ST222 2017 GAMES, DECISIONS AND BEHAVIOUR SHEET 1

(1) **Phone lists.** Some Warwick first year students are trying to reach fellow students to discuss homework problems from probability lectures.
   (a) Morsey, a first year MORSE student, has got the mobile numbers of 7 other MORSE students and 3 Math students. She wants to first call all the MORSE students on her list, than the Math students. How many different choices does she have with regard to the order of the 10 phone calls?
   (b) It turns out that the battery is empty and she has just enough cash to make 3 calls from a public phone. How many sets of 3 can she pick from the list of the 7 MORSE students? How many sets with at least one Math student can she pick from the list with all the 10 students?

(2) **Set theory** Let E, F, G be events.
   (a) Using intersections, unions and complements, give expressions for the following statements about the three events:
   "only E occurs", "at least one occurs", "at most one occurs".
   (b) Using elementary set operations, find the simplest expression for the events
   $(E \cup F) \cap (E \cup F^c)$ and $(E \cup F) \cap (E^c \cup F) \cap (E \cup F^c)$.

(3) **Representing events.** You have a black and a white standard die, that is, a hexahedron (aka cube) with numbers 1 to 6 on it. You observe the numbers facing up on each of the two dice and record their product. (e.g. if black shows 3 and white shows 4 record 12.) Let $\Omega = \{n \cdot m \,|\, n = 1, 2, \ldots, 6;\ m = 1, 2, \ldots, 6\}$. Say if the following events can be represented as subsets of Ω. If yes, provide the subset; if no, explain why.
   (a) Both dice show "6".
   (b) At least one of the dice shows"6".
   (c) The white die shows "4" and the black die shows "5".
   (d) At least one of the dice shows an even number.
   (e) Both dice show the same odd number.
   (f) The white die show a bigger number than the black die.

(4) Let Ω be a sample space (a set of possible outcomes). Let $\mathcal{F}$ be a finite algebra over Ω.
   (a) What are the three rules which $\mathcal{F}$ must obey in order to be an algebra?
   (b) Prove that
      - $\emptyset \in \mathcal{F}$
      - If $A, B, C \in \mathcal{F}$ then $A \cup B \cup C \in \mathcal{F}$.
      - If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \setminus B \in \mathcal{F}$. Here $A \setminus B = \{x : x \in A, x \notin B\}$ is the set of points contained in A but *not* in B.
      - $A \cap B = (A^c \cup B^c)^c$ (What is this identity called?)
   (c) What is the smallest algebra over Ω? Smallest here means the $\mathcal{F}$ with the fewest elements.
   (d) Let P be a probability measure on $(\Omega, \mathcal{F})$ and let $A, B, C \in \mathcal{F}$. Show that $P[A \cap B] \geq P[A] + P[B] - 1$.

(5) A bag contains n_r red, n_b blue and n_g green balls in it. Specify the event space related to a single draw and its atoms. If the draw is made at random what are the probabilities of each of the atoms? Now add n_y yellow balls. What is the new event space and its atoms. What are the probabilities of the atoms in the new space? Explain how you would use the axioms of probability to specify the probabilities of other events in this second event space.

(6) Let $\Omega = [0, 1]^2$, the unit square. Let $A \subset \Omega$ denote the circle with unit diameter. Give three examples of algebras over Ω that contain A.

(7) (a) **A discrete distribution.**

(i) Show that $P(\{n\}) = 2^{-n}$ defines a probability measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.
(Use $\mathbb{N} = \{1, 2, \ldots\}$.)

(ii) Pick an integer at random according to P. Let A be the event that the integer is an odd number. Compute the probability for A. (This can be done without explicitly calculating the limit of a series.)

(iii) Now, treat P as a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Give a formula describing the cumulative distribution function? What are the atoms?

(b) **A continuous distribution.** Define $F(x) := 0$ for $x < 0$, $F(x) = x$ for $0 \leq x < 1$ and $F(x) = 1$ for $x \geq 1$.

(i) Show that F is a cumulative distribution function.

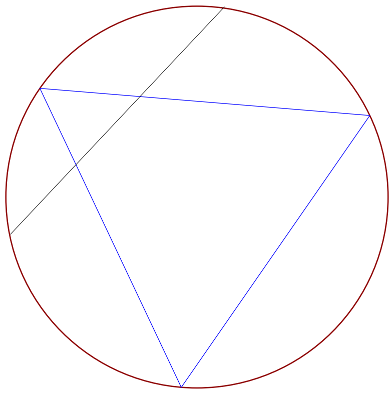
(ii) Define and describe the corresponding probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In particular, calculate the probability for an interval of length $c > 0$.

$\star$ (8) (*Not examinable*) **An additive measure that is not σ-additive.** Let $\Omega = [0, 1] \cap \mathbb{Q}$. Let $\mathcal{A}$ be the algebra of sets each of which is the finite union of disjoint sets A of one of the forms

$$(a, b) \cap \mathbb{Q}, \qquad (a, b] \cap \mathbb{Q}, \qquad [a, b) \cap \mathbb{Q}, \qquad [a, b] \cap \mathbb{Q}.$$

Define P on such sets by $P(A) = b - a$. Show that P is an additive set function on $\mathcal{A}$ and that P is not σ-additive.

$\star$ (9) (*Not examinable*) It is important to be precise when we talk about probabilities. Even when we have a simple geometric problem it is critical to specify precisely what we mean by a concept such as uniformity as this simple "paradox" due to Bertrand shows. Consider the following situation:



Given an equilateral triangle inscribed within a circle of unit radius, what is the probability that a random chord has a length greater than the sides of that triangle ($\sqrt{3}$)? One might assume that there is some natural interpretation of this simple problem, but we can obtain different answers with slightly different interpretations.

(a) Calculate the probability by assuming that a random chord has a midpoint chosen uniformly at random within the circle. Use the fact that the chord will be longer $\sqrt{3}$ if the midpoint lies within a circle of radius $\frac{1}{2}$ centred at the middle of the main circle.

(b) Calculate the probability by assuming that two points are selected uniformly at random around the perimeter of the circle. When will the chord obtained by joining these points be longer than a side of the inscribed triangle? Hint: without loss of generality, you may assume that the first point is a vertex of the triangle.

(c) A natural group-theoretic approach suggests that a "natural" random chord should have a midpoint at a radius chosen uniformly at random from $[0, 1]$. What is the probability that such a chord is longer than a side of the triangle? Hint: again, a geometric approach makes this simpler.

(d) Is this really a paradox? Why?