Subjective Probability

What is the probability of a nuclear war occurring next year?

- First, we must be precise about the question.
- We can't appeal to symmetry of geometry.
- We can't appeal meaningful to an infinite ensemble of experiments.
- We *can* form an individual, *subjective* opinion.

If we adopt this subjective view, difficulties emerge:

- How can we quantify degree of belief?
- Will the resulting system be internally consistent?
- What does our calculations actually tell us?

A Behavioural Definition of Probability

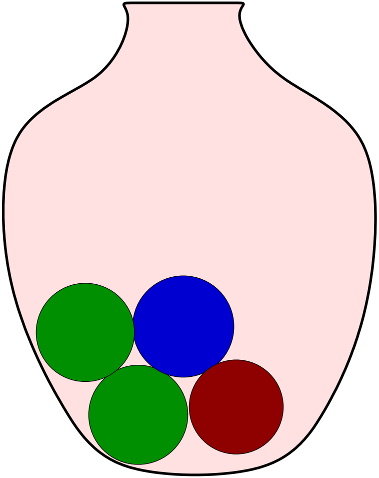
- Consider a *bet*, $b(M, A)$, which pays a reward M if A happens and nothing if A does not happen.
- Let $m(M, A)$ denote the maximum that *You* would be prepared to pay for that bet.
- Two events A_1 and A_2 are equally probable if $m(M, A_1) = m(M, A_2)$.
- Equivalently $m(M, A)$ is the minimum that *You* would accept to offer the bet.
- A value for $m(M, \Omega \setminus A)$ is implied for a rational being...

Personal probability must be a matter of action!

A Bayesian View of Symmetry

- If $A_1, \ldots, A_k$ are *disjoint/mutually exclusive*, *equally likely* and *exhaustive*
$$\Omega = A_1 \cup \cdots \cup A_k,$$
- then, for any i,
$$\mathbb{P}(A_i) = \frac{1}{k}.$$
- Think of the examples we saw before...

Balls in Urns

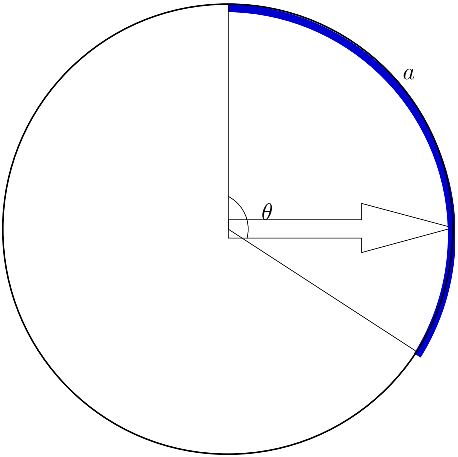


- Let $\mathcal{I}$ be (discrete) a set of colours.
- An urn contains n_i balls of colour i.
- The probability that a drawn ball has colour i is:

$$\frac{n_i}{\sum_{j \in \mathcal{I}} n_j}$$

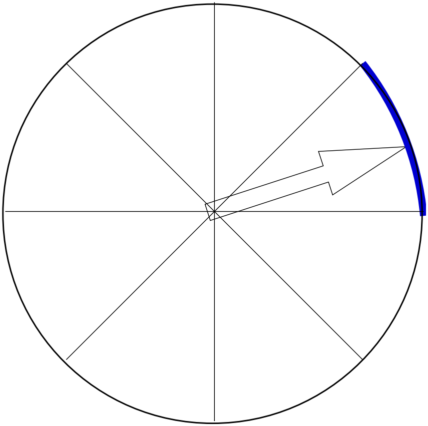
We assume that the colour of the ball does not influence its probability of selection.

Spinners



- $\mathbb{P}$[Stops in purple] $= a$
- Really a statement about physics.
- What do we mean by probability?

Discretised Spinners



- Each of k segments is equally likely:

$$\mathbb{P}[\text{Stops in purple}] = 1/k$$

- k may be very large.
- Combinations of arcs give rational lengths.
- Limiting approximations give real lengths.
- We can describe *most* subsets this way [ST213].

Example (Selling a house)

- You wish to sell a house, for at least £250,000.
- On Monday you receive an offer of X.
- You must accept or decline this offer immediately.
- On Tuesday you will receive an offer of Y.
- What should you do?

- $\Omega = \{(x, y) : x, y \geq £100,000\}$
- But, we only care about events of the form:

$$\{(i, j) : i < j\} \text{ and } \{(i, j) : i > j\}$$

- Including some others ensures that we have an algebra:

$$\{(i, j) : i = j\} \quad \{(i, j) : i \neq j\} \quad \{(i, j) : i \leq j\} \quad \{(i, j) : i \geq j\} \quad \emptyset \quad \Omega$$

Example (House selling again)

- The three atoms in this case were:

$$\{(i,j) : i > j\} \qquad \{(i,j) : i = j\} \qquad \{(i,j) : i < j\}$$

- No reason to suppose all three are equally likely.
- If our bidders are believed to be *exchangeable*

$$\mathbb{P}(\{(i,j) : i > j\}) = \mathbb{P}(\{(i,j) : i < j\})$$

- So we arrive at the conclusion that:

$$\mathbb{P}(\{(i,j) : i > j\}) = \mathbb{P}(\{(i,j) : i < j\}) \leq \frac{1}{2}$$
$$\mathbb{P}(\{(i,j) : i = j\}) \geq 0$$

- One strategy would be to accept the first offer if $i > k$...

Elicitation

What probabilities does someone assign to a complex event?

- We can use our behavioural definition of probability.
- The *urn* and *spinner* we introduced before have probabilities which we all agree on.
- We can use these to *calibrate* our personal probabilities.
- When does an *urn* or *spinner* bet have the same value as one of interest.
- There are some difficulties with this approach, but it's a starting point.

A First Look At Coherence

- Consider a collection of events $A_1, \ldots, A_n$.
- If
  - the elements of this collection are disjoint: $\forall i \neq j : A_i \cap A_j = \emptyset$
  - the collection is exhaustive: $\cup_{i=1}^n A_i = \Omega$

  then a collection of probabilities $p_1, \ldots, p_n$ for these events is *coherent* if:
  - $\forall i \in \{1, \ldots, n\} : p_i \in [0, 1]$
  - $\sum_{i=1}^n p_i = 1$

Assertion: A *rational being* will adjust their personal probabilities until they are coherent.

Dutch Books

- A collection of bets which:
  - definitely won't lead to a loss, and
  - might make a profit

  is known as a Dutch book.

  A rational being would not accept such a collection of bets.
- If a collection of probabilities is incoherent, then a Dutch book can be constructed.

  A rational being must have coherent personal probabilities.

Example (Trivial Dutch Books)

- Consider two cases of incoherent beliefs in the coin-tossing experiment:

  Case 1 $P(\{H\}) = 0.4, P(\{T\}) = 0.4.$
  Case 2 $P(\{H\}) = 0.6, P(\{T\}) = 0.6.$

- To exploit our good fortune, in case 1:
  - *Place* a bet of $\pounds X$ on *both* possible outcomes.
  - Stake is $\pounds 2X$; we win $\pounds X/\frac{2}{5} = \pounds 5X/2$.
  - Profit is $\pounds(5/2 - 2)X = X/2$.
- In case 2:
  - *Accept* a bet of $\pounds X$ on *both* possible outcomes.
  - Stake is $\pounds 2X$; we lose $\pounds X/\frac{3}{5} = \pounds 5X/3$.
  - Profit is $\pounds(2 - 5/3)X = X/3$.

In betting: Payoff is Stake/p

Strategy: Either way you gain from having placed/ accepted a bet (simultaneously on each possible outcome)

Example (A Gambling Example)

Consider a horse race with the following odds:

Horse	Odds
Padwaa	7-1
Nutsy May Morris	5-1
Fudge Nibbles	11-1
Go Lightning	10-1
The Coaster	11-1
G-Nut	5-1
My Bell	10-1
Fluffy Hickey	15-1

Odds: offered by a bookie

If you had £100 available, how would you bet?

Example

My own collection of bets looked like this:

Horse	Odds	Stake
Padwaa	7-1	£14.38
Nutsy May Morris	5-1	£19.17
Fudge Nibbles	11-1	£9.58
Go Lightning	10-1	£10.46
The Coaster	11-1	£9.58
G-Nut	5-1	£19.17
My Bell	10-1	£10.45
Fluffy Hickey	15-1	£7.19

Stakes: my choices

Outcome: profit of

$$16 \times £7.19 - £99.99 = £(115.04 - 99.99) = £(15.05)$$

Example

My own collection of bets looked like this:

Horse	Odds	Implicit P.	Stake
Padwaa	7-1	0.125	£14.38
Nutsy May Morris	5-1	0.167	£19.17
Fudge Nibbles	11-1	0.083	£9.58
Go Lightning	10-1	0.091	£10.46
The Coaster	11-1	0.083	£9.58
G-Nut	5-1	0.167	£19.17
My Bell	10-1	0.091	£10.45
Fluffy Hickey	15-1	0.063	£7.19

Outcome: profit of

$$16 \times £7.19 - £99.99 = £(115.04 - 99.99) = £(15.05)$$

Example

My own collection of bets **P** **S**

Horse	Odds	Implicit P.	Stake	S/P
Padwaa	7-1	0.125	£14.38	£115.04
Nutsy May Morris	5-1	0.167	£19.17	£115.02
Fudge Nibbles	11-1	0.083	£9.58	£114.96
Go Lightning	10-1	0.091	£10.46	£115.06
The Coaster	11-1	0.083	£9.58	£114.96
G-Nut	5-1	0.167	£19.17	£115.02
My Bell	10-1	0.091	£10.45	£115.06
Fluffy Hickey	15-1	0.063	£7.19	£115.04

Outcome: profit of

$$16 \times £7.19 - £99.99 = £(115.04 - 99.99) = £(15.05)$$

Similarly for the other horses. **Hence have sure (risk-free) profit!**

Efficient Markets and Arbitrage

- The *efficient market hypothesis* states that the prices at which instruments are traded reflects all available information.
- In the world of economics a Dutch book would be referred to as an arbitrage opportunity: a risk-free collection of transactions which guarantee a profit.
- The *no arbitrage principle* states that there are no arbitrage opportunities in an efficient market at equilibrium.
- The collective probabilities implied by instrument prices are coherent.

Elicitation

What does she believe?

We need to obtain and quantify our clients beliefs.
Asking for a direct statement about personal probabilities doesn't usual work:

- $\mathbb{P}(A) + \mathbb{P}(A^c) \neq 1$
- Recall the British economy: people confuse belief with desire.

A better approach uses *calibration*: comparison with a standard.

Key: use standard presenting probabilities in a way familiar to the person.

Example (General Election Results)

Which party you think will win most seats in the next general election?

- Conservative
- Labour
- Liberal Democrat
- Green
- Monster-Raving Loony

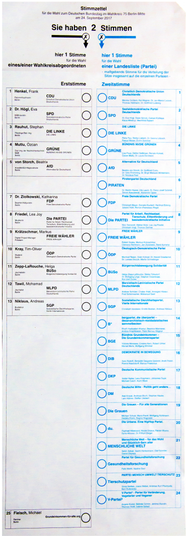
Consider the bet $b(\pounds 1, \text{Conservative Victory})$:

- You win £1 if the Conservative party wins.
- You win nothing otherwise.

Just for fun, not examinable!
Voting ballot Bundestagswahl September 2017

The Monster-Raving Loony party was a UK 1980s phenomenon… Germany more recently seems to have more and more of such movements - see ballot. A minimum of 5% of the votes is needed to be represented in Parliament, so there is a limit to the relevance of this.

Currently a coalition government is being formed by traditional parties: Christian democrats, Liberals, Greens, though having a 3 party coalition rather than the typical 2 is unusual, as is its nick name "Jamaica coalition".



Stimmzettel

für die Wahl zum Deutschen Bundestag im Wahlkreis 75 Berlin-Mitte am 24. September 2017

Sie haben 2 Stimmen

hier 1 Stimme für die Wahl eines/einer Wahlkreisabgeordneten

hier 1 Stimme für die Wahl einer Landesliste (Partei) - maßgebende Stimme für die Verteilung der Sitze insgesamt auf die einzelnen Parteien -

Erststimme			Zweitstimme		
1	Henkel, Frank	CDU	CDU	Christlich Demokratische Union Deutschlands	1
2	Dr. Högl, Eva	SPD	SPD	Sozialdemokratische Partei Deutschlands	2
3	Rauhut, Stephan	DIE LINKE	DIE LINKE	DIE LINKE	3
4	Mutlu, Özcan	GRÜNE	GRÜNE	BÜNDNIS 90/DIE GRÜNEN	4
5	von Storch, Beatrix	AfD	AfD	Alternative für Deutschland	5
			PIRATEN	Piratenpartei Deutschland	6
7	Dr. Ziolkowski, Katharina	FDP	FDP	Freie Demokratische Partei	7
8	Friedel, Lea Joy	Die PARTEI	Die PARTEI	Partei für Arbeit, Rechtsstaat, Tierschutz, Elitenförderung und basisdemokratische Initiative	8
9	Krätzschmar, Markus	FREIE WÄHLER	FREIE WÄHLER	FREIE WÄHLER	9
10	Kray, Tim-Oliver	ÖDP	ÖDP	Ökologisch-Demokratische Partei	10
11	Zepp-LaRouche, Helga	BüSo	BüSo	Bürgerrechtsbewegung Solidarität	11
12	Tawil, Mohamad	MLPD	MLPD	Marxistisch-Leninistische Partei Deutschlands	12
13	Niklaus, Andreas	SGP	SGP	Sozialistische Gleichheitspartei, Vierte Internationale	13
			B*	bergpartei, die überpartei – ökoanarchistisch-realdadaistisches sammelbecken	14
			BGE	Bündnis Grundeinkommen – Die Grundeinkommenspartei	15
			DiB	DEMOKRATIE IN BEWEGUNG	16
			DKP	Deutsche Kommunistische Partei	17
			DM	Deutsche Mitte - Politik geht anders...	18
			Die Grauen	Die Grauen – Für alle Generationen	19
			du.	Die Urbane. Eine HipHop Partei.	20
			MENSCHLICHE WELT	Menschliche Welt – für das Wohl und Glücklich-Sein aller	21
			Gesundheitsforschung	Partei für Gesundheitsforschung	22
			Tierschutzpartei	PARTEI MENSCH UMWELT TIERSCHUTZ	23
			V-Partei³	V-Partei³ - Partei für Veränderung, Vegetarier und Veganer	24
25	Fielsch, Michael	Grundeinkommen-für-alle.org			

Translations (attempted…) of some of the rather unusual party names:

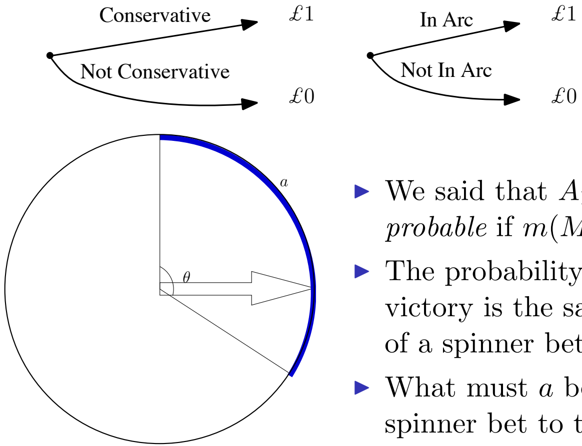
Die Partei - Partei für Arbeit, Rechtsstaat, Tierschutz, Eliteförderung und basisdemokratische Initiative

The Party - Party for work, constitutional stage, animal protection, promotion of elite and grassroot initiative

V-Partei - Partei für Veränderung, Vegetarier und Veganer

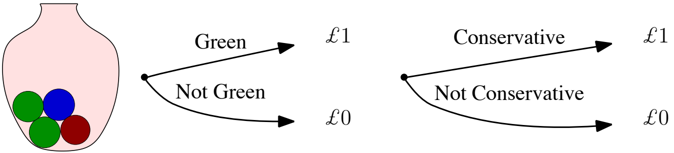
V-Party - For change, vegetarians and vegans

Behavioural Approach to Elicitation



- We said that A_1 and A_2 are *equally probable* if $m(M, A_1) = m(M, A_2)$.
- The probability of a Conservative victory is the same as the probability of a spinner bet of the same value.
- What must a be for us to prefer the spinner bet to the political one?

Eliciting With Urns Full of Balls



- If the urn contains:
  - n balls
  - g of which are green
- Increase g from 0 to n...
- Let $g^\star$ be such that
  - The real bet is preferred when $g = g^\star$.
  - The urn bet is preferred when $g = g^\star + 1$.
- This tells us that:
  - $\mathbb{P}(\text{C.}) \geq g^\star/n$
  - $\mathbb{P}(\text{C.}) < (g^\star + 1)/n$
- Nominal accuracy of $1/n$.

Why should subjective probabilities behave in the same way as our axiomatic system requires?

- We began with axiomatic probability.
- We introduce a subjective interpretation of probability.
- We wish to combine both aspects...

- We briefly looked at "coherence" previously.
- Now, we will formalise this notion.

Coherence Revisited

Definition

Coherence An individual, $\mathcal{I}$, may be termed *coherent* if her probability assignments to an algebra of events obey the probability axioms.

Assertion

A rational individual must be coherent.

A Dutch book argument in support of this assertion follows.

Theorem

Any rational individual, $\mathcal{I}$, must have $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$.

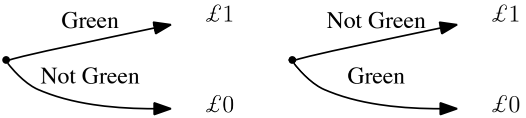
Proof: Case 1: $\mathbb{P}(A) + \mathbb{P}(A^c) < 1$
Consider an urn bet with n balls.

- Let $g^\star(A)$ and $g^\star(A^c)$ be preferred to bets on A and A^c.
- As $\mathbb{P}(A) + \mathbb{P}(A^c)$, for large enough n and $k > 0$:

$$g^\star(A) + g^\star(A^c) = n - k.$$

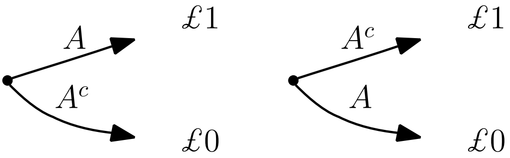
- (Think of an urn with *three* types of ball).
- Let $b^u(n, k)$ pay £1 if a "k from n" urn-draw wins.
- Bet $b(A)$ pay £1 if event A happens.
- Consider two systems of bets...

- System 1: $S_1^u = [b^u(n, g^\star(A)), b^u(n, g^\star(A^c) + k)]$



k>0 based on (irrational) assumption

- System 2: $S_1^e = [b(A), b(A^c)]$



- $\mathcal{I}$ prefers S_1^u to S_1^e and so should pay to win on S_1^u and lose of S_1^e... but everything cancels!

Case2: $\mathbb{P}(A) + \mathbb{P}(A^c) > 1$

- Now, our elicited urn-bets must have

$$g^\star(A) + g^\star(A^c) = n + k$$

- Consider an urn with $g^\star(A)$ green balls and $g^\star(A^c) - k$ blue.
- This time, consider two other systems of bets:

$$S_2^u = [b^u(n, g^\star(A)), b^u(n, g^\star(A^c) - k)]$$

$$S_2^e = [\, \mathrm{b(A), b(A^c)}]$$

- The stated probabilities mean, $\mathcal{I}$ will pay £c to win on S_2^e and lose on S_2^u.
- Again, everything cancels.

A rational individual won't pay for a bet which certainly returns £0. So $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$.

Theorem

A rational individual, $\mathcal{I}$, must set

$$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B)$$

for any $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$.

Proof: Case 1 $\mathbb{P}(A) + \mathbb{P}(B) < \mathbb{P}(A \cup B)$

- Urn probabilities must be such that:

$$g^\star(A) + g^\star(B) = g^\star(A \cup B) - k$$

- Let

$$s_3^e = [b(A), b(B)]$$

  and

$$S_3^u = [b^u(n, g^\star(A)), b^u(n, g^\star(B) + k)]$$

- $\mathcal{I}$ will pay £c to win with S_u^3 which they consider equivalent to $b(\{A \cup B\}$ and lose with S_e^3 ...

(This is only a sketch of the proof, see lecture notes for more.)

Caveat Mathematicus

There are several points to remember:

- Subjective probabilities are subjective.

  People need not agree.

- Elicited probabilities should be coherent.

  The decision analyst must ensure this.

- Temporal coherence is not assumed or assured.

  You are permitted to change your mind.

The latter is re-assuring, but how *should* we update our beliefs?