Model uncertainty: minimax confidence limits from models that fit the data

John Copas

University of Warwick, UK

jbc@stats.warwick.ac.uk

(with Shinto Eguchi, ISM)

Text book inference pretends that the model is known (fixed in advance) — ???

Weak assumptions (those we are prepared to make) $\rightarrow$ model M_0

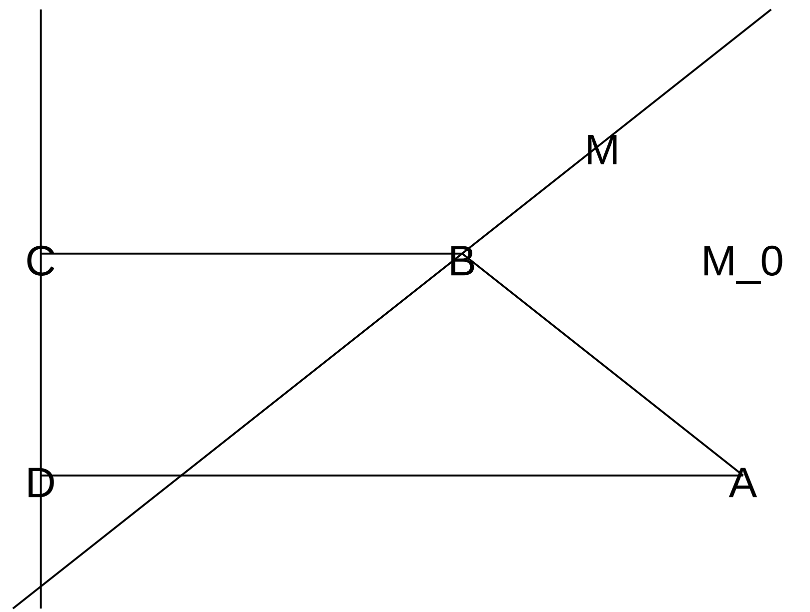
Strong assumptions (those we assume for inference) $\rightarrow$ model M

Data x, parameter of interest ϕ

eg 2×2 table, $p = (p_1, p_2, p_3, p_4)$ (3 df)

M_0 row totals fixed (2 df)

M row and column totals fixed (1 df)



$$A = \Pi(x|M_0),\ B = \Pi(x|M)$$

$$C = \Pi(\Pi(x|M)|\nabla\phi),\ D = \Pi(\Pi(x|M_0)|\nabla\phi)$$

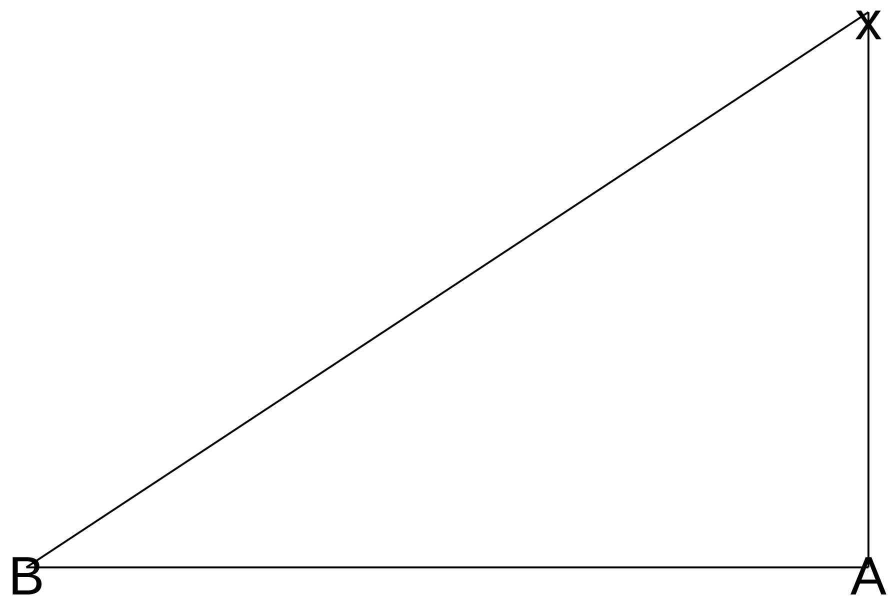
$$
\begin{aligned}
\Pi(x|M) &= \text{projection of } x \text{ onto model } M \\
&= \text{MLE fitted value of } p \text{ under} M
\end{aligned}
$$

Then

$$
\begin{aligned}
\Pi(\Pi(x|M_0)|M) &= \Pi(x|M) \\
\hat{\phi}_M &= \Pi(\Pi(x|M)|\nabla\phi) \\
&= \text{MLE of } \phi \text{ under } M \\
CI_M &= \text{CI for } \phi \text{ under } M \\
&= \hat{\phi}_M \pm z_\alpha \sigma_M \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_M^2 &= \text{Var}(\hat{\phi}_M) \\
z_\alpha &= \Phi(-\alpha/2)
\end{aligned}
$$



$$A = \Pi(x|M), \quad B = \Pi(x|M_0)$$

$$\mathrm{Var}(\hat{\phi}_{M_0} - \hat{\phi}_M) = \sigma^2_{M_0} - \sigma^2_M$$

$$z = \frac{\hat{\phi}_{M_0} - \hat{\phi}_M}{\sqrt{(\sigma_{M_0}^2 - \sigma_M^2)}} \sim_M N(0,1)$$

$$\begin{aligned} CI_M &= \hat{\phi}_M \pm z_\alpha \sigma_M \\ &= \hat{\phi}_{M_0} - z(\sigma_{M_0}^2 - \sigma_M^2)^{\frac{1}{2}} \pm z_\alpha \sigma_M \\ CI_{M_0} &= \hat{\phi}_{M_0} \pm z_\alpha \sigma_{M_0} \end{aligned}$$

- CI_M depends on M only through σ_M and z
- $\sigma_M \leftrightarrow$ “orientation” of M
- $z \leftrightarrow$ “translation” of M

Goodness-of-fit statistic $D\{\Pi(x|M), \Pi(x|M_0)\}$

Model M is "acceptable" if

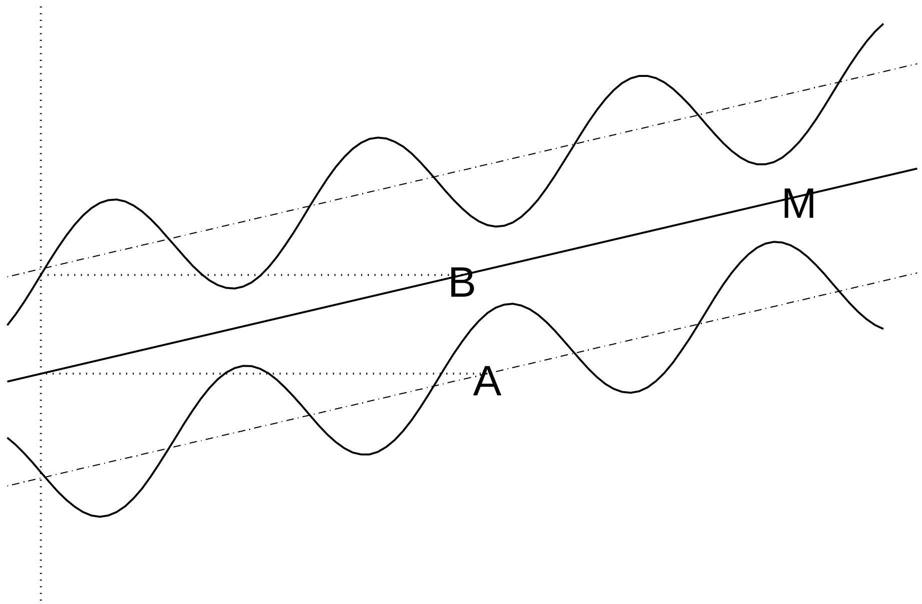
$$\Pi(x|M_0) \in \mathcal{A}_{M,D}$$

where

$$\mathcal{A}_{M,D} = \{\Pi(x|M_0) : D\{\Pi(x|M), \Pi(x|M_0)\} \leq d_M\}$$

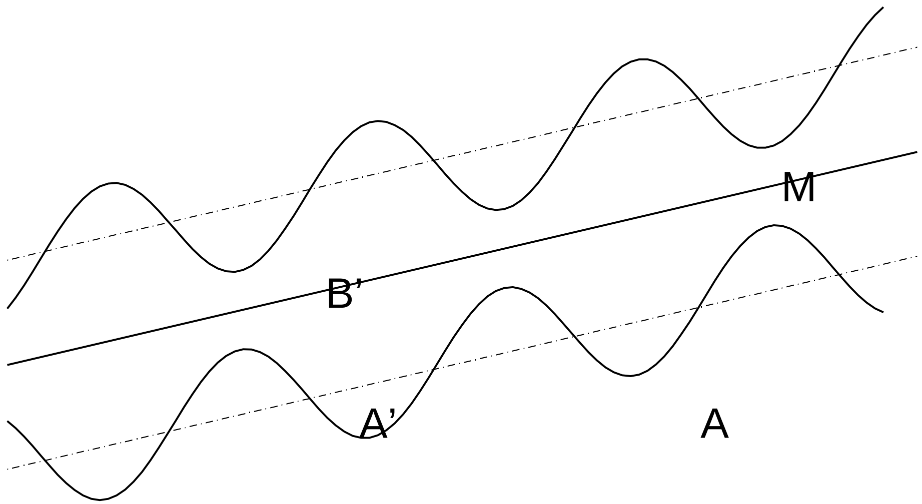
and

$$P_M(\mathcal{A}_{M,D}) = 1 - \alpha$$



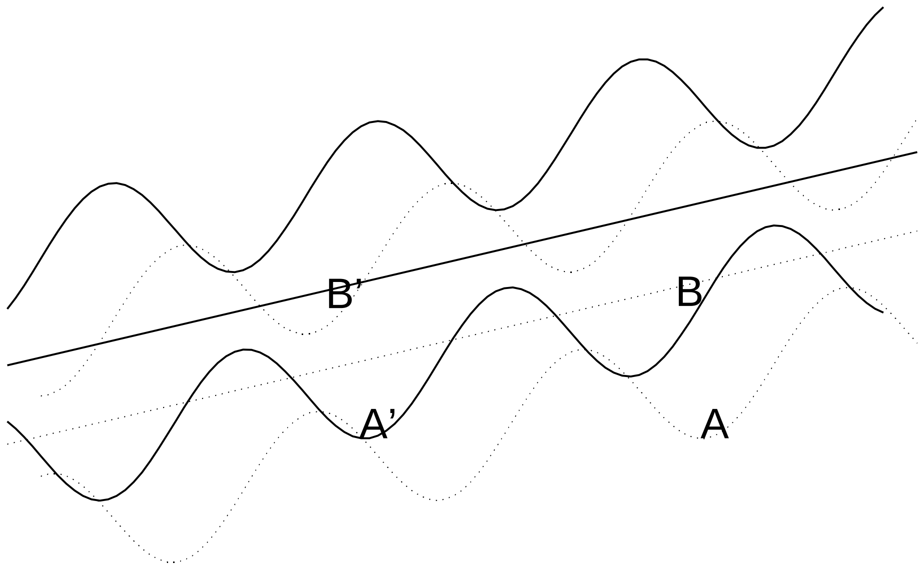
curves: boundaries of $\mathcal{A}_{M,D}$

lines: boundaries of $\{\Pi(x|M_0) : |z| \leq z_\alpha\}$



$$A' = \Pi(x'|M_0) \in \mathcal{A}_{M,D} \text{ with } |z'| > z_\alpha$$

$$A = \text{ actual } \Pi(x|M_0)$$



Model M translated into M' such that

$$\Pi(x|M_0) - \Pi(x|M) = \Pi(x'|M_0) - \Pi(x'|M)$$

Then

$$\max_M\{|z| : \Pi(x|M_0) \in \mathcal{A}_{M,D}\} \geq z_\alpha$$

We want to:

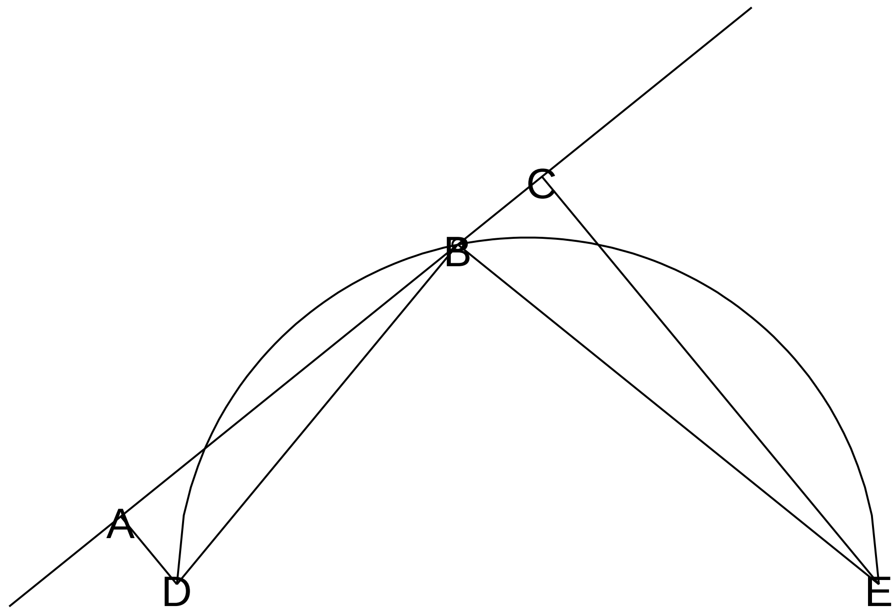
(1) Find, for fixed D,

$$\max_M\{\hat{\phi}_{M_0} + |z|(\sigma^2_{M_0} - \sigma^2_M)^{\frac{1}{2}} + z_\alpha \sigma_M : \Pi(x|M_0) \in \mathcal{A}_{M,D}\}$$

and

$$\min_M\{\hat{\phi}_{M_0} - |z|(\sigma^2_{M_0} - \sigma^2_M)^{\frac{1}{2}} - z_\alpha \sigma_M : \Pi(x|M_0) \in \mathcal{A}_{M,D}\}$$

(2) Find the min/max over D



$$|DE| = \sigma_{M_0},\ |DB| = \sigma_M,\ \tan(\angle ABD) = |z|/z_\alpha$$

$$z(\sigma_{M_0}^2 - \sigma_M^2)^{\frac{1}{2}} + z_\alpha \sigma_M \propto |AC|$$

Minimax solution is $z = \pm z_\alpha$ and

$$\sigma_M = (\sigma^2_{M_0} - \sigma^2_M)^{\frac{1}{2}} \;\; \Rightarrow \sigma_M = 2^{-\frac{1}{2}} \sigma_{M_0}$$

Hence minimax upper limit is

$$U = \hat{\phi}_{M_0} + 2^{\frac{1}{2}} z_\alpha \sigma_{M_0}$$

and maximin lower limit is

$$L = \hat{\phi}_{M_0} - 2^{\frac{1}{2}} z_\alpha \sigma_{M_0}$$

$$(L, U) = \hat{\phi}_{M_0} \pm 2^{\frac{1}{2}} z_\alpha \sigma_{M_0}$$

$$cf \;\; CI_{M_0} = \hat{\phi}_{M_0} \pm z_\alpha \sigma_{M_0}$$

$$\Rightarrow (L, U) \supseteq CI_{M_0}$$

Discussion

Interpretation of minimax confidence limits:

For any $\phi \in (L, U)$ and for any goodness-of-fit test,
$\phi \in CI_M$ for some well fitting model M

Goodness-of-fit is not a good enough reason for making stronger modelling assumptions than M_0 ie which go beyond those assumptions that we can honestly assume a priori

eg M_0 = linear regression, M = subset regression