More properties of preferences:

(Needed for v. Neumann – Morgenstern theorem below)

Archimedian property (Arch)

$\forall x, y, z \in \mathcal{A}$ with $x \succ y \succ z$ $\exists \alpha, \beta \in (0,1)$ such that $\alpha x + (1-\alpha) z \succ y \succ \beta x + (1-\beta) z$

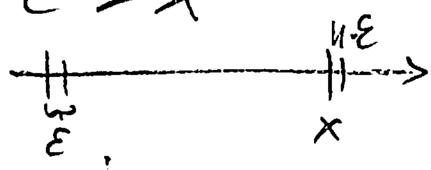
Interpretation: $x \succ y$ is not so strong that mixing it with z can't lead to reversal of preferences, x can not be incommensurably better than y. z can not be incommensurably worse than y.

Homework: Lexicographical order is not (Arch).
(see below)

Compare with Archimedian on $\mathbb{R}$:

$\forall \varepsilon > 0 \ \forall x \in \mathbb{R} \ \exists n \in \mathbb{N}: \ n \cdot \varepsilon > x$

aka "continuity axiom"



- Can act as substitute for continuity.
- continuity of $\succ$ implies (Arch)

Example for incommensurability

$x = (£1000, 1)$
$y = (£10, 1)$
$z = (\text{dead}, 1)$

$x \succ y \succ z$

$(1-\alpha)x + \alpha z \succ y$?

No! no matter how small α is.

Hence, Archimedian property is not true.
Interpretation: money and life/death are incommensurable

However, people do take small risks for gaining or saving money,
e.g. walk/drive/cycle across town to save £20.
Reason: denial

W5 — L3

Independence (Ind) (see last week L2)

$\forall x, y, z \in \mathcal{L}$ and $\forall \alpha \in (0,1]$:

$$x \succ y \;\Rightarrow\; (1-\alpha)z + \alpha x \succ (1-\alpha)z + \alpha y$$

In plain English: Adding the same stuff to both sides does not change the preference.

Practical example for situation where independence is not true: You are in the desert and hungry. You have nothing and you are being offered a choice:

x: dry bread

y: can with soup

Your preference will be $x \succ y$

But if another choice is added:

z: can opener

Your prefrence will be $x + z \prec y + z$

von Neumann – Morgenstern theorem is similar to the previous theorem about preferences and utilities in that it connects them, but goes beyond this by providing an explicit representation.

We previously showed theorems about the representation of preference relations through utility function. Now, we will add some assumption that allow to specify the form of such a representation more detailled.

<u>von Neumann–Morgenstern representation theorem:</u>

Let $\mathcal{A}$ be an action space on a countable outcome space Ω and let $\succ$ be a preference relation on $\mathcal{A}$ that also satisfies (Ind) and (Arch).

Then there is a function $u : \Omega \longrightarrow \mathbb{R}$ such that for all $x, y \in \mathcal{A}$ we have

$$x \succ y \iff \sum_{\omega \in \Omega} p_x(\omega)\, u(\omega) > \sum_{\omega \in \Omega} p_y(\omega)\, u(\omega)$$

(using the representations of the elements in $\mathcal{A}$ through $x = b(\omega, p_x(\omega); \omega \in \Omega)$, $y = b(\omega, p_y(\omega); \omega \in \Omega)$ with PMFs $(p_x(\omega))_{\omega \in \Omega}$ and $(p_y(\omega))_{\omega \in \Omega}$)

Furthermore, the representation is unique up to positive linear transformations.

<u>Note</u>: The sums are actually expectations w.r.t. the PMFs that belong to x and y.

<u>Note</u>: Writing the (countable) Ω as $\{\omega_1, \omega_2, \omega_3, \dots\}$ objects above can be written as $x = b(\omega_n, p_n ; n \in \mathbb{N})$, $(p_n)_{n \in \mathbb{N}}$, $\sum_{n \in \mathbb{N}} p_n\, u(\omega_n)$ etc.

Note on the extension to continuous Ω:

The theorem is also true there, but we can not use sums. The notation used so far is actually somewhat redundant. To describe an action we only need the probabilities for the outcomes. In other words, $\mathcal{A}$ can be identified with a subset $\mathcal{M}$ of the set $\mathcal{M}_1(\Omega, \mathcal{F})$ of probability measures on the outcome space Ω with a σ-algebra $\mathcal{F}$. Assume $\mathcal{M}$ is convex, i.e.

$$P, Q \in \mathcal{M} \Rightarrow \alpha P + (1-\alpha) Q \in \mathcal{M} \quad \forall \alpha \in (0,1)$$

Then the representation in the v. Neumann-Morgenstern theorem takes the form

$$U(P) = \int_\Omega u(\omega)\, P(d\omega)$$

Again, this is an expectation (w.r.t. the measure P).

This note is not necessary for this module, but useful for students who may want to link the theory studied here to the more advanced literature in (measure theoretical) mathematical economics or in Stochastic Finance (e.g. book by H. Föllmer and A. Schied, de Gruyter).

Ex: Lexicographical order on $\mathbb{R}^2$ (ct.)

Recall for $x, y \in \mathbb{R}^2$, $x=(x_1, x_2)$, $y=(y_1, y_2)$

$$\underbrace{x \succ y}_{\text{lexicographical order (binary relation on } \mathbb{R}^2)} \iff x_1 \underbrace{>}_{\text{regular "larger than" order on } \mathbb{R}} y_1 \vee (x_1 = y_1 \wedge y_1 > y_2)$$

lexicographical order (binary relation on $\mathbb{R}^2$) regular "larger than" order on $\mathbb{R}$

Proposition: The lexicographical order on $\mathbb{R}^2$ does not have a numerical representation.

Proof (indirect): Assume u was a numerical repres.
Then, $\forall r \in \mathbb{R}, r \geq 0: (r,2) \succ (r,1) \Rightarrow u(r,2) > u(r,1)$

$\forall r \geq 0 \; \exists q_r \in \mathbb{Q}: \quad u(r,2) \overset{(1)}{>} q_r \overset{(2)}{>} u(r,1) \quad (*)$

because $\mathbb{Q}$ is dense in $\mathbb{R}$

$r \mapsto q_r$ defines a map from $\mathbb{R}^+$ to $\mathbb{Q}$ $\quad (**)$

$\forall r > \tilde{r} \geq 0: \quad u(r,1) > u(\tilde{r},2)$

(1) in $(*)$ $\Rightarrow \exists q_r \in \mathbb{Q}: \; q_r > u(r,1)$

(2) in $(*)$ $\Rightarrow \exists q_{\tilde{r}} \in \mathbb{Q}: \quad u(\tilde{r},2) > q_{\tilde{r}}$

Hence, we have shown: $r > \tilde{r} \Rightarrow q_r > q_{\tilde{r}}$, and thereby also that $q_r = q_{\tilde{r}}$ implies $r = \tilde{r}$ (use previous statement with $r, \tilde{r}$ swapped), so the map is injective from $\mathbb{R}^+$ to $\mathbb{Q}$, a contradiction to the countability of $\mathbb{Q}$. $\square$

Question

Is the map in (**) well defined?

Answer: Yes it is, but it is not given explicitly and it is not unique. Let φ be such a map. For each $r \in \mathbb{R}^+$ its value $\varphi(r)$ was chosen from the countable set $(u(r,1), u(r,2)) \cap \mathbb{Q}$. Different choices would have resulted in a different map $\tilde{\varphi}$. Both φ and $\tilde{\varphi}$ are well defined maps. However, the non-uniqueness of the construction may result in other object not being well defined, as they may depend on the choice of φ. (As an explicit example, imagine you define a function f as follows: $f(x) = \varphi(x) + \varphi(x/2)$, where φ is the map defined in (**). Then f is not well defined. The mistake was to refer to φ as "the" map, because φ as in (**) is not unique, so there is no "the" map φ.)

It is, however, possible to make φ unique by including a (constructive) algorithmus to obtain q_r in (*). Can you think of any? Try, or read some suggestions on the following page.

– Not examinable – W5 – L3

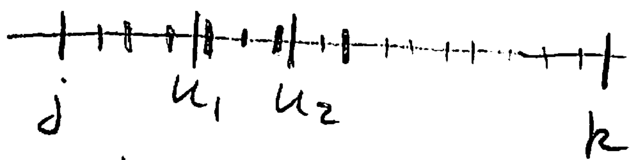
Task: For $u_1 < u_2$ find algorithm to select

$$q \in (u_1, u_2) \cap \mathbb{Q}$$

In other words, describe a way to determine explicitly $q \in \mathbb{Q}$ such that $u_1 < q < u_2$.

(Note that $u_1, u_2 \in \mathbb{R}$ but it is not known whether or not $u_1 \in \mathbb{Q}$ or $u_2 \in \mathbb{Q}$, so the primitive choice $q = (u_1 + u_2)/2$ does not solve the problem.)

Algorithm 1:



Define $j := \max\{n \in \mathbb{N} \mid n \leq j\}$, $k := \min\{n \in \mathbb{N} \mid n \geq k\}$

$\forall n \in \mathbb{N}_0$ $D_n := \{j + 2^{-n} \cdot m \cdot (k - j) \mid m = 0, \dots, 2^m\}$

$D_n \subset D_{n+1}$ $\forall n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} D_n$ is dense in $[j, k]$.

Hence, there exists $n_0 \in \mathbb{N}_0$ with

$$(u_1, u_2) \cap D_{n_0} = \emptyset \text{ but } (u_1, u_2) \cap D_{n_0+1} \neq \emptyset$$

Let q be the minimum of $(u_1, u_2) \cap D_{n_0+1}$.

(Actually, $(u_1, u_2) \cap D_{n_0+1}$ has only one element, which can be shown with a little more reasoning, but simply using the minimum also does the trick.)

$\square$

Algorithm 2:

The Archimedian axiom in $\mathbb{R}$ implies there is $n \in \mathbb{N}$ with $n(u_2 - u_1) > 1$. Let n_0 be the smallest n with this property. Since this means $n_0 u_2 > n_0 u_1 + 1$, there must be $m \in \mathbb{N} \setminus \{0\}$ with $n_0 u_2 > m > n_0 u_1$. Let m_0 be the smallest m with this property. Hence, $q := \frac{m_0}{n_0} \in \mathbb{Q}$ solves the task. $\square$

Algorithm 3:

Let $\tilde{u}_1 = u_1 + \frac{u_2 - u_1}{4}$ and $\tilde{u}_2 = u_2 - \frac{u_2 - u_1}{4}$

Write $\tilde{u}_2$ in decimal representation:

$$\tilde{u}_2 = \sum_{n=0}^{\infty} d_2^{(n)} \cdot 10^{k-n}, \quad k \in \mathbb{N}_0, \; d_2^{(n)} \in \{0, 1, \dots, 9\},$$

and represent $\tilde{u}_1$ accordingly using $d_1^{(n)} \in \{0, 1, \dots, 9\}$.

Let n_0 be the smallest n such that $d_2^{(n)} > d_1^{(n)}$.

Then $q := \sum_{n=0}^{n_0} d_2 \;\; 10^{k-n}$ fulfills $u_1 < \tilde{u}_1 \leq q \leq \tilde{u}_2 < u_2$ $\square$

Algorithm n:

A countably infinite number of algorithms can be constructed by replacing base 10 by other bases in algorithm 3.