GAMES, DECISIONS AND BEHAVIOUR
EXERCISE SHEET 1 – SOLUTIONS

1. **Phone lists.** Some Warwick first year students are trying to reach fellow students to discuss homework problems from probability lectures.
   (a) Morsey, a first year MORSE student, has got the mobile numbers of 7 other MORSE students and 3 Math students. She wants to first call all the MORSE students on her list, than the Math students. How many different choices does she have with regard to the order of the 10 phone calls?

   $7! \cdot 3!$

   (b) It turns out that the battery is empty and she has just enough cash to make 3 calls from a public phone. How many sets of 3 can she pick from the list of the 7 MORSE students? How many sets with at least one Math student can she pick from the list with all the 10 students?

   As we count "sets of 3" (rather than triplets) the order does not matter and we select without replacement. There are $\binom{7}{3} = \frac{7!}{3! \cdot 4!}$ sets of 3 from list of 7 MORSE students. To get the number N of sets with at least one math student partition the set into three disjoint subsets:
   $A_i = \{\text{sets of 3 with } i \text{ math students in it}\}$ $(i = 1, 2, 3)$.
   A_3 has 1 element, just the set with all the three available math students in it.
   A_1 can be constructed by first selecting the math student (3 options) and then multiplying that number with the number of options for picking 2 out of 7 MORSE students for the remaining places, i.e. $|A_1| = 3 \cdot \binom{7}{2} = 63$. Similarly, $|A_2| = 7 \cdot \binom{3}{2} = 21$. Hence, $N = 85$.
   Alternatively, $N = |\{\text{all possibilities}\}| - |\{\text{no math students}\}| = \binom{10}{3} - \binom{7}{3} = 85$.
   Comment: One might think that N could be obtained by first choosing one of the Math students and then select sets of 2 from the remaining total of 9 students, i.e. $3 \cdot \binom{9}{2}$. However, this leads to certain combinations being counted twice.

2. **Set theory** Let E, F, G be events.
   (a) Using intersections, unions and complements, give expressions for the following statements about the three events:
   "only E occurs", "at least one occurs", "at most one occurs".

   "only E occurs" means $E \cap F^c \cap G^c$.
   "at least one occurs" means $E \cup F \cup G$. Note, it could also be expressed as "not none occurs" which means $(E^c \cap F^c \cap G^c)^c$ and equal $E \cup F \cup G$ by DeMorgan's laws.
   Finally "at most one occurs" means $(E \cup F \cup G)^c \cup (E \cup F^c \cup G^c) \cup (E^c \cup F \cup G^c) \cup (E^c \cup F^c \cup G)$. This is just the straight forward translation of the expression, probably can be simplified or done more elegantly to start with.

   (b) Using elementary set operations, find the simplest expression for the events
   $(E \cup F) \cap (E \cup F^c)$ and $(E \cup F) \cap (E^c \cup F) \cap (E \cup F^c)$.

   $(E \cup F) \cap (E \cup F^c) = [(E \cup F) \cap E] \cup [(E \cup F) \cap F^c] = E$, because $E \subset (E \cup F)$ implies that $[(E \cup F) \cap E] = E$ and because $(E \cup F) \cap F^c = \emptyset$.
   Applying the previous identity to the first two expressions, $(E \cup F) \cap (E^c \cup F) \cap (E \cup F^c) = F \cap (E \cup F^c) = (F \cap E) \cup (F \cap F^c) = F \cap E$.
   Remark: Visualise to students by using diagrams. However, discuss that diagrams have limitations. Most of all, they are useless once you have more than three sets to deal with.

3. **Representing events.** You have a black and a white standard die, that is, a hexahedron (aka cube) with numbers 1 to 6 on it. You observe the numbers facing up on each of the two dice and record their product. (e.g. if black shows 3 and white shows 4 record 12.) Let $\Omega = \{n \cdot m \,|\, n = 1, 2, \ldots, 6;\ m = 1, 2, \ldots, 6\}$. Say if the following events can be represented as subsets of Ω. If yes, provide the subset; if no, explain why.
   (a) Both dice show "6".

   Yes: $\{36\}$.

   (b) At least one of the dice shows "6".

No, because the product does not uniquely define the factors. For example: $6 = 1 \cdot 6 = 2 \cdot 3$.

(c) The white die shows "4" and the black die shows "5".

No, because the product does not reveal which factor was obtained by which of the dice.

(d) At least one of the dice shows an even number.

Yes: The only way to get an odd product is when both factors are odd. So, the even numbers are obtained if and only if at least one of the dice shows an even number. The subset is $\{2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 36\}$.

(e) Both dice show the same odd number.

Yes: $\{1, 9, 25\}$.

(f) The white die show a bigger number than the black die.

No, for the same reason as in (c).

4. Let Ω be a sample space (a set of possible outcomes). Let $\mathcal{F}$ be a finite algebra over Ω.

(a) What are the three rules which $\mathcal{F}$ must obey in order to be an algebra?

Axiom 1: $\Omega \in \mathcal{F}$.
Axiom 2: If $A \in \mathcal{F}$, then $A^c = \Omega \setminus A = \{x \in \Omega : x \notin A\} \in \mathcal{F}$.
Axiom 3: If $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.
(*Note: In general, to do probability we really need σ-algebras. In this case replace the last condition with the stronger condition, if $A_1, A_2, \dots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.*)

(b) Show that the following follow from those three rules:

- $\emptyset \in \mathcal{F}$

  Follows from axioms 1 and 2.

- If $A, B, C \in \mathcal{F}$ then $A \cup B \cup C \in \mathcal{F}$.

  $A \cup B \cup C = (A \cup B) \cup C$. Use axiom 3.

- If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \setminus B \in \mathcal{F}$. Here $A \setminus B = \{x : x \in A, x \notin B\}$ is the set of points contained in A but *not* in B.

  Check $A \setminus B$ is equal to $(A^c \cup B)^c$ which is found in $\mathcal{F}$ by axioms 2 and 3.

- $A \cap B = (A^c \cup B^c)^c$ (What is this identity called?)

  This is one of De Morgan's laws. Check LHS=RHS for each of the four cases: For $x \in \Omega$, either $x \in A \cap B$, $x \in A \cap B^c$, $x \in A^c \cap B$ or $x \in A^c \cap B^c$. Make sure your answer *does not* make use of De Morgan's law in a different form.

(c) What is the smallest $\mathcal{F}$ which is an algebra over Ω? Smallest, here, means the set with the fewest elements.

There is a clue in the fact that this question can be answered without knowing anything about Ω. In fact, the smallest algebra over any sample space, Ω consists of $\mathcal{F} = \{\emptyset, \Omega\}$. It's easy to verify that this satisfies all three properties. However, this algebra isn't going to be very useful...

5. A bag contains n_r red, n_b blue and n_g green balls in it. Specify the event space related to a single draw and its atoms. If the draw is made at random what are the probabilities of each of the atoms? Now add n_y yellow balls. What is the new event space and its atoms. What are the probabilities of the atoms in the new space? Explain how you would use the axioms of probability to specify the probabilities of other events in this second event space.

Let R, B, G and Y denote the event of drawing a red, blue ,green or yellow ball and let

$$\begin{aligned} n_1 &= n_r + n_b + n_g \\ n_2 &= n_r + n_b + n_g + n_y \end{aligned}$$

The first event space is

$$\{\emptyset, \{R\}, \{B\}, \{G\}, \{R \cup B\}, \{R \cup G\}, \{B \cup G\}, \Omega\}$$

with atoms $\{R\}, \{B\}, \{G\}$ & $P\{R\} = n_r/n_1, P\{B\} = n_b/n_1$ and $P\{G\} = n_g/n_1$. The second event space is

$$\left\{ \begin{array}{c} \emptyset, \{R\}, \{B\}, \{G\}, \{Y\}.\{R \cup B\}, \{R \cup G\}, \{R \cup Y\}, \{B \cup G\}, \{B \cup Y\}, \{G \cup Y\}, \\ \{R \cup B \cup G\}, \{R \cup B \cup Y\}, \{R \cup G \cup Y\}, \{B \cup G \cup Y\}, \Omega \end{array} \right\}$$

with atoms $\{R\}, \{B\}, \{G\}, \{Y\}$. & $P\{R\} = n_r/n_2, P\{B\} = n_b/n_2, P\{G\} = n_g/n_2$.& $P\{Y\} = n_g/n_2$. To find the probability of an event simply sum the probabilities of its constituent atoms.

6. Let $\Omega = [0,1]^2$, the unit square. Let $A \subset \Omega$ denote the circle with unit diameter. Give three examples of algebras over Ω that contain A.

   Examples include $\{\emptyset, A, A^c, \Omega\}$, $\mathcal{P}(\Omega)$, the Borel measurable subsets of Ω, the Lebesgue measurable subsets of Ω, the set $\{\emptyset, A, B, A^c, B^c, A \setminus B, B \cup A^c, \Omega\}$ with B any strict, non-empty subset of A, ...

7. (a) **A discrete distribution.**

   (i) Show that $P(\{n\}) = 2^{-n}$ defines a probability measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. (Use $\mathbb{N} = \{1, 2, \ldots\}$.)

   Use geometric series to show that total mass is 1.

   (ii) Pick an integer at random according to P. Let A be the event that the integer is an odd number. Compute the probability for A. (This can be done without explicitly calculating the limit of a series.)

   $P(\{2n\}) = P(\{2n-1\})/2$ for all $n \in \mathbb{N}$. This implies for $A =$"odd number", $P(A) = 2\,P(A^c)$. Hence $P(A) = 2/3$ and $P(A^c) = 1/3$.

   (iii) Now, treat P as a probability measure on $\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Give a formula describing the cumulative distribution function? What are the atoms?

   $F(x) = 0$ for all $x < 1$. F has jumps in all $n \in \mathbb{N}$ (the atoms) with jump sizes 2^{-n}. In other words, $F(x) = \sum_{n=1}^{\infty} 2^{-n} \cdot 1_{[n,\infty)}(x)$. For illustration:

$$F(2) = 2^{-1} \cdot 1_{[1,\infty)}(2) + 2^{-2} \cdot 1_{[2,\infty)}(2) + \sum_{n=3}^{\infty} 2^{-n} \cdot 1_{[n,\infty)}(2)$$

   Hence,

$$F(2) = 1/2 \cdot 1 + 1/4 \cdot 1 + 0 = 3/4 = P(\{1,2\})$$

   (b) **A continuous distribution.** Define $F(x) := 0$ for $x < 0$, $F(x) = x$ for $0 \le x < 1$ and $F(x) = 1$ for $x \ge 1$.

   (i) Show that F is a cumulative distribution function.

   Straight forward.

   (ii) Define and describe the corresponding probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In particular, calculate the probability for an interval of length $c > 0$.

   Also obvious, but make sure to properly deal with any parts of the interval that may be outside the region $[0,1]$ where F actually assigns mass.

⋆ 8. (*Not examinable*) **An additive measure that is not σ-additive.** Let $\Omega = [0,1] \cap \mathbb{Q}$. Let $\mathcal{A}$ be the algebra of sets each of which is the finite union of disjoint sets A of one of the forms

$$(a,b) \cap \mathbb{Q}, \qquad (a,b] \cap \mathbb{Q}, \qquad [a,b) \cap \mathbb{Q}, \qquad [a,b] \cap \mathbb{Q}.$$

Define P on such sets by $P(A) = b - a$. Show that P is an additive set function on $\mathcal{A}$ and that P is not σ-additive.

Note that complements are taken with respect to the reference set $\mathbb{Q}$. We can show that $\mathcal{A}$ indeed is an algebra: $(A1)$ With $a = 0, b = 1$, $\Omega = [a,b] \cap \mathbb{Q} \in \mathcal{A}$, $(A2)$ If A and B are each finite unions of disjoint sets of one of the forms

$$(*) \qquad (a,b) \cap \mathbb{Q}, \qquad (a,b] \cap \mathbb{Q}, \qquad [a,b) \cap \mathbb{Q}, \qquad [a,b] \cap \mathbb{Q}$$

then their union can again be represented in this form. For this, modify B by removing all intersections with A. Note that the modified B is again a finite union of disjoint sets of one of the forms $(*)$. $(A3)$ For $A = [a,b] \cap \mathbb{Q}$, $A^c = ([0,a) \cup (b,1]) \cap \mathbb{Q} = ([0,a) \cap \mathbb{Q}) \cup ((b,1] \cap \mathbb{Q}) \in \mathcal{A}$. For half-open intervals it goes the same way. (Don't knock off points for not showing that $\mathcal{A}$ is an algebra, as it wasn't explicitly asked for and it's rather obvious anyway.)

To show that P defines an additive set function: Any $A \in \mathcal{A}$ can be represented as $\bigcup_{i=1}^n I_i$, with $n \in \mathbb{N}$, each I_i of one of the forms $(*)$.

For sets in $(*)$ P has already been defined, and we set $P(A) := \sum_{i=1}^n P(I_i)$. To see that this is well-defined, note that two representations of the same set $A \in \mathcal{A}$ would differ only in that one or more of the I_i's would be represented several sets of the forms in $(*)$. Let I_i be replaced by a partition given by

J_j $(j = 1, \dots, n_i)$, where each is of one of the forms $(*)$. But the resulting $P(I_i)$ would be the same. For example, for $I_i = [a, b]$, $J_1 = [a, a_1)$, $J_j = [a_{j-1}, a_j)$ $(j = 2, \dots, n_{i-1})$, $J_{n_i} = [a_{j_{n_i}}, b]$,

$$P(\bigcup_{j=1\dots,n_i} J_j) = (b - a_{j_{n_i}}) + (a_{j_{n_i}} - a_{j_{n_i}-1}) + \cdots + (a_2 - a_1) + (a_1 - a) = b - a = P(I_i),$$

and similarly for half-open intervals.

Additivity is now simple. If A and B can each be represented as finite unions of disjoint sets of the form $(*)$, and A and B are disjoint themselves, then $A \cup B$ and again be represented as a finite unions of disjoint sets of the form $(*)$ and the additivity follows right away. For n pairwise disjoint sets additivity follows by induction using such arguments.

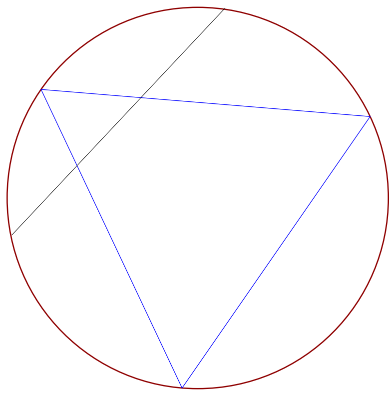
Proof that P is not σ-additive: For any $r \in \mathbb{Q}$, $a = b = r$ defines a set $A := [a, b] \cap \mathbb{Q}$ of the type described above and $A = \{r\}$. By definition, $P(\{r\}) = b - a = r - r = 0$. Let $r_n (n \in \mathbb{N})$ be a numbering of the countable set $[0, 1] \cap \mathbb{Q}$ so that $\cup_{n \in \mathbb{N}} \{r_n\} = \Omega$. Then

$$P(\cup_{n \in \mathbb{N}} \{r_n\} = P(\Omega) = 1, \text{but} \sum_{n \in \mathbb{N}} P(\{r_n\}) = \sum_{n \in \mathbb{N}} 0 = 0.$$

If P was σ-additive, they would have to be equal.

⋆ 9. (*Not examinable*) It is important to be precise when we talk about probabilities. Even when we have a simple geometric problem it is critical to specify precisely what we mean by a concept such as uniformity as this simple "paradox" due to Bertrand shows.

Consider the following situation:



Given an equilateral triangle inscribed within a circle of unit radius, what is the probability that a random chord has a length greater than the sides of that triangle ($\sqrt{3}$)?

One might assume that there is some natural interpretation of this simple problem, but we can obtain different answers with slightly different interpretations.

(a) Calculate the probability by assuming that a random chord has a midpoint chosen uniformly at random within the circle. Use the fact that the chord will be longer $\sqrt{3}$ if the midpoint lies within a circle of radius $\frac{1}{2}$ centred at the middle of the main circle.

$\frac{1}{4}$. The area of the smaller circle is $\pi \frac{1}{2}^2$ that of the unit circle is π. The ratio of these, $\frac{1}{4}$, is the probability that a point sampled uniformly at random in the larger circle is within the smaller circle.

(b) Calculate the probability by assuming that two points are selected uniformly at random around the perimeter of the circle. When will the chord obtained by joining these points be longer than a side of the inscribed triangle? Hint: without loss of generality, you may assume that the first point is a vertex of the triangle.

$\frac{1}{3}$. If one vertex of the equilateratal triangle coincides with the first point sampled then the chord will be longer than the sides of the triangle if and only if the second point lies between the other two vertices (consider the geometry). The vertices divide the perimeter of the circle into three segments of equal length, hence the probability of a point sampled uniformly around the perimeter of the circle lying between those two vertices is $\frac{1}{3}$.

(c) A natural group-theoretic approach suggests that a "natural" random chord should have a midpoint at a radius chosen uniformly at random from $[0, 1]$. What is the probability that such a chord is longer than a side of the triangle? Hint: again, a geometric approach makes this simpler.

$\frac{1}{2}$. Arrange the triangle so that it has a vertex opposite the sampled point (i.e. so that the sampled point, the centre of the circle and the vertex are collinear). Now, the centre of the triangle coincides with the centre of the circle and is $\frac{1}{3}$ of the length of the perpendicular from the base of the triangle to the vertex: i.e. the centre of the circle is twice as far from the vertex as it is from its base. Thus the base lies a distance $\frac{1}{2}$ from the centre of the circle. If the sampled point lies closer to the centre than this then it will be longer than the edge of the triangle otherwise it must be shorter.

(d) Is this really a paradox? Why?

No. There's a good reason that each of these three methods gives a different answer: they're all addressing a different question. Until we specify what we mean by "a random chord" the question is underspecified. Each of these solutions make a particular assumption about the nature of this randomness and each assumption has a unique correct solution.