ST222 2017 GAMES, DECISIONS AND BEHAVIOUR
EXERCISE SHEET 3 – SOLUTIONS

1. Joe owes £100, 000 to the Mafia. They want the money immediately. He has £60, 000. He has the opportunity to place a bet in a casino in which with probability $\frac{1}{2}$ he will make a profit of £40, 000 and with probability $\frac{1}{2}$ he will lose £60, 000.
   (a) What is his EMV decision?

   The two possible decisions are to play or not to play. Not playing has an expected reward of zero; playing has an expected reward of -£10,000. The EMV decision is not to play.

   (b) Does that coincide with common sense?

   Most people in this position would probably accept that having £60,000 is unlikely to be of any value whilst having £100,000 might keep them alive and would be prepared to indulge in the bet, even if it's an unfair bet.

   (c) What utility function would you advise Joe to use in this circumstance?

   Joe needs to win at least £40,000 in order to be able to pay his debts. This is his sole concern. If he assigns an outcome in which he wins at least £40, 000 a utility of 1 and other outcomes a utility of 0 then this represents these beliefs.

$$U(x) = \begin{cases} 0 & x < 40,000 \\ 1 & x \geq 40,000 \end{cases}$$

   (d) What is the expected utility of the two possible decisions? And what is the expected utility decision?

   The expected utility of not playing is zero: there is no possible way of achieving a utility other than zero.
   The expected utility of playing is $0.5[1 + 0] = 0.5$. It's not a coincidence that this is equal to the probability of achieving an outcome with a utility of one under the specified utility function.
   Consequently, maximising the expected utility in this case tells us that Joe should play.

2. The St. Petersburg Paradox shows that EMV decisions don't always lead to behaviour which we might consider sensible. It concerns the following game: You pay a fixed fee to enter, and then a fair coin will be tossed repeatedly until a tail first appears, ending the game. The pot starts at 1 dollar and is doubled every time a head appears. You win whatever is in the pot after the game ends.
   (a) Show that the expected monetary value of this game is infinite.

   The probability that the first tail occurs after n tosses is equal to the probability of receiving $n-1$ heads followed by a tail: $p(n) = 0.5^{n-1} \times 0.5^1 = 0.5^n$. The reward associated with the first tail occurring on the n^{th} toss is $R(n) = 2^{n-1}$. Hence, the expected reward is:

$$\begin{aligned}\mathbb{E}(R) = &\sum_{n=1}^{\infty} p(n)R(n) = \sum_{n=1}^{\infty} 0.5^n \times 2^{n-1} \\ =&0.5 \sum_{n=1}^{\infty} 0.5^{n-1} \times 2^{n-1} = 0.5 \sum_{n=1}^{\infty} 1^{n-1} = \infty\end{aligned}$$

   (b) Show that, if your utility function U is bounded: $\forall x : U(x) \leq A$, then the expected utility of this game is also bounded.

   This can be shown directly:

$$\mathbb{E}(U) = \sum_{n=1}^{\infty} p(n)U(R(n)) \leq \sum_{n=1}^{\infty} p(n)A \leq A \sum_{n=1}^{\infty} p(n) = A$$

(c) If your utility function is the risk-averse $U(x) = \log(x)$ then what is the expected utility of the game[1]?

$$\mathbb{E}(U) = \sum_{n=1}^{\infty} p(n)U(R(n)) = \sum_{n=1}^{\infty} 0.5^n \log(2^{n-1}) = \sum_{n=1}^{\infty} 0.5^n \times (n-1)\log(2)$$

$$= \log(2) \sum_{n=0}^{\infty} n \times 0.5^{n+1} = \log(2) \sum_{n=1}^{\infty} n(1-0.5)0.5^{n-1} = \log(2) = 0.69$$

Where the final lines involve recognising the sum as the expectation of a geometric random variable with parameter 0.5. So, you'd accept any amount exceeding $\pounds x$ where $\log(x) = 0.69 \Rightarrow x = 2$ as an alternative to playing the game. This is perhaps a more risk-averse utility than many people would choose.

3. An investor has \$1,000 to invest in speculative stocks. The investor is considering investing $\$a$ in stock A and $\$1,000 - a$ in stock B. An investment in stock A has a 0.6 chance of doubling in value, and a 0.4 chance of being lost. An investment in stock B has a 0.7 chance of doubling in value, and a 0.3 chance of being lost. Assume the stocks are independent of each other. The investor's utility function for a change in fortune, z, is $u(z) = \log(0.0007z + 1)$ for $-1,000 \le z \le 1,000$.

(a) As a function of a, what are the monetary values of all four potential scenarios? What are their probabilities?

Record the change of capital for all four scenarios (drop \$ for simplicity).
If both stocks loose the change to the investors capital is -1000, and if both win it is 1000.
If A looses and B wins, the change is $-a + (1000 - a) = -2a + 1000$.
If A wins and B looses, the change is $a - (1000 - a) = 2a - 1000$.

(b) What is the optimal value of a in terms the investor's expected utility?

The expected utility is

$$\begin{aligned}U(a) &= 0.4 \cdot 0.3 \cdot \log(0.0007 \cdot (-1000) + 1) + 0.6 \cdot 0.7 \cdot \log(0.0007 \cdot 1000 + 1) \\ &+ 0.6 \cdot 0.3 \cdot \log(0.0007 \cdot (2a - 1000) + 1) + 0.4 \cdot 0.7 \cdot \log(0.0007 \cdot (-2a + 1000) + 1) \\ &= 0.12 \log(0.3) + 0.42 \log(1.7) + 0.18 \log(0.0014a + 0.3) + 0.28 \log(1.7 - 0.0014a)\end{aligned}$$

Find the value for a that maximises $U(a)$ using calculus.

$$\begin{aligned}\frac{dU(a)}{da} &= \frac{0.18}{0.0014a + 0.3} \cdot 0.0014 + \frac{0.28}{1.7 - 0.0014a} \cdot (-0.0014) \\ &= \frac{0.000252}{0.0014a + 0.3} - \frac{0.000392}{1.7 - 0.0014a}.\end{aligned}$$

U has a unique critical value (i.e. $\frac{dU(a)}{da} = 0$ in $a* = 344.72$.
Since the second derivative

$$\frac{d^2U(a)}{da^2} = \frac{0.000252 \cdot 0.0014}{(0.0014a + 0.3)^2} - \frac{0.000392 \cdot 0.0014}{(1.7 - 0.0014a)^2} < 0,$$

a^* is indeed a maximum.

[1] *Hint: It may be useful to note that for* $p \in (0,1): \sum_{n=1}^{\infty} n(1-p)p^n = \frac{p}{1-p}$.

4. Suppose a decision maker has constant absolute risk aversion of the range $-\$100$ to $\$1,000$, that is, $u(x) = -ae^{-\lambda b} + b$, for all $x \in [-100, 1,000]$, for some constants $a, b \in \mathcal{R}$. We ask for her certainty equivalent for a gamble with prizes $\$0$ and $\$1,000$, each with probability 0.5, She says that her certainty equivalent for the gamble is 488. What, then, should she choose, if faced with the choice of:

(A) a gamble with prizes $-\$100$, $\$300$, and $\$1,000$, each with probability 1/3;

(B) a gamble with prizes $\$530$ with probability 3/4 and $\$0$ with probability 1/4;

(C) a gamble with are sure thing payment of $\$385$?

Since the decision maker has constant absolute risk aversion over the range $-\$100$ to $\$1000$, we have

$$u(z) = -a\,e^{\lambda z} + b, \text{ for all } z \in [-100, 1000].$$

We know that the certainty equivalent for a 50/50 gamble with prizes $\$0$ and $\$1000$ is $\$488$. Therefore,

$$u(488) = 1/2 \cdot u(0) + 1/2 \cdot u(1000).$$

Suppose $u(0) = 0, u(1000) = 1$. Then we have

$$\begin{aligned} 0 &= -a\,e^{\lambda \cdot 0} + b \\ 1 &= -a\,e^{\lambda \cdot 1000} + b \\ 1/2 &= -a\,e^{\lambda \cdot 488} + b \end{aligned}$$

The system implies that $a = b = 10.9207$ and $\lambda = 0.0000960369$. Therefore, we have

$$u(z) = 10.9207(1 - e^{-0.0000960369\,z}), \text{ for all } z \in [-100, 1000].$$

We now obtain
$u(-100) = -0.105384$, $u(300) = 0.310148$, $u(385) = 0.396411$ and $u(530) = 0.541949$. The expected utilities associated with gambles A, B and C are

$$\begin{aligned} U(A) = 1/3(u(-100) + u(300) + u(1000)) &= 0.401588 \\ U(B) = 3/4u(530) + 1/4u(o) &= 0.406462 \\ U(C) = u(385) &= 0.396411. \end{aligned}$$

Based on the maximum utility principle we conclude that the decision maker should choose gamble B.

5. You are being offered the choice between gamble A_1 and gamble A_2 and between gamble B_1 and gamble B_2 described below. Your preference is $A_1 \succ A_2$ and $B_2 \succ B_1$. Show that they are incompatible with the principle of maximising expected utility, no matter what your utility of money happens to be.

A_1 : £50, £50, and £50, each with probability 1/3;
A_2 : £100, £50, and £0, each with probability 1/3;
B_1 : £50, £0, and £50, each with probability 1/3;
B_2 : £100, £0, and £0, each with probability 1/3.

Calculate the difference in expected utilities between both pairs.

$$\begin{aligned} &u(A_1) - u(A_2) \\ &= u(50) - 1/3(u(100) + u(50) + u(0)) = -1/3u(100) + 2/3u(50) - 1/3u(0) \\ &u(B_1) - u(B_2) \\ &= 2/3u(50) + 1/3u(0) - (1/3u(100) + 2/3u(0)) = -1/3u(100) + 2/3u(50) - 1/3u(0) \end{aligned}$$

Since they are equal, you must prefer A_1 to A_2 if and only if you prefer B_1 to B_2.

6. Let x be a bet that gives you £10,000,001 for sure, let y be a bet that gives you £10,000,000 for sure and let z be a bet that gives you 50 years in prison for sure. Your preferences are $x \succ y \succ z$.

(a) State what the Archimedean axiom says for this situations.

There are $\alpha, \beta \in (0,1)$ such that $\alpha\,x + (1-\alpha)z \prec y \prec \beta\,z + (1-\beta)z$

(b) What does is actually mean in terms of people's behaviour?

The first relation means means that there is a convex combination of the bets x and z that you would prefer to bet y. The second relation means that there is a convex combination of bets x and z such that you would prefer y to it.

(c) Discuss whether or not this is realistic. In particularly, consider that according to empirical evidence, people can not distinguish between very small probabilities.

The first relation does not seem sensible. You would not want to take even a very small risk $(1-\alpha)$ of spending 50 years in prison only to increase the pay-off by £1. This is even more true in that, if you are a typical human being, your interpretation of the tiny risk would be higher than it actually is.

7. The lexicographical order relation on $\mathcal{R}^2$ is defined as follows

$$(x_1, x_2) \succ (y_1, y_2) \quad \iff \quad x_1 > y_1 \ \vee \ (x_1 = y_1 \ \wedge \ x_2 > y_2).$$

(This is using the notation $x = (x_1, x_2)$ for $x \in \mathcal{R}^2$.)

(a) Show that lexicographical order relation is complete and transitive.

Completeness: We need to show that for each pair $x, y \in \mathcal{R}^2$ one of the following is true: $x \succ y$, $y \succ x$ or $x \sim y$.

If $x_1 > y_1$ then $x \succ y$. If $x_1 < y_1$ then $y \succ x$. In the remaining case, $x_1 = y_1$, we proceed with comparing x_2 and y_2. If $x_2 > y_2$ then $x \succ y$. If $x_2 < y_2$ then $y \succ x$. Finally, if $x_2 = y_2$ then $x \sim y$.

Transitivity: Let $x, y, z \in \mathcal{R}^2$ with $x \succ y$ and $y \succ z$. We need to show that $x \succ z$.

If $x_1 > y_1$ or $y_1 > z_1$ then $x_1 > z_1$ and thereby $x \succ z$. Otherwise, $x_1 = y_1 = z_1$ and we proceed with the second coordinate. Since $x \succ y$, $x_2 \succ y_2$ and, since $y \succ z$, $y_2 \succ z_2$. Hence $x_2 > z_2$, which in this case implies $x \succ z$.

(b) Is it independent? Proof it or demonstrate that it is not true.

Yes. Let $x, y \in \mathcal{R}^2$ with $x \succ y$. Let further $z \in \mathcal{R}^2$ and $\alpha \in (0,1)$. Then

$$\alpha x + (1-\alpha) z \succ \alpha y + (1-\alpha) z$$

is equivalent with

$$(\alpha x_1 + (1-\alpha) z_1, \alpha x_2 + (1-\alpha) z_2) \succ (\alpha y_1 + (1-\alpha) z_1, \alpha y_2 + (1-\alpha) z_2) \qquad (*)$$

If $x_1 > y_1$ then $\alpha x_1 + (1-\alpha) z_1 > \alpha y_1 + (1-\alpha) z_1$, which implies (*).

Otherwise, since $x \succ y$ by assumption, $x_1 = y_1 \wedge x_2 > y_2$. In this case, $\alpha x_1 + (1-\alpha) z_1 = \alpha y_1 + (1-\alpha) z_1$, but $\alpha x_2 + (1-\alpha) z_2 > \alpha y_2 + (1-\alpha) z_2$, which also implies (*).

(c) Does it have the Archimedean property? Proof it or demonstrate that it is not true.

It is not true. For example, $x = (0,1), y = (0,0), z = (-1,0)$ fulfills $x \succ y \succ z$. Assume there were $\alpha, \beta \in (0,1)$ with

$$\alpha\, x + (1-\alpha) z \prec y \prec \beta\, z + (1-\beta) z \qquad (*)$$

That means,

$$(\alpha \cdot 0 + (1-\alpha) \cdot (-1), \alpha \cdot 1 + (1-\alpha) \cdot 0) \succ (0,0) \succ (\beta \cdot 0 + (1-\beta) \cdot (-1), \beta \cdot 1 + (1-\beta) \cdot 0)$$

which simplifies to

$$(\alpha - 1, \alpha) \succ (0,0) \succ (\beta - 1, \beta).$$

For the first relation to be true $\alpha - 1 \geq 0$. But that implies $\alpha \geq 1$, which is a contradiction to the assumptions on α.

(d) (*Not examinable*) Show that it is not continuous using the following definition for continuity: A preference relation on a topological space $\mathcal{A}$ is called *continuous* if for all $x \in \mathcal{A}$

$$\underline{\mathcal{B}} := \{y \in \mathcal{A} \,|\, x \succ y\} \qquad \text{and} \qquad \overline{\mathcal{B}} := \{y \in \mathcal{A} \,|\, y \succ x\}$$

are open subsets in $\mathcal{A}$.

For example, consider $x = (0,0)$. Then the corresponding set $\underline{\mathcal{B}}$ is $\{y \in \mathcal{R}^2 \,|\, (0,0) \succ y\}$ and its boundary is the y-axis. The lower half of the boundary $\{(0,v) \,|\, v < 0\}$ belongs to $\underline{\mathcal{B}}$ whereas the upper half (including the origin) does not belong to $\underline{\mathcal{B}}$. This shows that $\underline{\mathcal{B}}$ is not open. (*Note that it is not closed either.*)