$X_i \in \{0,1\}$ $i = 1, 2, \ldots, N$, independent, $P(X_i = 1) = P(X_i = 0) = \frac{1}{2}$

<u>Expected # of runs</u>

Z_r = # of runs of length r $= Z_r^{(0)} + Z_r^{(1)}$

$Z_r^{(0)}$ = " " " of 0 of length r

$Z_r^{(1)}$ = " " " of 1 " " "

$$Z_r^{(1)} = \sum_{i=1}^{N-r+1} 1_{\{\text{run of length } r \text{ starting in } i\}}$$

e.g.: $r = 3$

value	1	1	1	0	*	...	*	0	1	1	1	0	*	...	*	0	1	1	1
position	1	2	3	4	5	...	$i-2$	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$	...	...	$N-3$	$N-2$	$N-1$	N
			r									$i+r$				$N-r$			

start

in between

end

$$= 1_{\{X_1 = X_2 = \ldots = X_r = 1,\, X_{r+1} = 0\}}$$

$$+ \sum_{i=2}^{N-r} 1_{\{X_{i-1} = 0,\, X_i = X_{i+1} = \ldots = X_{i+r-1} = 1,\, X_{i+r} = 0\}}$$

$$+ 1_{\{X_{N-r} = 0,\, X_{N-r+1} = X_{N-r+2} = \ldots = X_N = 1\}}$$

$$E[1_A] = P(A)$$

$$\begin{aligned}
E[Z_r^{(1)}] &= P(X_1 = X_2 = \dots = X_r = 1,\ X_{r+1} = 0) \\
&\quad + \sum_{i=2}^{N-r} P(X_{i-1} = 0,\ X_i = X_{i+1} = \dots = X_{i+r-1} = 1,\ X_{i+r} = 0) \\
&\quad + P(X_{N-r} = 0,\ X_{N-r+1} = X_{N-r+2} = \dots = X_N = 1) \\
&= \left(\frac{1}{2}\right)^r \cdot \frac{1}{2} + \sum_{i=2}^{N-r} \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^r \cdot \frac{1}{2} \\
&= \left(\frac{1}{2}\right)^r + (N-r-1) \cdot \left(\frac{1}{2}\right)^{r+2}
\end{aligned}$$

$$E[Z_r^{(0)}] = E[Z_r^{(1)}]$$

$$E[Z_r] = 2 \cdot E[Z_r^{(1)}] = \left(\frac{1}{2}\right)^{r-1} + (N-r-1)\left(\frac{1}{2}\right)^{r+1}$$