

What is a Game

A *game* in mathematics is, roughly speaking, a problem in which:

- ▶ Several *agents* or *players* make 1 or more decisions.
- ▶ Each player has an objective / set of preferences.
- ▶ The outcome is influenced by the set of decisions.
- ▶ There may be additional non-deterministic uncertainty.
- ▶ The players may be in competition or they may be cooperating.
- ▶ Examples include: chess, poker, bridge, rock-paper-scissors and many others.

However, we will stick to simple two player games with each player simultaneously making a single decision.

Simple Two Player Games

- ▶ Player 1 chooses a move for a set $D = \{d_1, \dots, d_n\}$.
- ▶ Player 2 chooses a move from a set $\Delta = \{\delta_1, \dots, \delta_m\}$.
- ▶ Each player has a *payoff function*.
- ▶ If the players choose moves d_i and δ_j , then:
 - ▶ Player 1 receives reward $R(d_i, \delta_j)$.
 - ▶ Player 2 receives reward $S(d_i, \delta_j)$.
- ▶ The relationship between decisions and rewards is often shown in a payoff matrix:

	δ_1	...	δ_m
d_1	$(R(d_1, \delta_1), S(d_1, \delta_1))$...	$(R(d_1, \delta_m), S(d_1, \delta_m))$
\vdots			\vdots
d_n	$(R(d_n, \delta_1), S(d_n, \delta_1))$...	$(R(d_n, \delta_m), S(d_n, \delta_m))$

Payoff Matrices Again

It's sometimes useful to consider a single player's payoff as a function of the possible decisions.

Player 1 and player 2 have these payoff matrices:

	δ_1	...	δ_m
d_1	$R(d_1, \delta_1)$...	$R(d_1, \delta_m)$
\vdots			\vdots
d_n	$R(d_n, \delta_1)$...	$R(d_n, \delta_m)$

	δ_1	...	δ_m
d_1	$S(d_1, \delta_1)$...	$S(d_1, \delta_m)$
\vdots			\vdots
d_n	$S(d_n, \delta_1)$...	$S(d_n, \delta_m)$

Example (Rock-Paper-Scissors)

- ▶ Each player picks from the same set of decisions:

$$D = \Delta = \{R, P, S\}$$

- ▶ R beats S; S beats P and P beats R
- ▶ One possible payoff matrix is:

	R	P	S
R	(0,0)	(-1,1)	(1,-1)
P	(1,-1)	(0,0)	(-1,1)
S	(-1,1)	(1,-1)	(0,0)

Example (The Prisoner's Dilemma)

- ▶ Again, each player picks from the same set of decisions:

$$D = \Delta = \{\text{Stay Silent, Betray Partner}\}$$

- ▶ If they both stay silent they will receive a short sentence; if they both betray one another they will get a long sentence; if only one betrays the other the traitor will be released and the other will get a long sentence.

- ▶ One possible payoff matrix is:

	S	B
S	(1,1)	(5,0)
B	(0,5)	(4,4)

- ▶ Notice that each player wishes to minimise this payoff!

Example (Love Story)

- ▶ A boy and a girl must go to either of:

$$D = \Delta = \{\text{Football, Opera}\}$$

- ▶ They both wish to meet one another most of all.
- ▶ If they don't meet, the boy would rather see the football; the girl, the opera.
- ▶ A possible payoff matrix might be:

	F	O
F	(100,100)	(50,50)
O	(0,0)	(100,100)

Some Features of these Examples

- ▶ The rock-paper-scissors game is *purely competitive*: any gain by one player is matched by a loss by the other player.
- ▶ The RPS and PD problems are symmetric:

$$R(d, \delta) = S(\delta, d)$$

[Note that this makes sense as $D = \Delta$]

- ▶ $D = \Delta$ in all three of these examples, but it isn't always the case.

Uncertainty in Games

As the players don't know what action the other will take, there is uncertainty.

- ▶ Thankfully, the Bayesian interpretation of probability allows them to encode their uncertainty in a probability distribution.
- ▶ Player 1 has a probability mass function p over the actions that player 2 can take, Δ .
- ▶ Player 2 has a probability mass function q over the actions that player 1 can take, denoted D .

Expected Rewards

Just as in a decision problem, we can think about expected rewards:

- ▶ For player 1, the expected reward of move d_i is:

$$\begin{aligned}\bar{R}(d_i) &= \mathbb{E} [R(d_i, \delta_j)] \\ &= \sum_{j=1}^m q(\delta_j) R(d_i, \delta_j)\end{aligned}$$

- ▶ Whilst, for player 2, we have

$$\begin{aligned}\bar{S}(\delta_j) &= \mathbb{E} [S(d_i, \delta_j)] \\ &= \sum_{i=1}^n p(d_i) S(d_i, \delta_j)\end{aligned}$$

Some Interesting Questions

- ▶ When can a player act without considering what the opponent will do? i.e. When is player 1's strategy independent of p or player 2's of q ?
- ▶ When p or q is important, how can rationality of the opponent help us to elicit them?
- ▶ What are the implications of this?



Strategy in Separable Games

- ▶ Player 1's strategy should depend only upon r_1 as the decision they make doesn't alter the reward from r_2 .
- ▶ Player 2's strategy should depend only upon s_2 as the decision they make doesn't alter the reward from s_1 .
- ▶ So, player 1 should choose a strategy from the set:

$$D^* = \{d^* : r_1(d^*) \geq r_1(d_i) \quad i = 1, \dots, n\}$$

- ▶ And player 2 from:

$$\Delta^* = \{\delta^* : s_2(\delta^*) \geq s_2(\delta_j) \quad j = 1, \dots, m\}$$

The Prisoner's Dilemma is a Separable Game

- ▶ Let $r_1(S) = 0$ and $r_1(B) = 1$.
- ▶ Let $r_2(S) = -1$ and $r_2(B) = -5$.
- ▶ Now, $R(d, \delta) = r_1(d) + r_2(\delta)$.
- ▶ And $D^* = \{B\}$.
- ▶ Similarly for the second player, $\Delta^* = \{B\}$.
- ▶ This is the so-called paradox of the prisoner's dilemma: both players acting rationally and independently leads to the worst possible solution!

Rationality and Games

As in decision theory, a rational player should maximise their expected utility. We will generally assume that utility is equal to payoff; no greater complications arise if this is not the case.

- ▶ For a given pmf q , player 1 has:

$$\bar{R}(d_i) = \sum_{j=1}^m R(d_i, \delta_j) q(\delta_j)$$

- ▶ Whilst for given p , player 2 has:

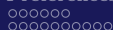
$$\bar{S}(\delta_j) = \sum_{i=1}^n S(d_i, \delta_j) p(d_i)$$

- ▶ We want p and q to be consistent with the assumption that the opponent is rational.
- ▶ We assume, that *rationality of all players is common knowledge*.

Common Knowledge: A Psychological Infinite Regress

In the theory of games the phrase *common knowledge* has a very specific meaning.

- ▶ Common knowledge is known by all players.
- ▶ That common knowledge is known by all players is known by all players.
- ▶ That common knowledge is common to all players is known by all players
- ⋮
- ▶ More compactly: common knowledge is something that is known by all players and the fact that this thing is known by all players is itself common knowledge.
- ▶ This is an example of an infinite regress.



Domination

- ▶ A move d^* is said to dominate all other strategies if:

$$\forall d_i \neq d^*, j : \quad R(d^*, \delta^j) \geq R(d_i, \delta_j)$$

- ▶ It is said to *strictly dominate* those strategies if:

$$\forall d_i \neq d^*, j : \quad R(d^*, \delta^j) > R(d_i, \delta_j)$$

- ▶ A move d' is said to be *dominated* if:

$$\exists i \text{ such that } d_i \neq d' \text{ and } \forall j : R(d', \delta_j) \leq R(d_i, \delta_j)$$

- ▶ It is said to be *strictly dominated* if:

$$\exists i \text{ such that } d_i \neq d' \text{ and } \forall j : R(d', \delta_j) < R(d_i, \delta_j)$$



Separability and Domination

Theorem (Dominant Moves Should be Played)

If a game has a payoff matrix such that player 1 has a dominant strategy, d^* then the optimal move for player 1 is d^* irrespective of q .

Proof:

- ▶ Player 1 is rational and hence seeks the d_i which maximises

$$\sum_j R(d_i, \delta_j)q(d_j)$$

- ▶ Domination tells us that $\forall i, j : R(d^*, \delta_j) \geq R(d_i, \delta_j)$
- ▶ And hence, that:

$$\sum_j R(d^*, \delta_j)q(d_j) \geq \sum_j R(d_i, \delta_j)q(d_j)$$

Rationality and Domination

If rationality is common knowledge and d^* is a strictly dominant strategy for player 1 then:

- ▶ Player 1, being rational, plays move d^* .
- ▶ Player 2, knows that player 1 is rational, and hence knows that he will play move d^* .
- ▶ Player 2 can exploit this knowledge to play the optimal move *given that player 1 will play d^** .
- ▶ Player 2 plays moves δ^* with δ^* such that:

$$\forall j : S(d^*, \delta^*) \geq S(d^*, \delta_j)$$

- ▶ If there are several possible δ^* then one may be chosen arbitrarily.

Example (A game with a dominant strategy)

Consider the following payoff matrix:

	δ_1	δ_2	δ_3	δ_4
d_1	(2,-2)	(1,-1)	(10,-10)	(11,-11)
d_2	(0,0)	(-1,1)	(1,-1)	(2,-2)
d_3	(-3,3)	(-5,5)	(-1,1)	(1,-1)

- ▶ If rational, player 1 must choose d_1 .
- ▶ Player 2 knows that player 1 will choose d_1 .
- ▶ Consequently, player 2 will choose δ_2 .
- ▶ (d_1, δ_2) is known as a discriminating solution.

Iterated Strict Domination

1. Let $D_0 = D$ and $\Delta_0 = 0$. Let $t = 1$
2. Player 1 checks D_{t-1} to see if it contains one or more strictly dominated moves. Let D'_t be the set of such moves.
3. Let $D_t = D_{t-1} \setminus D'_t$.
4. Player 1 checks D_{t-1} to see if it contains one or more strictly dominated strategies given that player 2 must choose a move from Δ_{t-1} . Let D'_t be the set of these strategies. Let $D_t = D_{t-1} \setminus D'_t$.
5. Player 2 updates Δ_{t-1} in the same way noting that player 1 must choose a move from D_t .
6. If $|D_t| = |\Delta_t| = 1$ then the game is solved.
7. If $|D_t| < |D_{t-1}|$ or $|\Delta_t| < |\Delta_{t-1}|$ let $t = t + 1$ and goto 2.
8. Otherwise, we have reduced the game to the simplest form we can by this method.

Example (Iterated Elimination of Dominated Strategies)

Consider a game with the following payoff matrix:

	L	C	R
T	(4,3)	(5,1)	(6,2)
M	(2,1)	(8,4)	(3,6)
B	(3,0)	(9,6)	(2,8)

Look first at player 2's strategies...

Example (Iterated Elimination of Dominated Strategies)

C is strictly dominated by R, leading to:

	L	R
T	(4,3)	(6,2)
M	(2,1)	(3,6)
B	(3,0)	(2,8)

Player 1 *knows* that player 2 won't play C...

Example (Iterated Elimination of Dominated Strategies)

Conditionally, both M and B are dominated by T:

	L	R
T	(4,3)	(6,2)

Player 2 *knows* that player 1 will play T and so, they play *L*.
Again, we have a deterministic “solution”.

Purely Competitive Games

- ▶ In a purely competitive game, one player's reward is improved only at the cost of the other player.
- ▶ This means, that if $R(d', \delta) = R(d, \delta) + x$ then $S(d', \delta) = S(d, \delta) - x$.
- ▶ Hence $R(d', \delta) + S(d', \delta) = R(d, \delta) + S(d, \delta)$.
- ▶ The sum over all players' rewards is the same for all sets of moves.
- ▶ It doesn't change the domination structure or the ordering of expected rewards if we add a constant to all rewards.
- ▶ Hence, any purely competitive game is equivalent to a game in which:

$$\forall \delta \in \Delta, d \in D : R(d, \delta) + S(d, \delta) = 0$$

a *zero-sum game*.



Payoff and Zero-Sum Games

- ▶ In a zero-sum game:

$$S(d_i, \delta_j) = -R(d_i, \delta_j)$$

- ▶ Hence, we need specify only one payoff.
- ▶ Payoff matrices may be simplified to specify only one reward⁶

Example (Rock-Paper-Scissors is a zero-sum game)

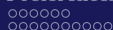
	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

- ▶ It can be convenient to use standard matrix notation, with $M = (m_{ij})$ and $R(d_i, \delta_j) = m_{ij}$.

⁶In the two player case, at least

What if no move is dominant?

- ▶ In the RPS game, like many others, no move is dominant (or dominated) for either player.
- ▶ If either player commits themselves to playing a particular move, the other player can exploit that commitment (if they knew what it was, that is).
- ▶ We need a strategy for dealing with such games.
- ▶ Perhaps the maximin approach might be useful here...



Maximin Strategies in Zero-Sum Games

- ▶ If a player adopts a maximin strategy, he believes that the opponent will always correctly predict their move.
- ▶ This means, the opponent will choose their best possible action based upon the player's act.
- ▶ In this case, player 1's expected payoff is:

$$R_{\text{maximin}}(d_i) = \min_j R(d_i, \delta_j)$$

- ▶ If this is the case, then player 2's payoff is:

$$-R_{\text{maximin}}(d_i) = \max_j -R(d_i, \delta_j)$$

- ▶ Hence $P1$ should play $d_{\text{maximin}}^* = \arg \max_{d_i} \min_j R(d_i, \delta_j)$.
- ▶ One could swap the two players to obtain a maximin strategy for player 2.

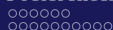
Example (RPS and Maximin)

- ▶ Let $M = (m_{ij})$ denote the payoff matrix for the RPS game.
- ▶ Then, $\min_j R(d_i, \delta_j) = \min_j m_{ij} = -1$ for all i .
- ▶ Thus any move is maximin for player 1.
- ▶ Player 1 expects to receive a payout of -1 whatever he does.
- ▶ If both players adopt a maximin view, then player 2 has the same expectation (by symmetry).
- ▶ How can we resolve this paradox?



What's Gone Wrong?

- ▶ The players aren't using all of the information available.
- ▶ They haven't used the fact that it is a zero sum game.
- ▶ They don't have compatible beliefs:
 - ▶ If P1 believes P2 can predict their move and P2 believes that P1 can predict their move then things inevitably go wrong.
 - ▶ It cannot be common knowledge that *both* players will adopt a maximin strategy!
- ▶ If a player really believes their opponent can predict their move then they can use randomization to make their action less predictable...



Mixed Strategies

- ▶ A *mixed strategy* for player 1 is a probability distribution over D .
- ▶ If a player has mixed strategy $\mathbf{x} = (x_1, \dots, x_n)$ then they will play move d_i with probability x_i .
- ▶ This can be achieved using a randomization device such as a spinner to select a move.
- ▶ A *pure strategy* is a mixed strategy in which exactly one of the x_i is non-zero (and is therefore equal to 1).
- ▶ A similar definition applies when considering player 2.