *coherent* forecasts under a proper scoring rule conforms to the use of (Levi's) extended-Bayes rule of *E-admissibility*, applied with a set of personal probabilities.**

  Recall the Heuristic Example of a *team of experts*.

A second way to use proper scoring rules to underwrite IP theory is analogous to the use of 1-sided (*lower* and *upper*), rather than 2-sided (*fair*) previsions.

- **Subject to a proper scoring rule for forecasting events, the forecaster gives lower and upper probabilistic forecasts for a particular set of events that characterize elements of an *IP*-model class *M* – e.g., ε*-contamination*, ALUP.**

- ***Coherence* of the set of *IP*-forecasts requires that these lower and upper forecasts are not dominated by any "*more informative*" *IP* model within the model class *M*, subject to the same *IP* scoring rule.**

We have illustrated this program using an IP Brier-style scoring rule and a simple geometric account of a "more informative" forecast set.

There remains much work to do in this research program:

- **What decision theory makes the *IP* scoring rule *proper*?**
- **Can we extend the Brier-style scoring rule approach to handle IP forecasting of (bounded) random variables?**
- **Can we extend our approach to work with discontinuous scoring rules?**
- **What are other indices of relative "*informativeness*" of a forecast score set – we use the partial order relation "is a subset under translation"?**
  How do other notions relate to rival explications of coherence?
- **What other differences between *IP-coherence*$_1$ and *IP-coherence*$_2$ relate to forecasts using (merely) finitely additive expectations?**
-
- **….**
- **…..**

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Appendix: Example_6 – A more complicated illustration of Proposition_4.

$\Omega = \{\omega_1, \omega_2, \omega_3\}$. Forecast sets F_j are of the form $\{ \{p_i, q_i\}$ for ω_i: $i = 1, 2, 3\}$.

(Step 1) $F_1 = \{ \{.25, .60\}, \{.20, .50\}, \{.10, .40\} \}$ $S_1 = \{(.60, .20, .10), (.25, .50, .10), (.25, .20, .40)\}$

Project score set S_1 to form set $P_1 = \{ (.6\overline{3}, .2\overline{3}, .1\overline{3}), (.30, .55, .15), (.30, .25, .45)\}$

(Step 2) Form new forecast and score sets F_2, S_2 based on the probabilities in set P_1

$F_2 = \{ \{.30, .6\overline{3}\}, \{.2\overline{3}, .55\}, \{.1\overline{3}, .45\} \}$ $S_2 = \{(.6\overline{3}, .2\overline{3}, .1\overline{3}), (.30, .55, .1\overline{3}), (.30, .2\overline{3}, .45)\}$

Project set S_2 to form set $P_2 = \{ (.63\overline{3}, .23\overline{3}, .13\overline{3}), (.30\overline{5}, .55\overline{5}, .15\overline{5}), (.30\overline{5}, .25\overline{5}, .45\overline{5})\}$

(Step 3) Form new forecast and score sets F_3, S_3 based on the probabilities in set P_2

$F_3 = \{ \{.30\overline{5}, .63\overline{3}\}, \{.23\overline{3}, .55\overline{5}\}, \{.13\overline{3}, .45\overline{5}\} \}$

$S_3 = \{(.63\overline{3}, .23\overline{3}, .13\overline{3}), (.30\overline{5}, .55\overline{5}, .13\overline{3}), (.30\overline{5}, .23\overline{3}, .45\overline{5})\}$

Project S_3 to form set $P_3 = \{ (.63\overline{3}, .23\overline{3}, .13\overline{3}), (.30\overline{740}, .55\overline{740}, .13\overline{740}), (.30\overline{740}, .23\overline{740}, .45\overline{740})\}$

(Step 4) Form new forecast and score sets F_4, S_4 based on the probabilities in set P_3

$F_4 = \{ \{.30\overline{740}, .63\overline{3}\}, \{.23\overline{3}, .55\overline{740}\}, \{.13\overline{3}, .45\overline{740}\} \}$

$S_4 = \{(.63\overline{3}, .23\overline{3}, .13\overline{3}), (.30\overline{740}, .55\overline{740}, .13\overline{3}), (.30\overline{740}, .23\overline{3}, .45\overline{740})\}$

Project S_4 to form set $P_4 \approx \{ (.63\overline{3}, .23\overline{3}, .13\overline{3}), (.308, .558, .134), (.308, .234, .458)\}$

(Step 5) Form new forecast and score sets F_5, S_5 based on the probabilities in set P_4

$F_5 = \{ \{.308, .63\overline{3}\}, \{.23\overline{3}, .558\}, \{.13\overline{3}, .458\} \}$

$S_5 = \{(.63\overline{3}, .23\overline{3}, .13\overline{3}), (.308, .558, .13\overline{3}), (.308, .23\overline{3}, .458)\}$

….. Converging to forecast set $F^* = \{ \{.308\overline{6}, .63\overline{3}\}, \{.23\overline{3}, .558\}, \{.13\overline{3}, .458\} \}$

and score set $S^* = \{(.63\overline{3}, .23\overline{3}, .13\overline{3}), (.308\overline{6}, .558, .13\overline{3}), (.308\overline{6}, .23\overline{3}, .458)\}$

F* is an ε-contamination model whose IP-Brier scores dominates F_1's scores.

F* has greater informativeness than forecast F_1, i.e., S* is isomorphic to a proper subset of S_1.