ST222 2017 GAMES, DECISIONS AND BEHAVIOUR
EXERCISE SHEET 4 – SOLUTIONS

1. **Ellsberg paradox.** There are some criticisms of utility-based decision making. This question (based upon the so-called Ellsberg paradox) illustrates one of them.

   An urn contains 90 balls. 30 balls are red and the remainder are black or yellow. You do not know how many are black and how many are yellow.

   In experiment 1 you are given a choice between two bets:
   - d_1: Win £100 if a red ball is drawn.
   - d_2: Win £100 if a black ball is drawn.

   In experiment 2 you are given a choice between two other bets:
   - d'_1: Win £100 if a red or yellow ball is drawn.
   - d'_2: Win £100 if a black or yellow ball is drawn.

   A friend asserts that they would prefer d_1 to d_2 but they would prefer d'_2 to d'_1 because in both cases this maximises a lower bound on their probability of winning.

   Show that this is inconsistent with the expected utility approach to decision making.

   Let $r = 30$ be the number of red balls; let y, b be the number of yellow and black balls, respectively. $b+y = 30$ and let $n = r+b+y$. Let R, Y and B be events corresponding to the probability of drawing a red, yellow or black ball, respectively

$$\mathbb{P}(R) = r/n \qquad \mathbb{P}(B) = b/n \qquad \mathbb{P}(Y) = y/n$$

   Whatever the utility function, the expected utility of d_1 is $rU(100)/n$ and that of d_2 is $bU(100)/n$. Preferring d_1 implies that:

$$rU(100)/n > bU(100)/n \Rightarrow r > b.$$

   Similarly, the expected utility of d'_1 is $[rU(100)+yU(100)]/n$ and that of d'_2 is $[bU(100)+yU(100)]/n$. Preferring d'_2 implies that:

$$[bU(100) + yU(100)]/n > [rU(100) + yU(100)]/n \Rightarrow b > r.$$

   It is clear that these two positions are incompatible – and notice that this doesn't depend upon which utility function is used. The difficulty here is how the uncertainty in the number of black and yellow balls is being handled. Really, it should be dealt with probabilistically.

   The important point about this example is that it shows that we often don't have all of the information which we need in order to make decisions: it is important to be aware of this and to attempt to encode all uncertainty within appropriate probability distributions. It is an unfortunate fact that this is not always straightforward to do this.

2. **Allais paradox.** In the lectures (Week 6 Tuesday and Week 8 Friday 4pm) we already saw another example for how utility-based decision is not always consistent with people's behaviour. Show that no utility function can explain the preferences observed in a majority of people. (To keep things shorter in the lecture, this was done under the additional assumption that $u(0) = 0$, but you can show it for the general case.)

   Use the notation and definitions of the bets from the lecture notes.

   The preferences mean:

$$x \succ x' \Leftrightarrow u(5) > 0.1u(25) + 0.89u(5) + 0.01u(0)$$
$$y' \succ y \Leftrightarrow 0.1u(25) + 0.9u(0) > 0.11u(5) + 0.89u(0)$$

Rearranging this yields

$$x \succ x' \Leftrightarrow 0.11u(5) - 0.1u(25) - 0.01u(0) > 0$$
$$y' \succ y \Leftrightarrow -0.11u(5) + 0.1u(25) = 0.01u(0) > 0$$

Since the left hand side of the second inequality equals the negative of the left hand side of the first inequality and the right hand side is 0 for both. Hence it is impossible to for both inequalities to be true simultaneuously. In other words, there is no utility function u that is compatible with these preferences.

3. **Separability.**

(a) Consider the following reward matrix in a zero-sum game:

$$\begin{pmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{pmatrix}$$

Determine under which conditions this game is separable. For the case where it is separable, determine the solution(s).

Separability for this 2×2 zero-sum game means that there are u, v, x and y such that

$$\begin{pmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{pmatrix} = \begin{pmatrix} \text{u+x} & \text{u+y} \\ \text{v+x} & \text{v+y} \end{pmatrix}$$

This condition can be expressed as a system of linear equations with 4 variables:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{u} \\ \text{v} \\ \text{x} \\ \text{y} \end{pmatrix} = \begin{pmatrix} \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{pmatrix}$$

Swapping second and third row and subtracting the first row from the (new) third row yields

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{u} \\ \text{v} \\ \text{x} \\ \text{y} \end{pmatrix} = \begin{pmatrix} \text{a} \\ \text{c} \\ \text{b-a} \\ \text{d} \end{pmatrix}$$

and subtracting the (new) second row from the forth yields

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \text{u} \\ \text{v} \\ \text{x} \\ \text{y} \end{pmatrix} = \begin{pmatrix} \text{a} \\ \text{c} \\ \text{b-a} \\ \text{d-c} \end{pmatrix}$$

So $b - a = d - c$ is a necessary condition for this game to be separable and the last row is redundant. Assuming the condition is fulfilled this is an underdetermined system. The game is separable. After choosing x the other parts of the solution are

$$y = b - a + x \qquad v = c - x \qquad u = a - x.$$

(b) Is the following zero-sum game separable?

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

If yes, determine the solution(s).

No, because $b - a = 1 - (-1) = 2$, but $d - c = -1 - 1 = -2$.

(c) Is the following zero-sum game separable?

$$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$$

If yes, determine the solution(s).

Yes, because $b - a = 1 - (-1) = 2$ and $d - c = 2 - 0 = 2$. Using the solution given above, fix x and then

$$y = 1 - (-1) + x = 2 + x \qquad v = 0 - x = -x \qquad u = -1 - x.$$

(d) Is the rock-paper-scissors game is a zero-sum game with payoff matrix

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

(*see also Lecture notes Week 6 Friday at 4pm*). Is it separable? If yes, determine the solution(s).

This game is not separable. Obviously, a solution would need to include $u = -x, v = -y, w = -z$, because of the diagonal. After reducing the remaining conditions using symmetry, only three conditions remain, which can be expressed in matrix form:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Simple matrix operations show that this has no solution. Hence the game is not separable.

4. **Game with dominant strategies.** Consider a game with the following payoff matrix:

	α	β	γ
a	(5,6)	(3,4)	(2,5)
b	(3,7)	(6,10)	(8,11)
c	(4,8)	(8,5)	(10,4)

Using iterated elimination of dominated strategies show that there is a single strategy which the players should, under the assumption that *rationality is common knowledge*, adopt deterministically.

Player 1 should always prefer c to b. b is eliminated. Knowing that player 1 will play a or c, player 2 optimises their strategy for the reduced payoff matrix:

	α	β	γ
a	(5,6)	(3,4)	(2,5)
c	(4,8)	(8,5)	(10,4)

Player 2 should prefer α to both β and γ as it always provides a higher payoff in this reduced game. Player 1 does addresses the further reduced payoff matrix:

	α
a	(5,6)
c	(4,8)

Finally, a always provides a higher payoff than c and player 1 will choose to play a.
Thus we end up with the rationally-justified solution that player 1 plays a and player 2, α.

5. **Prisoner's trilemma.** Consider a prisoner's trilemma in which, as well as the option of staying silent or betraying their partner, they can confess — admitting that both of them were involved in the crime. This leads to a payoff matrix:

	S	B	C
S	(-1,-1)	(-5,0)	(-5,-4)
B	(0,-5)	(-4,-4)	(-4,-4)
C	(-4,-5)	(-4,-4)	(-4,-4)

Using an argument involving domination and separability, explain what a rational player should do, and why.

For each player move C is dominated by move B: they should, therefore, never confess because simply betraying their partner will never lead to a worse position for themself, whilst it may improve their payoff. This leads to the following reduced payoff matrix:

	S	B
S	(-1,-1)	(-5,0)
B	(0,-5)	(-4,-4)

This is the usual prisoners dilemma. Using a separability argument, for each player betraying their partner is the optimal strategy. Actually, this strategy can also be justified by a domination argument.

6. **Zero-sum game.** Consider a zero sum game with the following payoff matrix (for player 1; remember player 2 has payoffs corresponding to the negative of those of player 1 in a zero sum game):

	δ_1	δ_2	δ_3	δ_4	δ_5
d_1	0	1	5	6	8
d_2	10	6	5	2	3

(a) What is player 1's maximin mixed strategy? *It is helpful to use a graphical method.*

First, notice that δ_5 is dominated by δ_4, so it does not need to be considered for optimal strategies. A mixed strategy for player 1 denoted by $\overline{x} = (x, 1-x)$ means d_1 is played with probability x and d_2 is played with probability $1-x$. The maximin strategy has associated probabilities $\overline{x}^\star = (x^\star, 1-x^\star)$ with $x^\star$ chosen to maximise the expected return obtained if player 2 makes the worst possible move.

The expected return R_k for player 1 against a pure strategy δ_k for player 2 is:

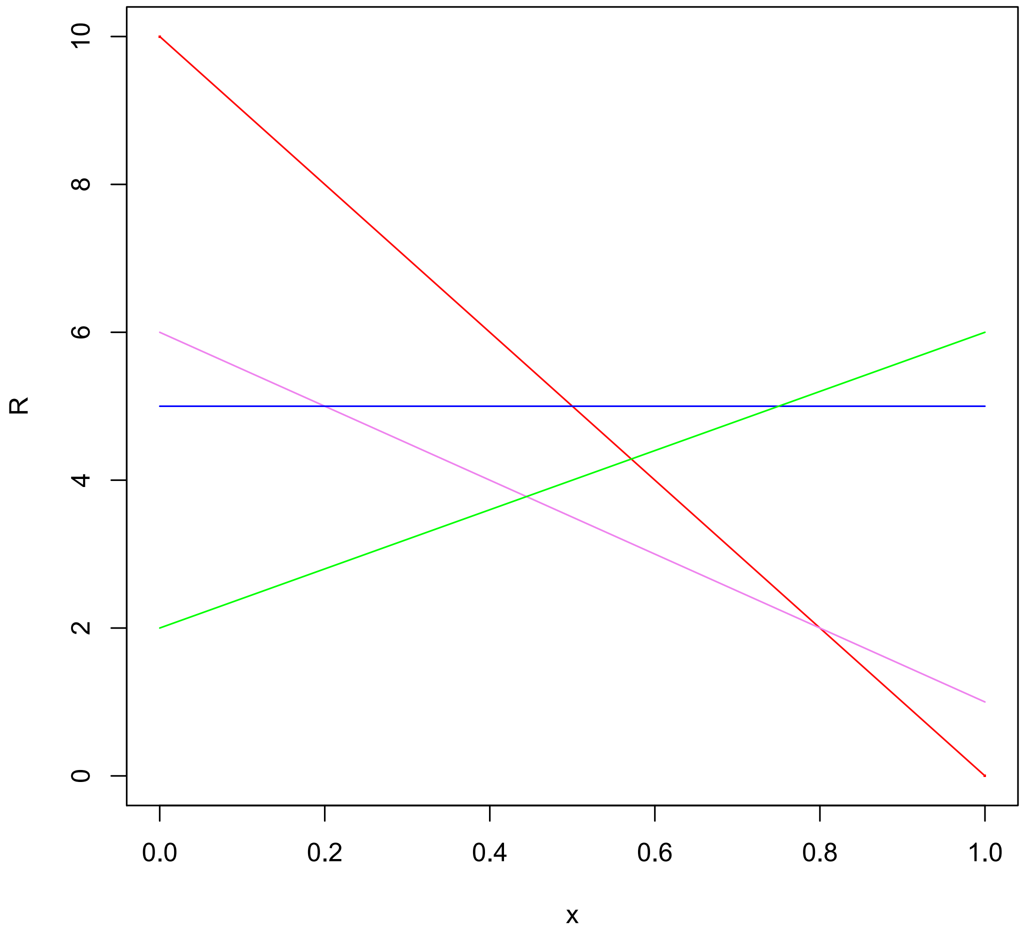
$$R_k = E[R(\overline{x}, \delta_k)] = xR(d_1) + (1-x)R(d_2)$$

The relevant strategies for player 2 are $k = 1, 2, 3, 4$ and the returns are:

$$\begin{aligned} R_1 &= 10(1-x) = -10 + 10 \\ R_2 &= x + 6(1-x) = -5x + 6 \\ R_3 &= 5x + 5(1-x) = 5 \\ R_4 &= 6x + 2(1-x) = 4x + 2 \end{aligned}$$

Plotting expected return against x for each of player 2's possible moves using red for δ_1, violet for δ_2, blue for δ_3 and green for δ_4 yields the figure below.

From the figure below it's clear that δ_2 and δ_4 have associated lines which intersect at the maximin point. Checking by calculation, it is clear that this intersection occurs at $x = 4/9$ at which point player 1 has an expected reward of $34/9$.



(b) What is player 2's maximin mixed strategy?

Player 2 may use the fact that the value of the game is the expected reward of player 1 at their maximin strategy (i.e. 34/9) and that player 1 only has to consider moves δ_2 and δ_4 (knowing that they will do better if player 2 plays any other strategy). Thus their maximin mixed strategy is $(0, y^\star, 0, 1 - y^\star)$ with $y^\star$ chosen to achieve a reward for player 1 of at most 34/9. Thus, $y^\star + 6(1 - y^\star) \leq 34/9$ and $6y^\star + 2(1 - y^\star) \leq 34/9$. And we must have $y^\star = 4/9$.

(c) What is the value of this game?

The value of the game is the expected reward of the first player with both adopting their maximin mixed strategy: 34/9.

Note: The weights in the mixed strategies happen to be the same in player 1 and player 2, but that is not generally true.

7. **(*) Plus one.** Each player chooses a number from $\{1, 2, ..., n\}$ and writes it down on a piece of paper; then the players compare the two numbers. If the numbers differ by one, the player with the higher number wins \$1 from the other player. If the players? choices differ by two or more, the player with the higher number pays \$2 to the other player. In the event of a tie, no money changes hands.

(a) What is the payoff matrix of the game?

(b) Use the concept of domination to reduce the game to a 3 by 3 game.

(c) Solve the reduced game.

For a solution see Example 3.3 (page 56) in Y. Peres: "Game theory, alive", Lecture notes, Department of Statistics at UC Berkeley, available at `www.stat.berkeley.edu/~peres/gtlect.pdf`. See also the same link at the module website, additional material on the right hand side.

8. **Row and column swapping.** Let M be the pay-off matrix for a zero sum game in which $D = \Delta$ and $|D| = |\Delta| = n$. Denote by $M((i_1, i_2), (j_1, j_2))$ the pay-off matrix obtained by permuting rows i_1 , i_2 and columns j_1 , j_2 in M. Suppose that for each relabelling of rows swapping i_1, i_2 there exists a relabelling of columns j_1, j_2 such that

$$M((i_1, i_2), (j_1, j_2)) = M.$$

(a) Show that if P1 has a unique maximin mixed strategy then it must be of the form $(1/n, ..., 1/n)$.

Permuting the order of the opponents strategies doesn't affect the maximin strategy of player 1. Consequently, if exchanging i_1 and i_2 always leads to the same payoff matrix (up to permutation of the opponent's strategies) then the mixed strategy must attach the same probability mass to moves d_{i_1} and d_{i_2}. As this applies for all i_1 and i_2, all moves must have the same associated probability mass. By normalisation/coherence this probability must be $1/n$ so that the total probability of making a move is 1.

(b) What is the value of this game if $M = (M_{ij})$ and $M_{ij} = -M_{ji}$ for all i, j?

We have an antisymmetric payoff matrix which means that the game is symmetric in the sense that $S(\delta, d) = R(d, \delta)$. By symmetry, player 2 has the same unique maximin solution as player 1. The value of the game is:

$$V = \sum_{i,j} p(d_i) M_{ij} p(d_j) = p^T M p = \frac{1}{n^2} \sum_{i,j} M_{i,j} = 0$$

where the final equality follows from the antisymmetry of M.