Modelling:
Mathematics as bridge between theory & application

"The instrument that mediates between theory and practice, between thought and observation, is mathematics; it builds the connecting bridge and makes it stronger and stronger. Thus it happens that our entire present-day culture, insofar as it rests on intellectual insight into and harnessing of nature, is founded on mathematics."

David Hilbert

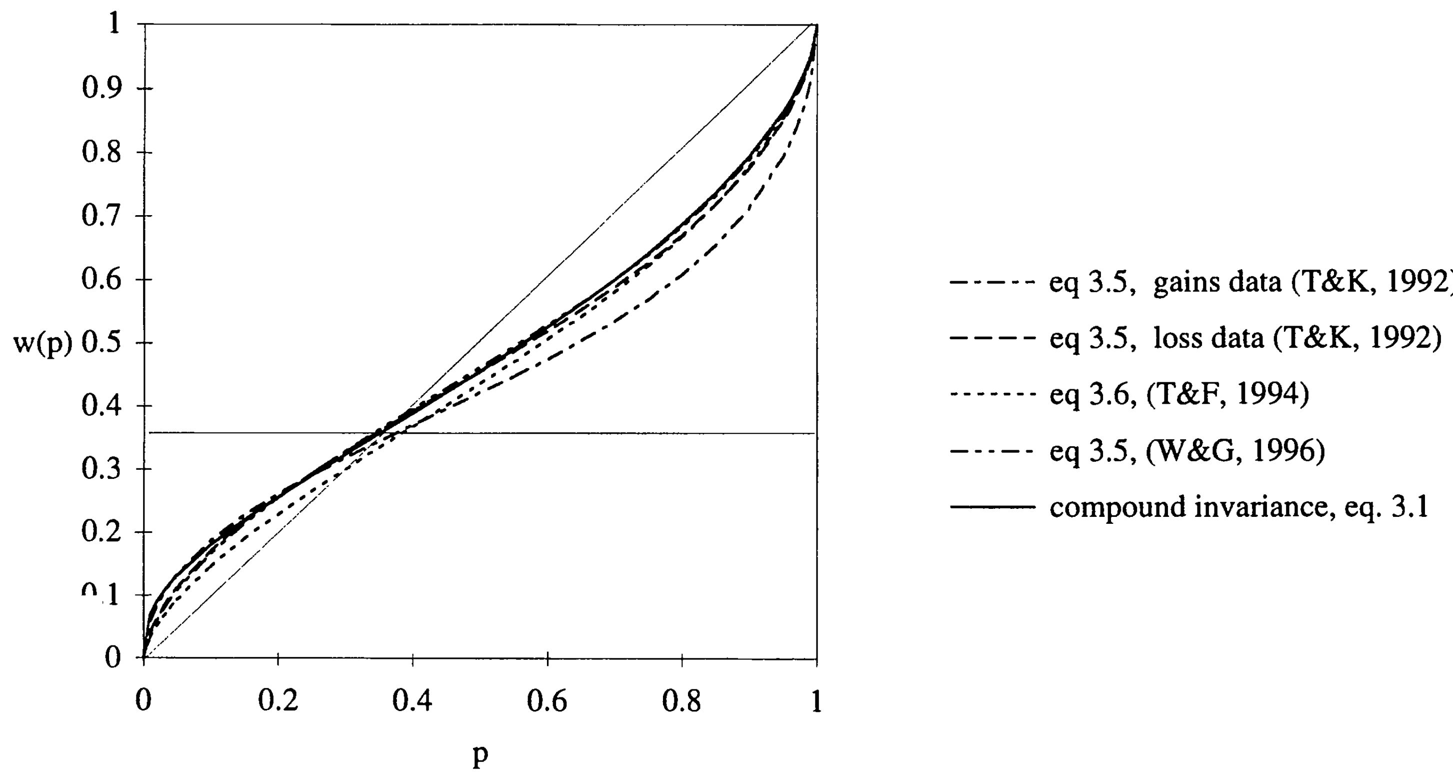
In Königsberg on 8 September 1930, David Hilbert addressed the yearly meeting of the Society of German Natural Scientists and Physicians (Gesellschaft der Deutschen Naturforscher und Ärzte). Generally regarded as the world's leading mathematician at the time, Hilbert was born and educated in Königsberg and spent the early years of his career there.
Full text of the speech in English and German at url below, including audio file:

http://math.sfsu.edu/smith/Documents/HilbertRadio/HilbertRadio.pdf

Ingredients: Prospect theory probability weighting function

Four-fold pattern of risk attitudes

regressive—intersecting the diagonal from above,
asymmetric—with fixed point at about 1/3,
s-shaped—concave on an initial interval and convex beyond that,
reflective—assigning equal weight to a given loss-probability as to a given gain-probability.



Ingredients: Prospect theory value function

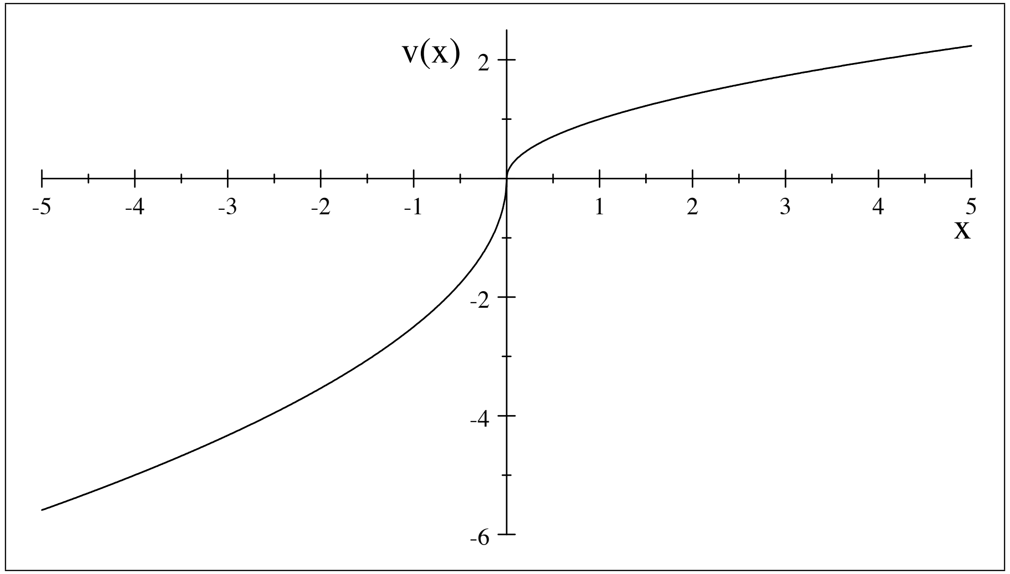
Definition *(Tversky and Kahneman, 1979). A utility function, $v(x)$, is a continuous, strictly increasing, mapping $v : \mathbb{R} \to \mathbb{R}$ that satisfies:*

1. $v(0) = 0$ *(reference dependence).* (*)

2. $v(x)$ *is concave for* $x \geq 0$ *(declining sensitivity for gains).*

3. $v(x)$ *is convex for* $x \leq 0$ *(declining sensitivity for losses).*

4. $-v(-x) > v(x)$ *for* $x > 0$ *(loss aversion).*



$$v(x) = \begin{cases} x^{\alpha} & x \geq 0 \\ -\lambda(-x)^{\beta} & x < 0 \end{cases}$$

$\alpha > 0$: degree of risk aversion in gains

$\beta > 0$: degree of risk seeking in losses

$\lambda > 0$: degree of loss aversion

(*) Note: reference point depends on context, so may be changed

Prospect theory: Value of a prospect

Given a prospect $b = (x, p; y, q)$

$$x, y \in \mathcal{R}, p, q \in [0, 1], p + q = 1$$

In EUT the aim is to maximise

$$U(b) = E[u(b)] = pu(x) + qu(y)$$

In PT the aim is to maximise

$$V(b) = w(p)u(x) + w(q)u(y)$$

- analogy to expectation
- not an expectation (distorted probabilities)
- can be generalised to more than two outcomes

Modelling: Prospect theory (PT)

Decision rule:

Optimise the value

$$V(b) = w(p)u(x) + w(q)u(y)$$

Non-EUT behaviour:

- *Certainty effect:* e.g. in Allais type problems
- *Reflection effect:* losses are associated with different behaviours than gains
- *Framing effect:* e.g. in disease example by K&T

Question:

Can prospect theory explain observed decision making behaviour that was not explicable by EUT?

D Kahneman and A Tversky, *Prospect Theory: An Analysis of Decisions under Risk,* Econometrica, Vol. 47, March 1979, Number 2, pp. 263-291.

Certainty effect: Common consequence - empirical study

(Similar to Allais paradox)

PROBLEM 1: Choose between

A:	2,500 with probability	.33,	B:	2,400 with certainty.
	2,400 with probability	.66,		
	0 with probability	.01;		

PROBLEM 2: Choose between

C:	2,500 with probability	.33,	D:	2,400 with probability	.34,
	0 with probability	.67;		0 with probability	.66.

Is there a connection between P1 and P2?

Certainty effect: Common consequence - empirical study

(Similar to Allais paradox)

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PROBLEM 2: Choose between

C:	2,500 with probability	.33,	D:	2,400 with probability	.34,
	0 with probability	.67;		0 with probability	.66.

P2 = P1 - (2400, .66)

Note the shifting a 0.01 probability from 0 gain to 2,400 gain (from A to B and from C to D).

Certainty effect: Common consequence - empirical study

PROBLEM 1: Choose between *(Similar to Allais paradox)*

A:	2,500 with probability	.33,	B:	2,400 with certainty.
	2,400 with probability	.66,		
	0 with probability	.01;		
$N = 72$	[18]	Subjects prefer B	[82]*	[ ] are %

PROBLEM 2: Choose between

C:	2,500 with probability	.33,	D:	2,400 with probability	.34,
	0 with probability	.67;		0 with probability	.66.
$N = 72$	[83]*	Subjects prefer C	[17]		

P2 = P1 - (2400, .66)

- EUT: A<B equivalent to C<D
- Empirical evidence contradicts EUT
- PT can explain A<B & C>D

Certainty effect: Common ratio - empirical study

(Also based on Allais)

PROBLEM 3:

A: (4,000,.80), or B: (3,000).

$N = 95$ [20] [80]* [ ] are %

PROBLEM 4:

C: (4,000,.20), or D: (3,000,.25).

$N = 95$ [65]* [35]

Is there a connection between P3 and P4?

Certainty effect: Common ratio - empirical study

(Also based on Allais)

PROBLEM 3:

A: (4,000,.80), or B: (3,000). **Subjects prefer B**

$N = 95$ [20] [80]* [ ] are %

PROBLEM 4:

C: (4,000,.20), or D: (3,000,.25). **Subjects prefer C**

$N = 95$ [65]* [35]

P4 = 0.25 * P3

- EUT: A<B equivalent to C<D
- Empirical evidence contradicts EUT

Common ratio: EUT applied to example with P3 & P4

PROBLEM 3:

A: (4,000,.80), or B: (3,000).

$N = 95$ [20] [80]*

[ ] are %

PROBLEM 4:

C: (4,000,.20), or D: (3,000,.25).

$N = 95$ [65]* [35]

$$x = 4000, y = 3000, \lambda = 4/5, p_1 = 1, p_2 = 0.25$$

$$R_i = (x, \lambda p_i) \qquad S_i = (y, p_i) \qquad i = 1, 2$$

$$R_1 \succ S_1 \;\Leftrightarrow\; R_2 \succ S_2$$

EUT

Common ratio: General form under EUT

Given a pair of prospects:

$$R = (x, \lambda p) \qquad S = (y, p) \qquad (x > y, 0 < \lambda < 1)$$

EUT says: $R \succ S \;\Leftrightarrow\; u(x)\lambda p > u(y)p \;\Leftrightarrow\; \lambda > \dfrac{u(y)}{u(x)}$

Note: This in **independent** of the probability p

Common ratio: General form under EUT

Given a pair of prospects:

$$R = (x, \lambda p) \qquad S = (y, p) \qquad (x > y, 0 < \lambda < 1)$$

EUT says: $R \succ S \;\Leftrightarrow\; u(x)\lambda p > u(y)p \;\Leftrightarrow\; \lambda > \frac{u(y)}{u(x)}$

(Assuming $u(0) = 0$ to keep argument concise)

Note: This in **independent** of the probability p

Hence, for two pairs of prospects:

$$R_i = (x, \lambda p_i) \qquad S_i = (y, p_i) \qquad i = 1, 2$$

$$R_1 \succ S_1 \;\Leftrightarrow\; R_2 \succ S_2$$

Common ratio: EUT applied to example with P3 & P4

PROBLEM 3:

A:	(4,000,.80),	or	B:	(3,000).
$N = 95$	[20]			[80]*

PROBLEM 4:

C:	(4,000,.20),	or	D:	(3,000,.25).
$N = 95$	[65]*			[35]

$$x = 4000, y = 3000, \lambda = 4/5, p_1 = 1, p_2 = 0.25$$

$$R_i = (x, \lambda p_i) \qquad S_i = (y, p_i) \qquad i = 1, 2$$

$$R_1 \succ S_1 \;\Leftrightarrow\; R_2 \succ S_2$$

EUT

Common ratio: EUT applied to example P3 & P4

PROBLEM 3:

A:	(4,000,.80),	or	B:	(3,000).
$N = 95$	[20]			[80]*

PROBLEM 4:

C:	(4,000,.20),	or	D:	(3,000,.25).
$N = 95$	[65]*			[35]

Observed

$$R_1 \prec S_1$$

$$R_2 \succ S_2$$

$$x = 4000, y = 3000, \lambda = 4/5, p_1 = 1, p_2 = 0.25$$

$$R_i = (x, \lambda p_i) \qquad S_i = (y, p_i) \qquad i = 1, 2$$

$$R_1 \succ S_1 \;\Leftrightarrow\; R_2 \succ S_2$$

EUT

EUT does not describe the observed behaviour.

Common ratio: General form under PT

Given two prospects:

$$R = (x, \lambda p) \qquad S = (y, p) \qquad (x > y, 0 < \lambda < 1)$$

Prospect theory (PT) says:

$$R \succ S \;\Leftrightarrow\; v(x)w(\lambda p) > v(y)w(p) \;\Leftrightarrow\; \frac{w(\lambda p)}{w(p)} > \frac{v(y)}{v(x)}$$

Note: This in **not independent** of the probability p

Common ratio: PT applied to example P3 & P4

$$R_i = (x, \lambda p_i) \qquad S_i = (y, p_i) \qquad i = 1, 2$$

$$x = 4000, y = 3000, \lambda = 4/5, p_1 = 1, p_2 = 0.25$$

Observed $\quad R_1 \prec S_1 \quad$ and $\quad R_2 \succ S_2$

Common ratio: PT applied to example P3 & P4

$$R_i = (x, \lambda p_i) \qquad S_i = (y, p_i) \qquad i = 1, 2$$

$$x = 4000, y = 3000, \lambda = 4/5, p_1 = 1, p_2 = 0.25$$

Observed $\quad R_1 \prec S_1 \quad$ and $\quad R_2 \succ S_2$

In PT: $\quad \dfrac{w(\lambda p_1)}{w(p_1)} < \dfrac{v(y)}{v(x)} \quad$ and $\quad \dfrac{w(\lambda p_2)}{w(p_2)} > \dfrac{v(y)}{v(x)}$

In other words: $\quad \dfrac{w(0.8)}{w(1)} < \dfrac{v(y)}{v(x)} < \dfrac{w(0.2)}{w(0.25)}$

If there is a value function and a probability weighting function that fulfils this, then PT **can describe** the observed behaviour?

Common ratio: PT applied to example P3 & P4

Is there a value function and a probability weighting function such that the inequality below is correct?

$$\frac{w(0.8)}{w(1)} < \frac{v(y)}{v(x)} < \frac{w(0.2)}{w(0.25)} \qquad (*)$$

Common ratio: PT applied to example P3 & P4

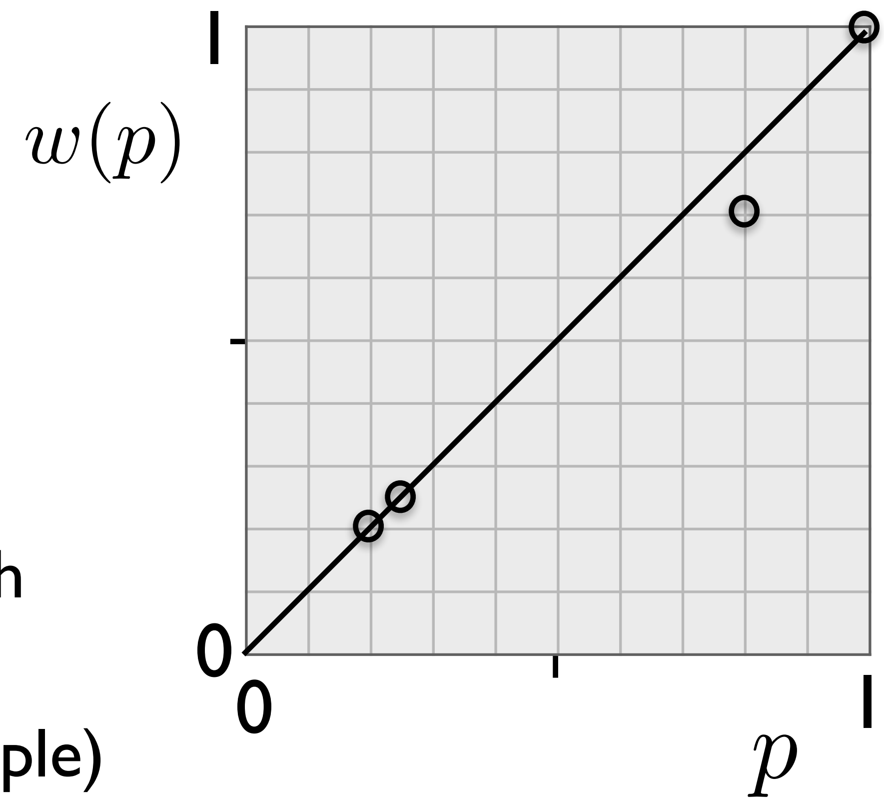
Is there a value function and a probability weighting function such that the inequality below is correct?

$$\frac{w(0.8)}{w(1)} < \frac{v(y)}{v(x)} < \frac{w(0.2)}{w(0.25)} \qquad (*)$$

Choose, for example: $v = Id$

$$w(1) = 1, w(0.8) = 0.7$$

$$w(0.25) = 0.25, w(0.2) = 0.2$$



- Probability weighting function with such values exists
- Only a small deformation (in this example)

Common ratio: PT applied to example P3 & P4

Is there a value function and a probability weighting function such that the inequality below is correct?

$$\frac{w(0.8)}{w(1)} < \frac{v(y)}{v(x)} < \frac{w(0.2)}{w(0.25)} \qquad (*)$$

Choose, for example: $v = Id$

$$w(1) = 1, w(0.8) = 0.7, w(0.25) = 0.25, w(2) = 0.2$$

Then (*) means $\frac{0.7}{1} < \frac{3000}{4000} < \frac{0.2}{0.25}$

In other words, $0.7 < 0.75 < 0.8$ which is true.

Common ratio: PT applied to example P3 & P4

Is there a value function and a probability weighting function such that the inequality below is correct?

$$\frac{w(0.8)}{w(1)} < \frac{v(y)}{v(x)} < \frac{w(0.2)}{w(0.25)} \qquad (*)$$

Choose, for example: $v = Id$

$$w(1) = 1, w(0.8) = 0.7, w(0.25) = 0.25, w(0.2) = 0.2$$

Then (*) means $\frac{0.7}{1} < \frac{3000}{4000} < \frac{0.2}{0.25}$

In other words, $0.7 < 0.75 < 0.8$ (which is true).

And these values can be realised by functions suitable for PT.

Hence, PT **can describe** the observed behaviour!

Modelling: Prospect theory (PT)

Question:

Can prospect theory explain behaviour deviating from EUT?

- Certainty effect

Answer: Yes, PT **can describe** the observed behaviour!

Used probability weighting.

Value function was not relevant for this.

Reflection effect: Observations

Observation:
For losses, people's risk aversion changes to risk seeking.

PREFERENCES BETWEEN POSITIVE AND NEGATIVE PROSPECTS

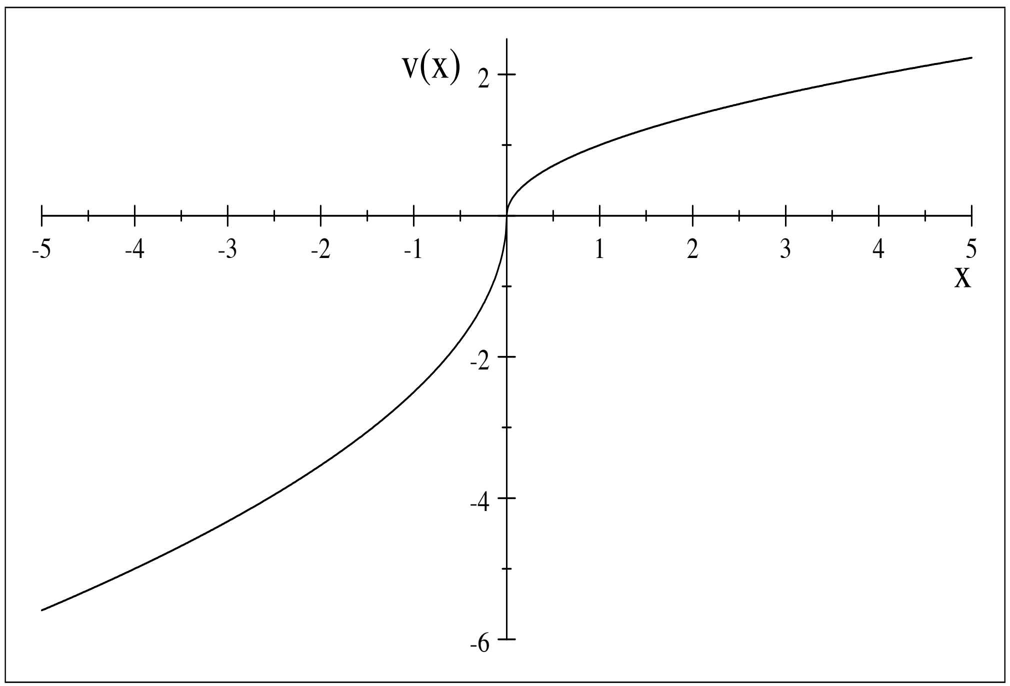
Positive prospects				Negative prospects			
Problem 3:	(4,000, .80)	<	(3,000).	Problem 3′:	(−4,000, .80)	>	(−3,000).
$N = 95$	[20]		[80]*	$N = 95$	[92]*		[8]
Problem 4:	(4,000, .20)	>	(3,000, .25).	Problem 4′:	(−4,000, .20)	<	(−3,000, .25).
$N = 95$	[65]*		[35]	$N = 95$	[42]		[58]
Problem 7:	(3,000, .90)	>	(6,000, .45).	Problem 7′:	(−3,000, .90)	<	(−6,000, .45).
$N = 66$	[86]*		[14]	$N = 66$	[8]		[92]*

Certainty effect implies
risk averse preferences:
Sure lower gain preferred to higher probable gain.

Certainty effect implies
risk seeking preferences:
Probable higher loss preferred to sure lower loss.

Reflection effect: Solution

PT replaces utility function by asymmetric value function:



$$v(x) = \begin{cases} x^\alpha & x \geq 0 \\ -\lambda(-x)^\beta & x < 0 \end{cases}$$

$\alpha > 0$: degree of risk aversion in gains

$\beta > 0$: degree of risk seeking in losses

$\lambda > 0$: degree of loss aversion

Framing effect: EUT vs PT

- EUT assumes that decision making is invariant to the manner of representation.
- Empirical evidence suggest this is incorrect (e.g. disease example with life vs death framings)
- PT is more flexible (e.g. gains and losses modelled differently)

Modelling: Prospect theory (PT)

Question:

Can prospect theory explain behaviour deviating from EUT?

- Reflection effect
- Framing effect

Answer:

Yes, through value function (convex in loss domain, concave for gains)

D Kahneman and A Tversky, *Prospect Theory: An Analysis of Decisions under Risk*, Econometrica, Vol. 47, March 1979, Number 2, pp. 263-291.

Examples: Empirically shown deviations from normative theory

- Gambler's fallacy, inverse gambler's fallacy, belief in hot hand
- Random sequences generation biases (starting value, runs)
- Clustering illusion
- Anchoring bias (with related and unrelated information)
- Framing effect
- Availability bias
- Certainty effect (Allais paradox)
- Disjunction effect
- Base rate neglect
- Relevant in finance, e.g. Disposition effect

Heuristics & biases: Affect heuristic

Affect plays an important role in guiding judgments and decisions.

Here, *affect* means the specific quality of 'goodness' or 'badness'

- experienced as a feeling state (conscious or subconscious)
- demarcating a positive or negative quality of a stimulus

Affective responses occur rapidly and automatically

Reliance on such feelings: A*ffect heuristic* (P. Slovic)

Heuristics & biases: Affect heuristic

Affective responses occur rapidly and automatically

Potentially anyone may be subject to this, e.g.:

- political/ideological obsessions
- exam anxiety (quite common, see also (*))
- stress among traders (e.g. *https://en.wikipedia.org/wiki/Nick_Leeson*)
- jealousy (e.g. Hamlet)
- many more…

http://www2.warwick.ac.uk/fac/sci/statistics/courses/studying/examstress/

Example: Weather & stock market

"*The discovery that the **weather in New York City has a long history of significant correlation with major stock indexes** supports the view that investor psychology influences asset prices.*"

Saunders (1993)

Example: Weather & stock market

"Psychological evidence and casual intuition predict that sunny weather is associated with upbeat mood. This paper examines the relation between morning sunshine at a country's leading stock exchange and market index stock returns that day at 26 stock exchanges internationally from 1982-97. ***Sunshine is strongly significantly correlated with daily stock returns.***

After controlling for sunshine, rain and snow are unrelated to returns. There were positive net-of-transaction costs profits to be made from substantial use of weather-based strategies, but the magnitude of the gains was fairly modest. These findings are difficult to reconcile with fully rational pricesetting."

Hirshleifer and Shumway: *Good Day Sunshine: Stock Returns and the Weather,* The Journal of Finance react-text: 53 58(3), 2001

Further links and resources about behavioural finance

Robert Grosse, (2012) "Bank regulation, governance and the crisis: a behavioral finance view", Journal of Financial Regulation and Compliance,
Vol. 20 Iss: 1, pp.4 - 25

http://www.fca.org.uk/your-fca/documents/occasional-papers/occasional-paper-1
check out annex in particularly

Interview with Bank director Greg B Davies on the use of behavioural finance theory in real world banking:

www.seeitmarket.com/interview-greg-b-davies-barclays-behavioural-finance-13577/

Robert Shiller lecture on behavioural finance and prospect theory:

https://www.youtube.com/watch?v=chSHqogx2CI

“Behavioral Finance” (review) *http://www.sfb504.uni-mannheim.de/publications/dp03-14.pdf*

Markus Glaser, Markus Nöth, and Martin Weber (University of Mannheim)

“The Behavior of Individual Investors” (review) http://faculty.haas.berkeley.edu/odean/

Brad M. Barber and Terrance Odean (UC Davis, UC Berkeley)