W2 - L3

Explicit derivation of the addition rule for subjective proba based on bets

Ω outcome space, $\mathcal{F}$ (σ-) algebra, $A, B \in \mathcal{F}$.
$P_{you}(A)$, $P_{you}(B)$, $P_{you}(A \cup B)$ can be defined as explained above (or see W2-L1).
Then we have additivity:

$$P_{you}(A \cup B) = P_{you}(A) + P_{you}(B)$$

Proof: This is equivalent to showing the corresponding equivalence for bets used to define P_{you}, i.e.,

combination of bets

$$b(M, A \cup B) \sim b(M, A) \,\&\, b(M, B)$$

We show this by demonstrating that both sides have the same payoff for all $\omega \in \Omega$.
Consider all cases:

$\omega \in A$: $b(M, A \cup B)$ has payoff M because $\omega \in A \cup B$
$b(M, A)$ has payoff M
$b(M, B)$ has payoff 0 because $A \cap B = \emptyset$
So $b(M, A) \,\&\, b(M, B)$ has payoff $M + 0 = M$ ✓

$\omega \in B$: Similar ✓

$\omega \in (A \cup B)^c$: $b(M, (A \cup B)^c)$ has payoff 0.
$b(M, A) \,\&\, b(M, B)$ has payoff $0 + 0 = 0$ ✓ □

W2-L3

Assuming this has been normalised so $m(M,\Omega)=1$ (see W2-L1) this also implies a rule for the complementary event:

$$1 = m(M,\Omega) = m(M, A\cup A^c) = m(M,A)+m(M,A^c)$$

$$\Rightarrow \quad m(M,A^c) = 1 - m(M,A)$$

and $P_{you}(A^c) = 1 - P_{you}(A)$

W2 – L3

Motivation for the use of conditional proba in the context of subjective proba and decisions

- Updating beliefs

  Current proba. belief P_0

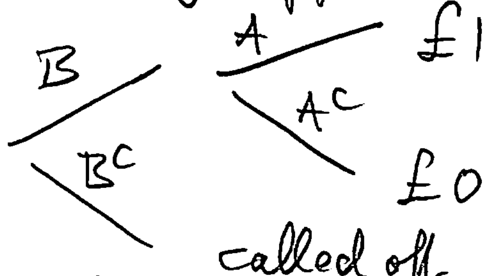
  Then C happens, which can change your beliefs so you instead adopt $P = P_0(\cdot \mid C)$

  This can also be iterated multiple times.

  See e.g. machine learning, Bayesian updates.

- "Called off bets"

  We need a framework that includes bets which only happens under certain conditions.



$b(A \mid B)$

The bet on A only takes place if B happened.

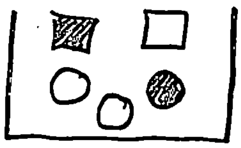
In calculations we will make use of $P(A \mid B)$.

W2 - L3

Review of basic definitions and properties of conditional proba

Ω outcome space, $\mathcal{F}$ (σ-) algebra, P proba, $P(B)>0$

$A, B \in \mathcal{F}$, $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Ex: 

$A =$ "white"
$B =$ "square"

$$P(A|B) = \frac{P(\text{white and square})}{P(\text{square})} = \frac{1/5}{2/5} = \frac{1}{2}$$

$$\neq$$

$$P(B|A) = \frac{P(\text{white and square})}{P(\text{white})} = \frac{1/5}{3/5} = \frac{1}{3}$$

Not symmetric!

Conditional proba can be used to define independence of events.

$A, B \in \mathcal{F}$ with $P(B) > 0$ are independent if $P(A|B) = P(A)$

$A, B \in \mathcal{F}$ with $P(A) > 0$ are independent if $P(B|A) = P(B)$

You may be familiar with the definition

$A, B \in \mathcal{F}$ are independent if $P(A \cap B) = P(A) \cdot P(B)$

It is easy to prove that for A, B with $P(A) > 0$, $P(B) > 0$ all three above definitions of independence are equivalent.

W2 - L3

Let $E_1, \dots, E_n \in \mathcal{F}$ be a partition of Ω,
i.e., $\bigcup_{i=1}^{n} E_i = \Omega$ and $E_i \cap E_j = \emptyset \ \forall i \neq j$ "pairwise disjoint"

Then for all $A \in \mathcal{F}$

$$P(A) = P\left(A \cap \bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{n} (A \cap E_i)\right)$$

also pairwise disjoint

$$= \sum_{i=1}^{n} P(A \cap E_i) \quad \text{because:}$$

$$= \sum_{i=1}^{n} P(E_i) \cdot P(A \mid E_i)$$

by definition of condit. proba.

This is called <u>total proba theorem</u>.

$P(A \mid B)$ is not symmetric, but there is the <u>Bayes theorem</u>. By definition of condit. proba:

$$P(A \mid B) = \frac{P(A)}{P(B)} \cdot P(B \mid A)$$

So if $P(A) > 0$, $P(B) > 0$, $P(A^c) > 0$ then

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c)}$$

(To prove this use $E_1 = A$, $E_2 = A^c$)