Representation of preferences as utilities W5-L2

<u>Def</u>: A binary relation $\succ$ on an action space $\mathcal{A}$ is called <u>preference relation</u> if it has properties (C), (A) and (NT).

<u>Def</u>: A <u>numerical representation</u> of a binary relation $\succ$ is a function

$$u : \mathcal{A} \longrightarrow \mathbb{R} \text{ such that}$$

$$\forall x, y \in \mathcal{A} \quad x \succ y \iff u(x) > u(y)$$

<u>Note</u>: If $\succ$ has a numerical representation then is it a preference relation.

<u>Theorem</u>: Let $\mathcal{A}$ be a finite action space with preference relation $\succ$. Then there is a numerical representation of $\succ$.

<u>Proof</u>: See next page. (Note that it is special case of next theorem.)

<u>Note</u>: u is not unique.

For example, if u is a numerical presentation and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $\tilde{u} = f \circ u$ is also a numerical presentation

(e.g. $f(x) = x + c$, $f(x) = 2x$, $f(x) = x^2 \ldots$)

Proof of theorem

B is finite, so it can be represented as a set

$B = \{x_1, x_2, \dots, x_n\}$

$\forall x \in B$ define $W(x) = \{y \in B \mid x \succ y\}$

("worse than x" - set)

Index set $N(x) \subseteq \{1, 2, \dots, n\}$ such that

$W(x) = \{x_i \mid i \in N(x)\}$ (indices of the "worse than x" - set)

Define $$u(x) = \sum_{i \in N(x)} \frac{1}{2^i}$$

(i) Assume $x \succ y$ and show $u(x) > u(y)$.

$$u(x) = \underbrace{\sum_{i \in N(x)-N(y)} \frac{1}{2^i}}_{>0} + \underbrace{\sum_{N(y)} \frac{1}{2^i}}_{=u(y)} > u(y)$$

because $y \in N(x)$ and $y \notin N(y)$

for this we need (A) !

(ii) Assume $u(x) > u(y)$ and show $x \succ y$.

Because of (C),

$$x \succ y \;\vee\; x \sim y \;\vee\; y \succ x$$

If $x \sim y$ was true then $W(x) = W(y)$, because:

$$z \in W(x) \Rightarrow z \prec x \sim y \underset{(T)}{\Rightarrow} z \prec y \Rightarrow z \in W(y)$$

$$z \in W(y) \Rightarrow z \prec y \sim x \underset{(T)}{\Rightarrow} z \prec x \Rightarrow z \in W(x)$$

Hence, $u(x) = u(y)$ (by definition of u)

Contradiction.

If $y \succ x$ was true, using (i) implied $u(y) > u(x)$. Contradiction.

So it has to be the remaining option $x \succ y$.

$\square$

W5-L2

What about infinite $\mathcal{A}$?
Countable? Think of proof, does it go through?
Continuous? Trace back to countable ...

<u>Def</u> $\mathcal{Z} \subset \mathcal{A}$ is called <u>order dense</u> if
$\forall x,y \in \mathcal{A}$ with $x \succ y$ $\exists z \in \mathcal{Z}$ with $x \succeq z \succeq y$
(e.g. $\mathbb{Q} \subset \mathbb{R}$)

<u>Theorem</u>: Let $\mathcal{A}$ be an action space with preference relation $\succ$. If $\mathcal{A}$ has a countable order dense subset then there is a numerical representation of $\succ$.

<u>Cor</u>: If $\mathcal{A}$ is countable then $\succ$ has a numerical representation. In particular if $\mathcal{A}$ is finite.

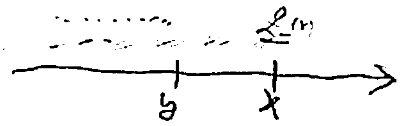
<u>Proof of theorem</u>:

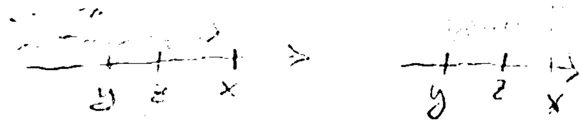
Let $\mathcal{Z}$ be an order dense subset of $\mathcal{A}$.

$\forall x \in \mathcal{A}$ $\quad \overline{\mathcal{Z}}(x) = \{ z \in \mathcal{Z} \mid z \succ x \}$

$$\underline{\mathcal{Z}}(x) = \{ z \in \mathcal{Z} \mid z \prec x \}$$

$$x \succeq y \Rightarrow \overline{\mathcal{Z}}(x) \subseteq \overline{\mathcal{Z}}(y) \wedge \underline{\mathcal{Z}}(x) \supseteq \underline{\mathcal{Z}}(y)$$





$x \succ y \Rightarrow$ at least one of the subset relationships is strict, because (*)

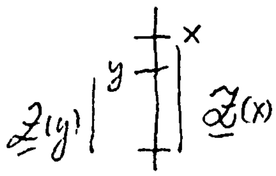
(using properties of $\succ$) $\exists z \in \mathcal{Z} : x \succ z \succsim y \lor x \succsim z \succ y$ (could be both)

$$\Rightarrow z \in \underline{\mathcal{Z}}(x) \setminus \underline{\mathcal{Z}}(y) \lor z \in \overline{\mathcal{Z}}(y) \setminus \overline{\mathcal{Z}}(x)$$

Take a $(p_z)_{z \in \mathcal{Z}}$ PMF with $p_z > 0 \ \forall z \in \mathcal{Z}$

Define $$u(x) = \underbrace{\sum_{z \in \underline{\mathcal{Z}}(x)} p_z}_{\overset{(1)}{\geq} \sum_{z \in \underline{\mathcal{Z}}(y)} p_z} - \underbrace{\sum_{z \in \overline{\mathcal{Z}}(x)} p_z}_{\overset{(2)}{\leq} \sum_{z \in \overline{\mathcal{Z}}(y)} p_z} \quad \forall x \in A$$

For $x \succ y$ we have: (1) because $\underline{\mathcal{Z}}(y) \subseteq \underline{\mathcal{Z}}(x)$, (2) because $\overline{\mathcal{Z}}(y) \supseteq \overline{\mathcal{Z}}(x)$



Using the definition again (for $u(y)$) and (*) we have that at least one of (1) and (2) is strict, hence $u(x) > u(y)$ $\square$

<u>Note</u>: The theorem was about a <u>sufficient</u> condition for the existence of a numerical representation. It can also be shown that the order dense set is <u>necessary</u>. The proof, however, is a rather tricky construction of such a set.