Ex   Zero-sum game with payoff matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{seperable?}$$

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \overset{!}{=} \begin{pmatrix} u+x & u+y \\ v+x & v+y \end{pmatrix}$$

I $u + x = a$ II $u + y = 0$

III $v + x = 0$ IV $v + y = d$

$\overset{\text{II}}{\underset{\text{III}}{\Rightarrow}}$ $\quad v = -x \qquad u = -y$

Plug in $\overset{\text{I, IV}}{\Rightarrow}$ $\quad -y + x = a \qquad -x + y = d$

$$x - y = -d$$

$\Rightarrow \quad a = -d$

$-y + x = a$
$-x + y = -a$ } equiv
$y = x - a$

In example :* $a = 1$; $d = 2$

$\Rightarrow a \neq -d$

$\Rightarrow$ not separable

* $a > 0, d > 0$
$\Rightarrow$ No dominant moves

Comments and extensions to slides

Prisoner's dilemma

Lecture notes/slides show, by giving an explicit form for r_1, r_2, s_1, s_2, that the prisoner's game is separable. Note that while this is correct for the matrix used for rewards of the prisoner's game in the lecture notes/slides, there are other representations of this game in the literature which are not necessarily separable. In fact, we will define a general form of the game below and show that separability has implications on the choice of the coefficients, in other words, necessary conditions for separability.

General form of the R-matrix:

$$\begin{bmatrix} b & d \\ a & c \end{bmatrix} \text{ with } 0 \leq a < b < c < d$$

Need to find $u = r_1(d_1)$, $v = r_1(d_2)$, $x = r_2(\delta_1)$, $y = r_2(\delta_2)$

$$\begin{bmatrix} b & d \\ a & c \end{bmatrix} = \begin{bmatrix} u+x & u+y \\ v+x & v+y \end{bmatrix}$$

This corresponds to a system of equations

I $u+v=b$ II $u+y=d$ III $v+x=a$ IV $v+y=c$

which imply

I-III $u-v=b-a$ II-IV $u-v=d-c$

I-II $x-y=b-d$ III-IV $x-y=a-c$

So, ~~two~~ necessary conditions for the game to be separable are:

$$\left.\begin{array}{r} b-a=d-c \\ \text{and } b-d=a-c \end{array}\right\} \text{actually equiv.}$$

An example for representations of the prisoner's game that does not fulfill these conditions is

$$\begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix}$$

and there are many more. (In fact, it is more typical for this story to be represented by a non-separable game than by a separable game.)

Similar reasoning can be done for S (player 2).

We may come back to this and further discussion on the prisoner's game (and dilemma) in homework questions.

Example for a game with dominant moves

Dominant moves can be used to construct optimal strategies using the assumption of common knowledge of rationality. Not all games have dominant moves. Here is a reward matrix $(R(d_i, \delta_j), S(d_i, \delta_j))$ of a game that does have some ...

		δ_1	δ_2	δ_3	Player 2
	d_1	(4,3)	(5,1)	(6,2)	
Player 1	d_2	(2,1)	(8,4)	(3,6)	
	d_3	(3,0)	(9,6)	(2,8)	

$D = \{d_1, d_2, d_3\}$

$\Delta = \{\delta_1, \delta_2, \delta_3\}$

Player 1 has no dominant moves: no $d_i \in D$ is dominant

Player 2 has dominant move:

δ_2 is dominated by δ_3 because (checking <u>second</u> values in pairs!)

$$S(d_i, \delta_2) < S(d_i, \delta_3) \quad \text{for } i = 1, 2, 3$$

Player 2 eliminates δ_2 because it would be irrational to use it. Player 1 knows because of the common knowledge of rationality that Player 2 will not use δ_2. Hence considers the reduced game with the D unchanged, but $\Delta = \{\delta_1, \delta_3\}$. Now, Player 1 has dominant moves: $d_1 > d_2$ and $d_1 > d_3$ and removes both d_2 and d_3.

The remaining game is just with $D = \{d_1\}$, so the matrix is

$$\begin{array}{c|cc} & \delta_1 & \delta_3 \\ \hline d_1 & (4,3) & (6,2) \end{array}$$

Player 2 knows Player 1 use d_1, so Player 2 uses δ_1. The strategy is (d_1, δ_1) with pay-offs 4 for Player 1 and 3 for Player 2.

Notes:
- not all games have dominant moves
- not all games with dominant moves lead to a unique strategy
- the assumption of common knowledge of rationality may not be always be true for humans

<u>Example for a zero-sum game with dominant moves:</u>

Now, $S = -R$. R is given by matrix:

	δ_1	δ_2	δ_3	δ_4	δ_5
d_1	4	5	6	4	4
d_2	4	2	3	4	4
d_3	2	4	5	5	5

Reduce move by removing moves that are dominated by others (or redundant)

Note: Player 1 wants big number in the matrix, Player 2 wants <u>small</u> ones ($S = -R$).

More on Friday.