# ST222 2017 GAMES, DECISIONS AND BEHAVIOUR
# EXERCISE SHEET 2 – SOLUTIONS

1. **Balls in bags.** A bag contains $n_r$ red, $n_b$ blue and $n_g$ green balls in it. Specify the event space related to a single draw and its atoms. If the draw is made at random what are the probabilities of each of the atoms? Now add $n_y$ yellow balls. What is the new event space and its atoms. What are the probabilities of the atoms in the new space? Explain how you would use the axioms of probability to specify the probabilities of other events in this second event space.

   Let $R, B, G$ and $Y$ denote the event of drawing a red, blue ,green or yellow ball and let

   $$\begin{aligned} n_1 &= n_r + n_b + n_g \\ n_2 &= n_r + n_b + n_g + n_y \end{aligned}$$

   The first event space is

   $$\{\emptyset, \{R\}, \{B\}, \{G\}, \{R \cup B\}, \{R \cup G\}, \{B \cup G\}, \Omega\}$$

   with atoms $\{R\}, \{B\}, \{G\}$ & $P\{R\} = n_r/n_1, P\{B\} = n_b/n_1$ and $P\{G\} = n_g/n_1$. The second event space is

   $$\left\{ \begin{array}{c} \emptyset, \{R\}, \{B\}, \{G\}, \{Y\}.\{R \cup B\}, \{R \cup G\}, \{R \cup Y\}, \{B \cup G\}, \{B \cup Y\}, \{G \cup Y\}, \\ \{R \cup B \cup G\}, \{R \cup B \cup Y\}, \{R \cup G \cup Y\}, \{B \cup G \cup Y\}, \Omega \end{array} \right\}$$

   with atoms $\{R\}, \{B\}, \{G\}, \{Y\}$. & $P\{R\} = n_r/n_2, P\{B\} = n_b/n_2, P\{G\} = n_g/n_2$.& $P\{Y\} = n_g/n_2$. To find the probability of an event simply sum the probabilities of its constituent atoms.

2. **Client's believes.** A client tells you that for two disjoint events $B$ and $C$,

   $$0 \leq P(B) \leq P(C) \leq P(B \cup C) \leq 1$$

   but that

   $$P(B) + P(C) > P(B \cup C)$$

   use the ball in bag method to construct a Dutch Book against your client's agent who is instructed to trade with these probabilities. Note that this together with the Dutch Book given in the notes demands that for any three elicited probabilities your client should always set probabilities such that $P(B)+P(C) = P(B \cup C)$ whenever $B$ and $C$ are disjoint.

   Suppose

   $$0 \leq P(B) \leq P(C) \leq P(B \cup C) \leq P(B) + P(C) \leq 1$$

   Choose $n$ such that we can express

   $$P(B) = \frac{r_1}{n}, P(C) = \frac{r_2}{n}, P(B \cup C) = \frac{r_3}{n}$$

   where from hypothesis $r_1 \leq r_2 \leq r_3$. Now construct a bag with $r_1$ white balls, $r_3 - r_1$ yellow balls, $r_1 + r_2 - r_3$ orange balls and $n - r_1 - r_2$ black balls. Then from the above we know

   $$\begin{aligned} b(B) &\sim b(n, r_1) \text{ a white draw} \\ b(C) &\sim b(n, r_2) \text{ a yellow or orange draw} \\ b(B \cup C) &\sim b(n, r_3) \text{ a white or yellow ball} \end{aligned}$$

   Suppose your agent holds the gamble a $b(B \cup C)$. Then she is happy to exchange this for $b(n, r_3)$. Also for large enough stakes she will be prepared to pay a small amount £$c$ to trade this gamble for a win for a gamble that wins not only on the draw of a yellow or white ball but an orange ball - bet $b(n, r_1 + r_2)$. Clearly bet $b(n, r_1 + r_2)$ is logically the same as holding simultaneously $b(n, r_1)$ and $b(n, r_2)$. So from the equivalences above your agent is happy to exchange these two bets for $b(B)$ and $b(C)$ respectively.. But since $B$ and $C$ are disjoint and so cannot happen together, holding $b(B)$ and

$b(C)$ simultaneously is logically equivalent to holding $b(B \cup C)$. So your agent has been induced to pay £$c$ for no gain! Finally if

$$P(B) + P(C) > 1$$

Then since $B$ and $C$ are disjoint $C \subseteq B^c$. So in particular your agent when holding gambles on $B$ and $B^c$ will also set.

$$P(B) + P(B^c) > 1$$

Now construct the Dutch Book similar to as shown in the lecture notes.

3. **Clinical trials.** A new treatment for a disease is being tested, to see whether it is better than the standard treatment. The existing treatment is effective on 50% of patients. It is believed initially that there is a 2/3 chance that the new treatment is effective on 75% of patients, and a 1/3 chance that the new treatment is effective on 50% of patients. In a pilot study, the new treatment is given to 20 patients selected at random, and turns out to be effective for 14 of them.

   (a) Given this information, what is the probability that the new treatment is better than the standard treatment?

   Let $E$ be the event that the new treatment is actually effective and let $X$ be the number of patients that the drug is effective for. We are told the prior probability of $E$ to be $P(E) = 2/3$. Using Binomial distributions with $p = 0.75$ and $p = 5$ we can calculate conditional probability for the observed event $\{X = 14\}$.

$$P(X = 14|E) = \binom{20}{14} \cdot 0.75^{14} \cdot 0.25^6 = 0.1686$$
$$P(X = 14|E^c) = \binom{20}{14} \cdot 0.5^{14} \cdot 0.5^6 = 0.0370$$

   The posterior probability that the treatment is effective conditional on the study results is

$$P(E|X = 14) = \frac{P(X = 14|E)P(E)}{P(X = 14|E)P(E) + P(X = 14|E^c)P(E^c)}$$
$$= \frac{0.1686 \cdot 2/3}{0.1686 \cdot 2/3 + 0.0370 \cdot 1/3} = 0.9011$$

   (b) A second study is done later, giving the new treatment to 20 new patients selected at random. Given the results of the first study, what is the PMF for how many of the new patients the new treatment will be effective on? *(Letting $p$ be the answer to the previous question your answer can be left in terms of $p$.)*

   Let Y be the number of patients that the drug will be effective on in this second study. This can be determined based on combining the results of two different Binomial distributions: if the new treatment is actually effective vs. the new treatment is not. Defining $p = P(E)$,

$$P(Y = y) = P(Y = y|E)P(Y = y|E^c)P(E^c)$$
$$= \binom{20}{k} \cdot 0.75^k \cdot 0.25^{20-k} \cdot p + \binom{20}{k} \cdot 0.5^k \cdot 0.5^{20-k} \cdot (1 - p)$$
$$= \binom{20}{k} \cdot (0.75^k \cdot 0.25^{20-k} \cdot p + 0.5^{20} \cdot (1 - p))$$

4. **Insurance policy.** Remember the insurance example described in lectures. In reality, most insurance policies have an excess. Consider the following decision problem.

   The policy costs $c$, your worldly goods are of value $v$ and the insurance policy has an excess $e$ (meaning that if you make a claim you receive an amount $e$ less than the amount stolen; you may assume that $e < v/10$).

   The three possible outcomes are:

$x_1 =$ {No thefts.} $\qquad x_2 =$ {Small theft, loss $0.1v$}

$x_3 =$ {Serious burglary, loss $v$}

You may assume that the probabilities elicited in lectures still hold:

$$p(x_1) = 0.946 \qquad p(x_2) = 0.043 \qquad p(x_3) = 0.011$$

and that a small theft is of goods worth $v/10$ whilst a serious burglary costs you everything ($v$).

(a) Write down the loss function for this problem.

If you don't buy insurance, you lose the stolen goods if there is a theft and nothing otherwise. If you do buy insurance you lose $c$ whatever happens and if there is a theft you also lose $e$ as the insurance payment is $e$ less than the value of the stolen goods, hence the loss function, in tabular form, is:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $d_1$ | $c$ | $c+e$ | $c+e$ |
| $d_2$ | 0 | $v/10$ | $v$ |

(b) Calculate the expected loss associated with each decision.

$$\begin{aligned}\bar{L}(d_1) =& cp(x_1) + (c+e)[p(x_2) + p(x_3)] \\ =& c + (1 - p(x_1))e \\ =& c + 0.054e \\ \bar{L}(d_2) =& 0p(x_1) + 0.1vp(x_2) + vp(x_3) \\ =& 0.0153v\end{aligned}$$

(c) Obtain inequalities in terms of $c, e$ and $v$ which describe the region of parameter space in which buying insurance is the optimal strategy in an EMV sense.

If $c + 0.054e < 0.0153v$ then $\bar{L}(d_1) < L(\bar{d_2})$ and the EMV-optimal decision is $d_1$: buying insurance.

5. **Investing into a machine.** A manager must decide whether to lease a small or large machine for one year. The small machine will cost them £10,000; the large machine £30,000. The small machine can produce up to 500 units a month; the large machine 3000 units.

If they find a distributor (an event which they believe has a probability $p$) then there will be a market for 2200 units per month over the year; otherwise just 400 units.

Whichever machine they lease, they will manufacture precisely the number of units which they sell. The profit per unit sold is £5.

(a) For what values of $p$ is the optimal EMV solution to lease the small machine?

The outcome space has two elements: $x_1$: a distributor is found and $x_2$: no distributor is found. The decision space also has two elements: $d_1$: lease a small machine and $d_2$: lease a large machine.

Begin by calculating the loss function (note that each month you sell as many units as you can unless the machine cannot produce enough in which case you sell as many as you can make:

$$\begin{aligned}L(d_1, x_1) =& 10000 - 12 \times 500 \times 5 \\ =& -20000 \\ L(d_2, x_1) =& 30000 - 12 \times 2200 \times 5 \\ =& -102000 \\ L(d_1, x_2) =& 10000 - 12 \times 400 \times 5 \\ =& -14000 \\ L(d_2, x_2) =& 30000 - 12 \times 400 \times 5 \\ =& 6000\end{aligned}$$

Now, calculate the expected loss associated with each decision:

$$\begin{aligned}\bar{L}(d_1) =& pL(d_1, x_1) + (1-p)L(d_1, x_2)\\ =& -20000p - 14000(1-p)\\ =& -14000 - 6000p\\ \bar{L}(d_2) =& pL(d_2, x_1) + (1-p)L(d_2, x_2)\\ =& -102000p + 6000(1-p)\\ =& 6000 - 108000p\end{aligned}$$

When $\bar{L}(d_1) < \bar{L}(d_2)$ the optimal decision is $d_1$; otherwise, it is $d_2$. In terms of $p$, this means:

$$\begin{aligned}-14000 - 6000p <& 6000 - 108000p\\ 102000p <& 20000\\ p <& 20/102 = 0.19608\end{aligned}$$

So we should lease a small machine only if $p < 0.19608$.

(b) Consider a third option. The manager has the option to indulge in industrial espionage. For a cost of £10,000 they can determine whether or not a distributor will be found in advance of making the decision.

We need to consider the expected loss of this third decision. Allowing $d_3$ to denote the decision corresponding to indulging in industrial espionage and then leasing the large machine if it indicators that a distributor will be found and the small machine if it does not, we obtain:

$$\begin{aligned}L(d_3, x_1) =& 10000 + L(d_2, x_1)\\ =& -92000\\ L(d_3, x_2) =& 10000 + L(d_1, x_2)\\ =& -4000\end{aligned}$$

Thus the expected loss associated with this decision is:

$$\bar{L}(d_3) = -92000p - 4000(1-p) = \qquad 4000 - 88000p$$

What is the new EMV strategy as a function of $p$?

Plotting the expected loss of each possible decision against $p$ produces figure 1. We wish to make the decision corresponding to the lowest curve at each value of $p$. We know that $d_1$ is better than $d_2$ when $p < 0.19608$ from the previous part. We need to determine when $d_3$ is the best strategy.

$$\begin{aligned}\bar{L}(d_1) =& \bar{L}(d_3)\\ -14000 - 6000p =& -4000 - 88000p\\ 82000p =& 10000\\ p =& 10/82 = 0.122\\ \bar{L}(d_2) =& \bar{L}(d_3)\\ 6000 - 108000p =& -4000 - 88000p\\ 10000 =& 20000p \Rightarrow p = \frac{1}{2}\end{aligned}$$

By inspection we have three possible optimal decisions depending upon the value of $p$:

| | $p < 0.122$ | $p \in [0.122, 0.5]$ | $p > 0.5$ |
|---|---|---|---|
| EMV Decision | $d_1$ | $d_3$ | $d_2$ |

This matches the intuition that:

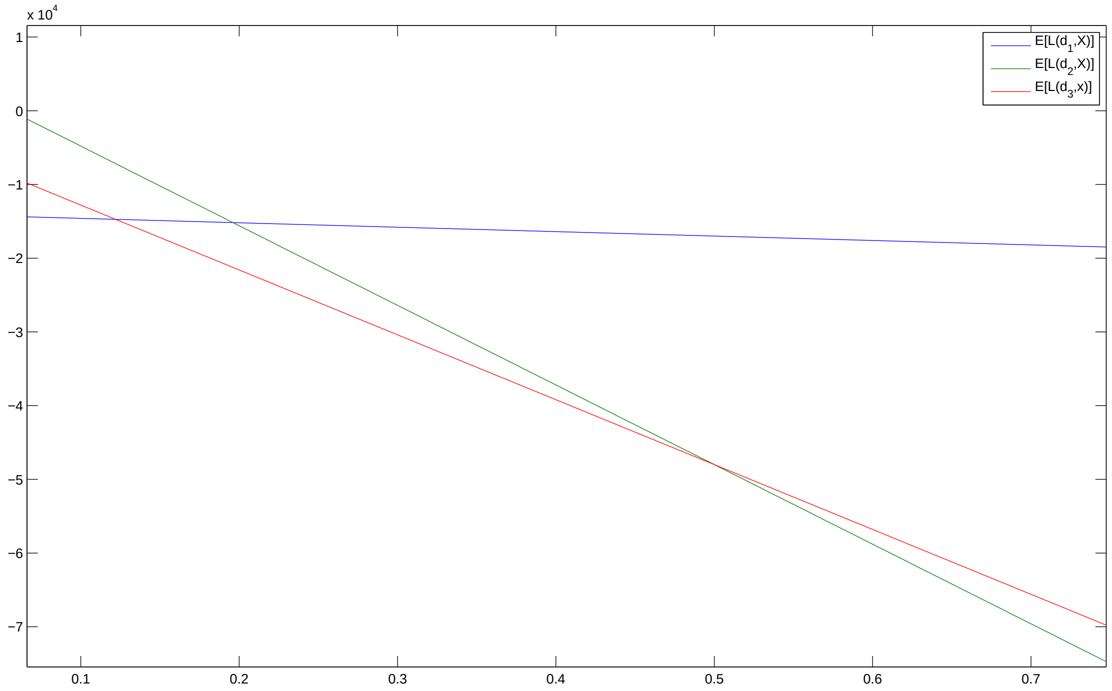
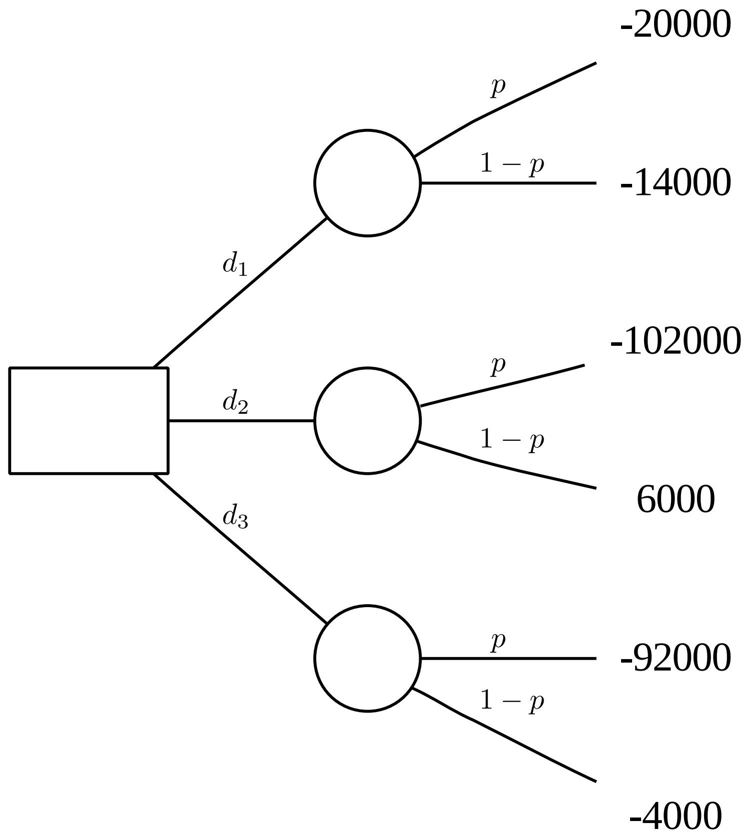



FIGURE 1. Expected loss for the three possible decisions for Q5(b)

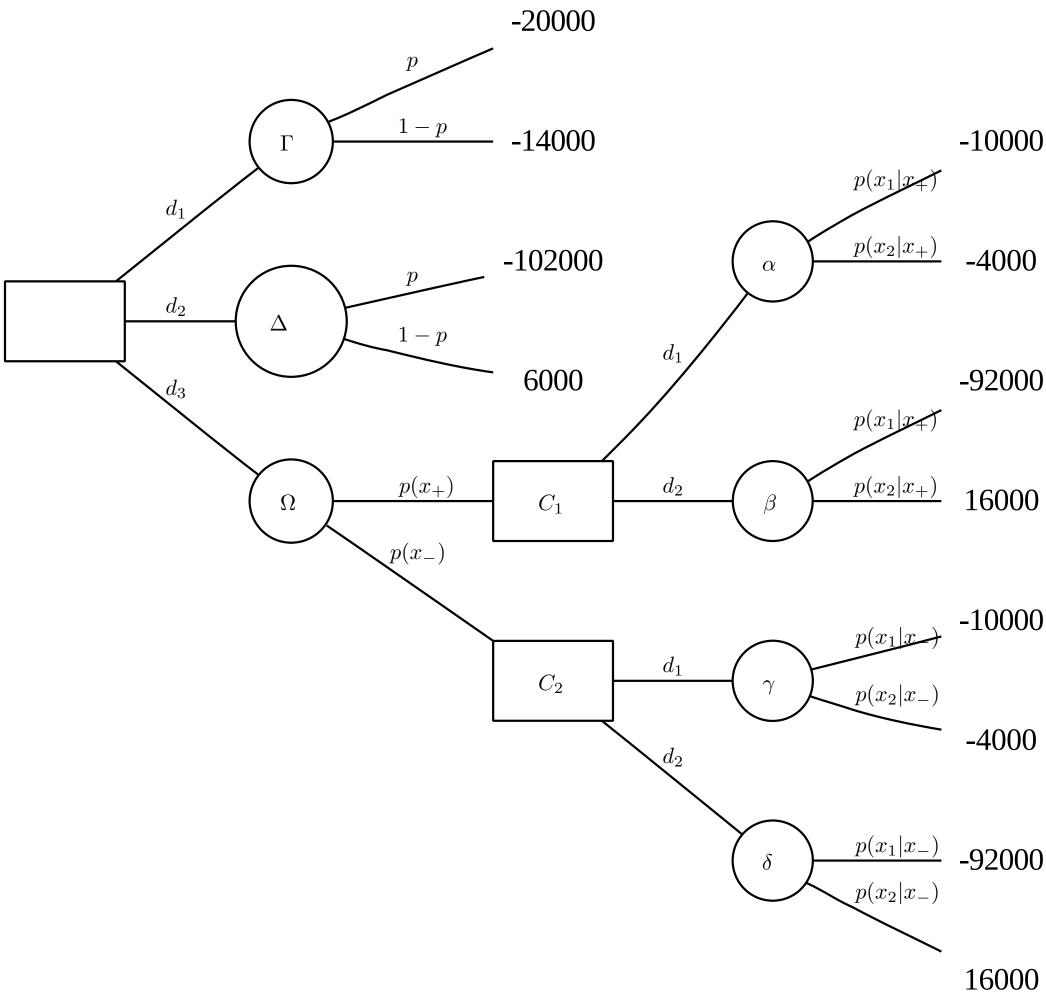
- If there is a very small probability of finding a distributor it's not worth paying to find out — you should just act as though a distributor won't be found.
- If there is a very large probability then you should just act as though one should be found.
- If there is a reasonable degree of uncertainty about whether a distributor will be found then the information provided by espionage is valuable enough to be worth paying £10,000 for.

(c) Draw a decision tree for problem 5b.




(d) If the espionage has a probability $q$ of being successful if a distributor is found and probability $r$ of being successful if they are not (i.e. with these probabilities, the espionage provides correct information and with probabilities $1-q$, $1-r$ the spy gets the wrong answer) then how does this change things?

(i) Draw a new decision tree for the revised problem.




This tree shows all of the important features, but we haven't yet calculated all of the probabilities or any of our expectations. We have let $x_+$ and $x_-$ be events corresponding to espionage implying that a distributor will or will not be found.

$\star$(ii) What is the EMV strategy?

From the previous question, we know that the expected value of the first two decisions are:

$$\Gamma = 1000 \times [6 - 108p]$$
$$\Delta = 1000 \times [-14 - 6p]$$

We know already that $p(x_+|x_1) = q$ (and, of course, that $p(x_-|x_1) = 1 - q$) and that $p(x_+|x_2) = 1 - r$ (and $p(x_-|x_2) = r$). We can use the law of total probability to calculate the first two probabilities that we need:

$$p(x_+) = pp(x_+|x_1) + (1-p)p(x_+|x_2)$$
$$= pq + (1-p)(1-r)$$
$$p(x_-) = pp(x_-|x_1) + (1-p)p(x_-|x_2)$$
$$= p(1-q) + (1-p)r$$

We then use Bayes' rule to calculate the other important probabilities:

$$\begin{aligned}
p(x_1|x_+) =& p(x_+|x_1)p(x_1)/p(x_+) \\
=& \frac{pq}{pq+(1-p)(1-r)} \\
p(x_2|x_+) =& p(x_+|x_2)p(x_2)/p(x_+) \\
=& \frac{(1-p)(1-r)}{pq+(1-p)(1-r)} \\
p(x_1|x_-) =& p(x_-|x_1)p(x_1)/p(x_-) \\
=& \frac{p(1-q)}{p(1-q)+(1-p)r} \\
p(x_2|x_-) =& p(x_-|x_2)p(x_2)/p(x_-) \\
=& \frac{(1-p)r}{p(1-q)+(1-p)r}
\end{aligned}$$

We are now in a position to calculate the expected values of making decision $d_1$ or $d_2$ after conducting espionage:

$$\begin{aligned}
\alpha =& \frac{-10pq-4(1-p)(1-q)}{pq+(1-p)(1-r)} \times 1000 \\
\beta =& \frac{-92pq+16(1-p)(1-q)}{pq+(1-p)(1-r)} \times 1000 \\
\gamma =& \frac{-10p(1-q)-4r(1-p)}{p(1-q)+r(1-p)} \times 1000 \\
\delta =& \frac{-92p(1-q)+16r(1-p)}{p(1-q)+r(1-p)} \times 1000
\end{aligned}$$

At this point things begin to get complicated. The decision we should make at $C_1$ and $C_2$ depends upon the value of the unknown parameters $p, q, r$. So we have to make a decision conditional upon these parameters.

At $C_1$ we should choose $d_1$ if $\alpha < \beta$:

$$\begin{aligned}
-10pq-4(1-p)(1-r) <& -92pq+16(1-p)(1-r) \\
82pq <& 20(1-p)(1-r) \\
p/(1-p) <& 20(1-r)/82q
\end{aligned}$$

otherwise we should make decision $d_2$ at $C_1$.

Similarly at $C_2$ we should choose $d_1$ if $\gamma < \delta$:

$$\begin{aligned}
-10p(1-q)-4r(1-p) <& -92p(1-q)+16r(1-p) \\
82p(1-q) <& 20r(1-p) \\
p/(1-p) <& 20r/82(1-q)
\end{aligned}$$

We need to consider, *a priori*, 4 regimes to determine the value of $\Omega$:

Regime 1: $p/(1-p) < 20(1-r)/82q$ and $p/(1-p) < 20r/82(1-q)$

Regime 2: $p/(1-p) < 20(1-r)/82q$ and $p/(1-p) > 20r/82(1-q)$

Regime 3: $p/(1-p) > 20(1-r)/82q$ and $p/(1-p) < 20r/82(1-q)$

Regime 4: $p/(1-p) > 20(1-r)/82q$ and $p/(1-p) > 20r/82(1-q)$

We make our lives simpler if we notice than in regime 1 and regime 4 we ignore the advice we get completely. If we aren't going to use the advice then we shouldn't be prepared to pay for it so in either of these regimes $\Omega$ is worse than one of $\Gamma$ and $\Delta$ and may be discounted. Regime 2 is the case in which we believe the advice to be true and act accordingly. Regime 3 illustrates the amusing fact that if the advice is bad enough it may be useful to us as we

can choose to do the opposite to whatever it is that is advised! Finally, for regime 2 and regime 3 we know the decisions which we make at $C_1$ and $C_2$ and can hence calculate the value of $\Omega$ in each of these regimes. For each regime, we can compare $\Omega$ with $\Gamma$ and $\Delta$ as a function of the arguments to develop a final decision rule:
Let $\Omega_2$ denote the value of $\Omega$ in regime 2 – the expected value of deciding to ask:

$$\Omega_2 = p(x_+)\alpha + p(x_-)\delta = (p[-10q - 92(1-q)] + (1-p)[-r(1-q) + 16r]) \times 1000$$

Whilst, in regime 3:

$$\Omega_3 = p(x_+)\beta + p(x_-)\gamma = (p(-92q - 10(1-q)) + (1-p)(16(1-r) - 4r)) \times 1000$$

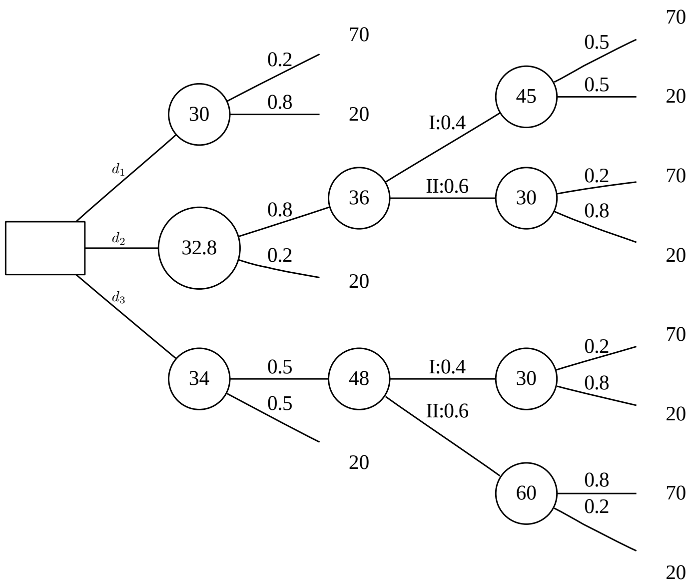
We can compare these with $\Gamma$ and $\Delta$ to complete our set of decision rules.

*This may seem unduly complicated, but the process can be automated completely and it shows that a single decision tree can encode the solution of an entire problem. If we want to see how varying the parameters alters the conclusions we reach it's easier to solve the problem in one go like this than to deal with a completely new decision tree every time.*

6. **Maximising life expectation in medical treatment decision.** On his twentieth birthday a patient is brought in to a hospital with an illness which is either Type I (with probability 0.4) or Type II (with probability 0.6). Independent of the type of the illness, without treatment he will die on that day with probability 0.8 and otherwise survive and have normal life expectancy. The surgeon may take one of three possible courses of action:
   - $d_1$ : not to treat the patient;
   - $d_2$ : to give the patient drug L at once;
   - $d_3$ : to operate on the patient at once.

   She cannot both operate and administer the drug. Both operating and administering the drug are dangerous to the patient. Independent of the type of illness, operating will kill the patient with probability 0.5, while the drug will kill him with probability 0.2. Should the patient survive the effects of the drug, if he has Type I illness then it will cure him with probability 0.5 and otherwise they will die. If he has Type II illness then it will have no effect. Should the patient survive the effects of the operation, if he has Type II illness then it will cure him with probability 0.8 and otherwise they will die. If he has Type I illness then it will have no effect. In all cases survival will give the patient 70 years life expectancy (measured from birth).
   (a) Draw a decision tree to represent the surgeon's decision problem.
   (b) Calculate her best strategy assuming she wishes to maximize her patient's life expectancy.

   Below is the decision tree with expected values shown. I demonstrates that the optimal strategy is $d_3$, to operate immediately. Even this decision only provides an expected life expectancy of 34.



7. (*Not examinable*) **Pasadena game.** Philosophers of probability have observed some strange goings-on in Pasadena. Similarly to the St Petersburg game, a fair coin is tossed until it lands heads for the first time. But the payoffs are less generous. In St Petersburg they were $2^n$, where $n$ is the number of trials to the first heads. In Pasadena, they reduced and on top that rewards take turns with losses: the payoff is $(-1)^{n-1}2^n/n$. (Assume money is linear in utility; if not replace currency unit with utiles.) How much would you play to play this game? What is the answer according to the EMV approach in decision theory? Is the question well-posed?
*Note: Philosophers are still arguing about the Pasadena game among themselves. We will not post a solution to these questions, but suggest you compare your thoughts to theirs, e.g. in Alan Hájek and Harris Nover, "Perplexing Expectations", Mind 115 (July 2006) and Alan Hájek, "Unexpected Expectations", Mind 123 (April 2014).*

See in references above.