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Risk aversion and block exercise of executive stock options

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ABSTRACT

It is well documented that executives granted stock options tend to exercise early and in a few large transactions or 'blocks'. Standard risk-neutral valuation models cannot explain these patterns, and attempts to capture the exercise behavior of risk averse executives have been limited to the special case of one option. This paper solves for the optimal exercise behavior for a risk averse executive who is granted multiple stock options. We show that utility-based models do not predict block exercise behavior. Rather, the risk averse executive exercises stock options individually at a sequence of increasing price thresholds. When, in addition, the executive faces frictions such as costly exercise, he faces a trade-off between exercising little and often to maximize return, and exercising larger quantities on fewer occasions to minimize effort. This generates realistic block exercise behavior and yields new predictions. In particular, executives should begin by exercising large blocks of options, but the block sizes should become smaller over time. Our framework also allows us to study the impact of multiple exercise dates on estimates of the cost of options to the company. We find that assuming the executive can only exercise on a single occasion underestimates the cost of the options compared with allowing for optimal exercise behavior.

In this paper we model the exercise behavior of executives who are granted executive stock options (ESOs). It is well documented that executives receiving stock options tend to exercise options in a few large transactions. That is, exercise typically takes place in a small number of large 'blocks'. Huddart and Lang (1996) find that the mean fraction of options from a single grant exercised by an employee at one time varied from 0.18 to 0.72 over employees at a number of companies. Similarly, Aboody (1996) reports yearly mean percentages of options exercised over the life of 5 and 10 year options, showing exercises are spread over the life of the options. This paper provides a model which is consistent with this empirical observation and, in addition, generates some new testable implications.

Standard Black Scholes American options models[1] do not adequately capture the exercise behavior of executives[2] since they advocate all options should be exercised at the *same* stock price threshold, on a single occasion. This prediction is

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[1] We refer to standard Black Scholes/Merton (1973) models for tradeable options as complete market models. These models assume the executive can trade the stock (and thus hedge all risks). We reserve the term risk-neutral to refer to models where an agent prices via a linear utility function.

[2] Rubinstein (1995) points out many features distinguishing ESO's from tradeable options. Also see Murphy (1999) for an overview and statistics on ESO grants.

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clearly violated in practice. It is also well understood that restrictions on trading the stock of their company leads executives exposed to unhedgeable or idiosyncratic risk. There is a large literature[3] which models the impact of risk aversion and such unhedgeble risks on option exercise and value. The general conclusion is that these features lead to early exercise, as observed in practice.

However, these models typically assume a grant of only one option or that all options are exercised at the same time. That is, nothing can be said about the nature of option exercises over time. The exception to this is the model of Jain and Subramanian (2004), to which we compare our model and results shortly.

We consider the problem of how a risk averse executive facing an incomplete market will exercise a grant of multiple options. In general, his exercise policy will depend on the current stock price, his current wealth, the number of options remaining, and the option's time-to-maturity. Since this will result in a complex multi-dimensional problem, we make a number of simplifying assumptions in order to capture the main aspects of the problem. First we assume the options have infinite maturity.[4] This removes the dependence of the exercise strategy on the time-to-maturity and is key in enabling thresholds to be derived in closed-form. Additionally, Huddart and Lang (1996) find that 'time is not a key predictor of exercise'. We revisit this assumption in Section 4 and demonstrate it does not materially affect results. Second, we assume the executive has exponential utility. Since wealth factors out in this case, we are again reducing the dimension of the problem.[5] Under these assumptions we ultimately derive an exercise boundary which relates the number of unexercised options and the level of stock price. Although the executive cannot trade in the company stock and thus faces unhedgeable risks in an incomplete market, we allow him to reduce this risk by taking a position in the market asset (which is correlated with the stock of the company). The executive chooses the quantities and times to exercise, as well as his position in the market asset in order to maximize his expected utility of wealth.

Our main result is to show that risk aversion and incomplete markets causes the executive to exercise options *individually* over time. Options are exercised at a sequence of increasing stock price thresholds. We give these thresholds in closed-form as the solution to a set of recursive equations. All options are exercised at thresholds which are lower than the single exercise threshold which holds under Black Scholes assumptions because the executive is facing unhedgeble risks which induce him to unwind his option exposure earlier. The options are exercised gradually because each additional option is worth less to a risk averse agent. The thresholds are decreasing with risk aversion (so that the options are exercised sooner) and increasing with the correlation between the company stock and market asset. The ability to invest in a correlated asset allows the executive to partially hedge the risk of the options and allows him to delay exercise.

Our main result shows that in the continuous time limit, block exercise disappears. We show risk aversion and incomplete markets can still form the basis of an explanation of observed block exercise behavior, if, in addition, we introduce a friction that exercise is costly. The risk averse executive will balance the benefit of exercising optimally with the desire to minimize costs by exercising less frequently. We demonstrate that the outcome of this trade-off restores the block exercise observed in practice and that only very small frictions are required. We also find that the block size is non-increasing across exercise dates. For example, an optimal strategy could be to exercise in proportions (40%, 30%, 20%, 10%) or (50%, 30%, 10%, 10%) but not (40%, 10%, 20%, 30%) or (10%, 20%, 20%, 50%). This conclusion is a testable implication of the model.

We also investigate the implications of our model for the expensing of stock options. Following much debate, firms are required under SFAS 123R[6] to recognize as an expense the fair value of their stock option grants using 'an appropriate valuation technique' which recognizes the timing of option exercise. Our costly exercise utility-based model shows exercise patterns differ with risk aversion, hedging capabilities, costs of exercise and with the volatility and dividends on the company stock. The existing literature (Carpenter (1998) and Bettis et al. (2005)) assumes the executive only exercises on one occasion. We show that restricting the executive to exercise at a single threshold rather than the optimal sequence of thresholds can lead to a significant underestimate of the company cost.

We now briefly contrast our model with the literature. Kadam et al. (2005) analyze the exercise of a single infinite maturity option by a manager with exponential utility and obtain an explicit solution. Our model extends theirs to consider multiple options and to consider partial hedging by the executive. Carpenter et al. (2008) investigate features of exercise strategies of a manager with general utility but restrict the manager to exercising options on a single occasion. Jain and Subramanian (2004) examine intertemporal exercise of multiple options by a risk averse executive in a binomial framework. They do not allow for partial hedging or vesting in the model. They show numerically how the expected proportion of an option grant exercised in the first year varies with model parameters. For instance, the proportion exercised increased with risk aversion. However, since their analysis is numerical, and the grid inevitably coarse, they

[3] Such models use a certainty equivalent option value corresponding to a given utility specification. The early literature assumes the executive has no non-option investments or outside portfolio choice and can simply invest in riskfree bonds. Contributions were made by Huddart (1994), Lambert et al. (1991), Kulatilaka and Marcus (1994) and Hall and Murphy (2002). Carpenter (1998) allows for outside investments but does not optimize in the presence of the option, rather assuming that the executive invests in a Merton (1971) style portfolio. Models have been developed whereby the executive also invests into a risky portfolio of other assets, including Detemple and Sundaresan (1999), Henderson (2005) and Ingersoll (2006).

[4] Other papers to exploit tractability afforded by an infinite maturity include Kadam et al. (2005) and Sircar and Xiong (2007).

[5] If we retained wealth dependence in the problem by using CRRA utility, we would expect similar exercise behavior. Instead of exercise taking place when the stock price reaches a boundary, it would occur when the stock price scaled by wealth reaches a boundary.

[6] Statement of Financial Accounting Standards 123 revised, 2004, 'Share-based Payment'.

cannot distinguish that there are separate thresholds for each option. One of the main advantages of our continuous time formulation and closed-form solution is that we are able to show that risk aversion and incomplete markets *alone* do not predict block exercise, and we can provide explicit solutions for the set of individual exercise thresholds. We also extend our model to consider the impact of frictions, and show such frictions will restore block exercise patterns.

There are other factors contributing to the observed exercise patterns of ESOs which are not part of our model.[7] We mention in particular the common practice of vesting.[8] This is a period of time after the grant during which ESOs cannot be exercised. For example, a common vesting structure is 25% over four years, see Huddart and Lang (1996). Whilst vesting can provide a partial explanation for empirically observed block exercise behavior, it cannot provide the full story. By focusing on risk aversion and costly exercise in this paper, we show block exercise will occur even in the absence of vesting. We elaborate on the role of vesting in block exercise in Section 4.

There are also several features of option grants which we do not incorporate into our model. The option strike can be reset by companies when they are out-of-the-money (Acharya et al., 2000) and additional options can be granted when options are exercised (called reloading, see Dybvig and Lowenstein (2003)). Reloading of options would give an incentive to exercise earlier. Sircar and Xiong (2007) treat both of these features in a model where the executive is risk-neutral. In common with most of the literature, our model also ignores incentive effects (whereby the executive can influence the stock price process itself) and dilution effects. We also do not include the risk that the executive leaves the firm but discuss the implications of this in Section 4.

1. Optimal exercise and hedging policies

Consider a risk averse executive[9] who is granted n call options on the stock of his company and is restricted from trading the stock itself. Each option has strike K and is American-style so that it can be exercised at any time. We assume for simplicity the options have no vesting period, or equivalently, are already vested. Aboody (1996) finds about 10% of options in his sample had no vesting period. We discuss the impact of vesting in Section 4. We assume the options have infinite maturity. In practice, ESOs typically have 10 year maturities and we demonstrate numerically in Section 4 that there is little difference between the exercise threshold at 10 years to maturity and those from an infinite maturity model.

The company stock price V follows a geometric Brownian motion

$$dV = (\nu - \delta)V\,dt + \eta V\,dW \tag{1}$$

with expected return ν, dividend yield $\delta > 0$, and volatility η. Interest rates are zero.

We will assume the executive cannot trade the stock V.[10] The executive faces unhedgeable risk as he is unable to transact in the company stock to undo the effect of the options. This puts the executive in the situation of an incomplete market. We assume the executive is risk averse, and has negative exponential utility denoted by $U(x) = -(1/\gamma)\,e^{-\gamma x}$.

The executive selects exercise times $\tau^n \leqslant \cdots \leqslant \tau^1$ where τ^j denotes the exercise time when there are j options remaining.[11] We allow for a maximum of n dates corresponding to exercising the options one-at-a-time. Options must be exercised in whole units. Allowing for dates to coincide (so $\tau^k = \tau^{k-1}$) recognizes that it may be optimal to exercise multiple options at the *same* time.

We assume that upon exercise of a single option, the executive pays K for the stock and immediately sells it for the current stock price, receiving a cash payout of the difference.[12] Thus at each exercise time τ^j, the executive receives the cash payoff $(V_{\tau^j} - K)^+;\ j = 1, \ldots, n$.

It is unlikely that executives receiving stock options make their exercise decisions in isolation of the rest of their portfolio. That is, executives would take into consideration their other holdings when deciding whether to exercise some of their options grant. We determine simultaneously the optimal exercise policy and optimal portfolio choice of the executive, assuming the executive has access to the market asset M with dynamics

$$dM = \mu M\,dt + \sigma M\,dB \tag{2}$$

where μ is the expected return and σ the volatility. Denote by $\lambda = \mu/\sigma$, the market's instantaneous Sharpe ratio. Let $dB\,dW = \rho\,dt$ so that M is correlated with the company stock with $\rho \in [-1, 1]$. We assume the company stock V follows (1)

[7] These include adverse private information known by the executive, liquidity needs of the executive and termination from the company, see Carpenter (1998). Heath et al. (1999) and Core and Guay (2001) also consider psychological reasons for early exercise and propose that executives exercise in response to stock price trends.

[8] Blackout periods (see Reda et al., 2005) when options cannot be exercised due to events such as earnings announcements would have a similar effect.

[9] We refer to any individual receiving stock options as part of their compensation, including employees, executives and CEO's.

[10] In practice, executives cannot short sell stock as they are prohibited by Section 16-c of the Securities and Exchange Act (see Carpenter, 1998). Since unconstrained executives granted a large parcel of ESOs would not choose to be long the stock, we assume (without loss of generality) that the appropriate restriction is that executives cannot *trade* the stock. Bettis et al. (2001) provide some evidence of the use of zero-cost collars and equity swaps by insiders to undo some of their incentives. However such deals must be reported to the SEC and generally only involve high-ranking executives.

[11] Note if the kth from last option is never exercised then $\tau^k = \tau^{k-1} = \cdots = \tau^1 = \infty$.

[12] Huddart and Lang (1996) provide evidence of such cash exercises, possibly due to taxes on capital gains. Ofek and Yermack (2000) also find most executives sell the shares acquired through option exercise. Note, however, prior to May 1991, the SEC imposed a six-month trading restriction on stock acquired through option exercise.

with the additional equilibrium (CAPM) condition

$$\nu = \mu\rho\eta/\sigma \tag{3}$$

relating the (excess) return on V with its correlation with the market portfolio.[13]

Allowing the executive to trade in the market asset enables him to partially hedge the risk of his options. He holds a cash amount θ_s in M at time s. Let

$$X_t = X_0 + \int_0^t \theta_s \frac{dM}{M} + \sum_{\tau_i \leqslant t} (V_{\tau_i} - K)^+ \tag{4}$$

be the executive's wealth at time t. Wealth comprises both the cash received from any option exercises up to (and including) that time, as well as the gains from trade from the position in the market asset M. The executive selects n exercise times $\tau^n \leqslant \cdots \leqslant \tau^1 < \infty$ as well as holdings θ_s in the market asset M.

The optimization problem at an intermediate time t, and with $k \leqslant n$ options remaining unexercised can be expressed as

$$H^k(t,x,\nu) = \sup_{t \leqslant \tau^k \leqslant \cdots \leqslant \tau^1} \sup_{(\theta_s)_{t \leqslant s < \tau^1}} \mathbb{E}_t[\tilde{U}(\tau^1, X_{\tau^1}) | X_t = x, V_t = \nu] \tag{5}$$

where

$$\tilde{U}(t,x) = -\frac{1}{\gamma} e^{-\gamma x} e^{(1/2)\lambda^2 t} = e^{(1/2)\lambda^2 t} U(x) \tag{6}$$

We want to find $H^n(0,x,\nu) = G^n(x,\nu)$, the value to the executive today of having the n options to exercise in the future.

We first remark on the problem formulation in (5). Although the executive is in fact solving an optimization problem over an infinite horizon (since the options have infinite maturity), the formulation in (5) evaluates wealth at time τ^1 using $\tilde{U}$ which depends on the Sharpe ratio of the market asset. This is because after τ^1 (when all options have been exercised), all wealth is assumed to be optimally invested in M and this is taken into account via the term $e^{(1/2)\lambda^2 t}$ in $\tilde{U}$. Thus $\tilde{U}$ does not represent the preferences of the executive directly, but rather is an implied or induced utility. By using $\tilde{U}$, we can solve the problem *as if* all wealth is liquidated at time τ^1 but the formulation reflects optimal investment behavior from then on. The term $e^{(1/2)\lambda^2 t}$ represents compensation for the opportunity cost of not exercising.[14]

We call $\tilde{U}(t,x)$ in (6) the horizon-unbiased exponential utility (see Henderson, 2007). It can also be interpreted as the choice which does not cause any artificial bias in the exercise times of the executive.[15] Appendix A contains further details of the horizon-unbiased exponential utility. For an account for general utilities, see Henderson and Hobson (2007).

We will show below that the executive's optimal behavior is to exercise options at a sequence of constant threshold levels $\tilde{V}^j; j = 1, \ldots, n$ which are obtained in recursive form in the following proposition. Proofs of all propositions are contained in Appendix A.

Proposition 1. *Define $\beta = 1 + 2\delta/\eta^2$. The exercise times $\tau^n \leqslant \cdots \leqslant \tau^1$ are characterized as the first passage times of stock price V to constant thresholds $\tilde{V}^j$; $j = 1, \ldots, n$ such that*

$$\tau^j = \inf\{t : V_t \geqslant \tilde{V}^j\}, \quad j = 1, \ldots, n.$$

The constant exercise thresholds $\tilde{V}^n, \ldots, \tilde{V}^1$ solve

$$C_{\gamma(1-\rho^2),\beta,K,\Lambda^{j-1}}(\tilde{V}^j) = 0, \quad j = 1, \ldots, n$$

where

$$C_{g,\xi,\kappa,G}(x) = x - \kappa - \frac{1}{g} \ln\left[1 + \frac{g}{\xi}(1 - Gx^{\xi})x\right] \tag{7}$$

and the constants $\Lambda^j(\gamma(1-\rho^2), \beta, K)$ are given by $\Lambda^0 = 0$ and

$$\Lambda^j = \left(\frac{1}{\tilde{V}^j}\right)^{\beta} (1 - e^{-\gamma(1-\rho^2)(\tilde{V}^j - K)^+}(1 - \Lambda^{j-1}(\tilde{V}^j)^{\beta})), \quad j = 1, \ldots, n-1$$

The time-independent functions $G^j(x,\nu)$ for $j = 1, \ldots, n$ are given by

$$G^j(x,\nu) = -\frac{1}{\gamma} e^{-\gamma x} \left[1 - (1 - e^{-\gamma(1-\rho^2)(\tilde{V}^j - K)^+}(1 - \Lambda^{j-1}(\tilde{V}^j)^{\beta}))\left(\frac{\nu}{\tilde{V}^j}\right)^{\beta}\right]^{1/(1-\rho^2)}$$

[13] Note that whilst this condition is standard, it could be relaxed and is not a necessary assumption for a closed-form solution. The main difference would be a change in the value of the parameter β.

[14] If there were no investment opportunities in the market asset, then $\lambda = 0$ would mean that $\tilde{U}(t,x) = U(x)$, as expected.

[15] Note the choice of $\tilde{U}$ in (6) is not necessary to obtain a closed-form solution in our model. An arbitrary discount factor could be introduced; however, this would create artificial incentives for early or late exercise, depending on whether the discount factor was greater than or smaller than $-\frac{1}{2}\lambda^2$.

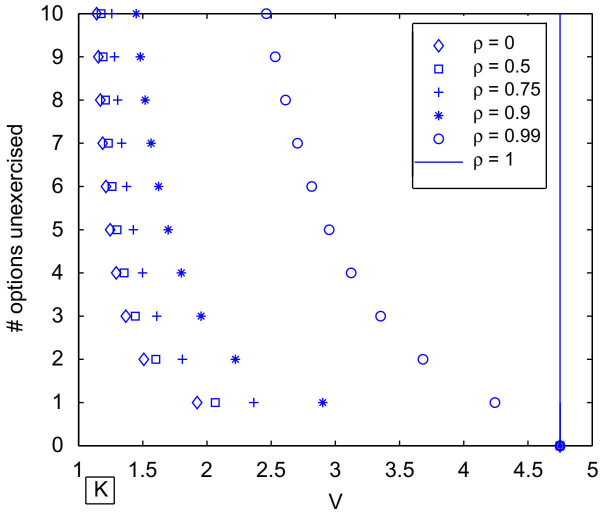

Fig. 1. *Exercise thresholds for model with portfolio choice of Proposition* 1: Plots of exercise thresholds for an executive with $n = 10$ options with $K = 1$. Risk aversion is fixed at $\gamma = 1$. The graph plots thresholds for various values of correlation. The threshold $\tilde{V}_{BS}$ arises under the Black Scholes assumptions that the stock V is tradeable by the executive, or equivalently, that $\rho = 1$. We take $\eta = 0.5$ and fix $\delta = 0.0335$ giving $\beta = 1.27$.

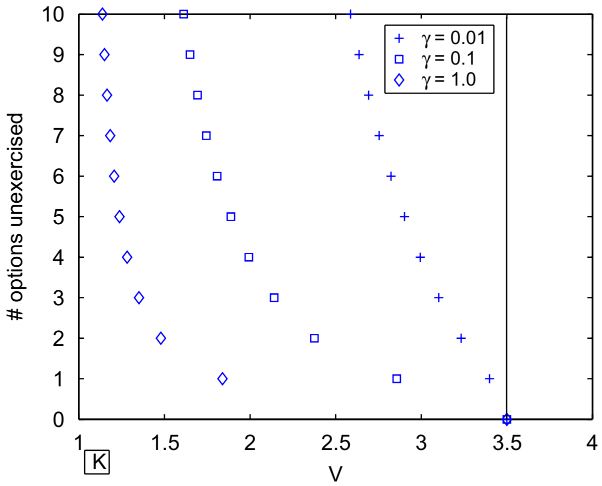

Fig. 2. *Exercise thresholds for model without outside portfolio choice*: Plots of exercise thresholds for an executive with $n = 10$ options with $K = 1$ and $\lambda = 0$, $\rho = 0$ so $\nu = 0$ and $\beta_0 = \beta$. Thresholds for three values of risk aversion are plotted, for $\gamma = 0.01, 0.1$ and 1. The solid line represents the threshold $\tilde{V}_{\langle\gamma=0\rangle}$ if the executive were risk-neutral. We take $\delta = 0.05, \eta = 0.5$ giving $\beta = 1.4$.

Fig. 1 gives exercise thresholds for $n = 10$ options with $K = 1$ for different values of correlation ρ between the stock and the asset M. Risk aversion is held fixed. Similarly, Fig. 2 displays various exercise thresholds for different levels of risk aversion but in the special case where there are no hedging opportunities available ($\lambda = 0, \rho = 0$). We observe in both cases, the executive should optimally exercise the 10 options at 10 distinct stock price thresholds. In fact, we can show in general that these thresholds are ordered, ruling out the possibility that multiple options are exercised at once.

Proposition 2. *For* $\gamma(1-\rho^2) > 0$, *the constant exercise thresholds defined in Proposition* 1 (*Case* 1), *satisfy* $\tilde{V}^n < \cdots < \tilde{V}^j < \tilde{V}^{j-1} < \cdots < \tilde{V}^1$.

The risk averse executive exercises his options one-at-a-time at an increasing sequence of stock price thresholds. We also see from the figures that the sequence of thresholds is convex. We explain why this is the form of the optimal exercise strategy. First, contrast the strategy of a risk-neutral executive. In this case the problem is a variant on the standard perpetual American option of Merton (1973). In this model, options are exercised when they are sufficiently in-the-money, that is, when the stock price is sufficiently above the strike K. Denote by $\tilde{V}_{\langle\gamma=0\rangle}$ the single exercise threshold in the risk-neutral model and $\beta_0 = 1 - 2(\nu - \delta)/\eta^2$. We obtain $\tilde{V}_{\langle\gamma=0\rangle} = (\beta_0/(\beta_0 - 1))K$ for $\beta_0 > 1$ and $\tilde{V}_{\langle\gamma=0\rangle} = \infty$ for $\beta_0 \leqslant 1$.[16] The solid vertical line in Fig. 2 indicates the threshold $\tilde{V}_{\langle\gamma=0\rangle}$. In this case, all n options are exercised at this single threshold.

The risk averse executive's exercise strategy differs from the risk-neutral strategy – all the exercise thresholds arising from the incomplete model are below the risk-neutral threshold and each option is exercised at a different threshold. The thresholds are all lower than the risk-neutral threshold because the executive dislikes being exposed to the unhedgeable risk of the stock price falling, and thus exercises earlier to avoid prolonging his exposure to risk. We can see on the figure that as risk aversion increases, each individual threshold is lowered. As we would expect, the risk-neutral threshold is precisely that obtained in the limit as risk aversion approaches zero.

Corollary 3. *As* $\gamma \to 0$,

Case 1: *If* $\beta_0 > 1$, *for* $j = 1, \dots, n$, $\lim_{\gamma\downarrow 0} \tilde{V}^j = \tilde{V}_{\langle\gamma=0\rangle} = (\beta_0/(\beta_0 - 1))K$.

Case 2: *If* $\beta_0 \leqslant 1$, $\lim_{\gamma\downarrow 0} \tilde{V}^j = \tilde{V}_{\langle\gamma=0\rangle} = \infty$ *for* $j = 1, \dots, n$ *and the executive waits indefinitely.*

Why does the executive unwind his option position gradually rather than all at once? The proposition and the figures show that the executive exercises each remaining option at a higher stock price threshold than the previous option. This is because unlike in a risk-neutral setting where identical options are all worth the same, each additional option is worth less to a risk averse agent. This results in a risk averse agent exercising each option at a different stock price threshold. The nth option is worth less than the $(n-1)$th option to the agent, and so he exercises the nth option earlier at a lower price threshold. Whilst the number of options remaining n is large, the thresholds will be quite close together, because incremental options are worth little to the agent when he already has a large (risky) position. However, as fewer options remain, the executive's option position (and hence exposure to risk) is smaller, and he is willing to wait longer between exercises. This results in the thresholds taking a convex shape.

We contrast also the exercise behavior under the standard perpetual American option model where the Black Scholes assumptions hold so the executive can trade in the stock V itself. Denote by $\tilde{V}_{\mathrm{BS}}$ the single exercise threshold under the perpetual American Black Scholes model. Standard arguments for perpetual American options (Merton, 1973) give $\tilde{V}_{\mathrm{BS}} = (\beta/(\beta - 1))K$. In fact, this threshold is the same as the one obtained in the limit as correlation approaches one in the model with risk aversion.[17]

Corollary 4 (*Complete market case*). *As* $\rho \to 1$, *the constant exercise thresholds* $\tilde{V}^j$, $j = 1, \dots, n$ *satisfy* $\lim_{\rho\to 1} \tilde{V}^j = \tilde{V}_{\mathrm{BS}}$.

The solid line in Fig. 1 represents the single threshold $\tilde{V}_{\mathrm{BS}}$. The individual thresholds $\tilde{V}^n < \cdots < \tilde{V}^1$ all lie below $\tilde{V}_{\mathrm{BS}}$. This is because although the executive can partially hedge in the market, he is exposed to the remaining unhedgeable risk, causing him to exercise options earlier to avoid the risk that the stock price will fall. We can see on the figure that a higher correlation results in a higher threshold $\tilde{V}^i$ at which the ith remaining option is exercised. The executive is exposed to less unhedgeable/idiosyncratic risk as ρ increases, and is willing to wait longer to exercise at a higher threshold. This is because the market in which trading is performed is providing a better hedge.

In fact, the only difference in the model with investment opportunities versus the model without such opportunities is that the executive's risk aversion γ is scaled down by $1 - \rho^2$. This scaled risk aversion coefficient represents the effective risk aversion in the model with the market asset M. The executive's opportunity to invest in the market means he is less exposed to the risk of the options. The more highly correlated the investment, the more risk he can hedge away and the more his risk aversion is scaled down. In fact, an executive with risk aversion γ and with the opportunity to invest in an asset with correlation ρ with his company stock chooses the same exercise thresholds as an executive without the investment opportunity, but with a (lower) risk aversion coefficient of $\gamma(1 - \rho^2)$. The implication is that for a given risk aversion level, γ, an executive with investment opportunities exercises each option at a higher threshold than the same executive without investment opportunities. The hedging opportunity allows the executive to delay exercising his options.[18]

We can be more precise about the hedging that the market is enabling the executive to undertake since in solving for the thresholds, we also solve for the holdings in the market asset.

Corollary 5. *Denote by* $\theta^i \equiv (\theta_s)_{\tau^{i+1} \leqslant s \leqslant \tau^i}$ (*where* $\tau^{n+1} = 0$). *The cash amount* θ^i *held in the market asset* M *when there are* i *options remaining is given by*

$$\theta^i = -\frac{\lambda G^i_x}{\sigma G^i_{xx}} - \frac{\rho\eta v G^i_{xv}}{\sigma G^i_{xx}} = \frac{\lambda}{\sigma\gamma} - \frac{\rho\eta\beta\Lambda^i v^\beta}{\sigma\gamma(1-\rho^2)(1-\Lambda^i v^\beta)}$$

[16] The form of this threshold is obtained by standard perpetual American option arguments, see Merton (1973). This condition translates to imply the option is exercised early provided $\nu < \delta$. Note the more familiar condition that a perpetual American call is exercised early if and only if there are dividends which arises only if the stock is *traded* whereby under the pricing measure $\nu = r$. Since $r = 0$, $\nu = 0$ and the condition becomes $\delta > 0$.

[17] Since the CAPM relationship in (3) holds, there is no difference between being able to hedge with the stock itself, and being able to hedge with a market which is perfectly correlated with the stock.

[18] Evidence of Hemmer et al. (1996) is consistent with this observation; however, in their paper, hedging refers to the firm adjusting other compensation (in addition to stock options) to offset to some degree changes in the value of options. Their conclusion is that such a reduction in the risk to the executive does reduce the extent to which executives exercise early.

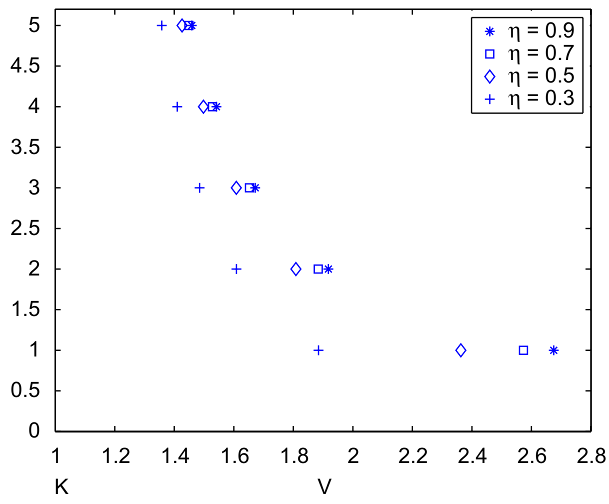

Fig. 3. *Effect of volatility*: Exercise thresholds for an executive with $n = 5$ options with $v = K = 1$ in the model with portfolio choice given in Proposition 1. Risk aversion is fixed at $\gamma = 1$ and correlation at $\rho = 0.75$. Thresholds are plotted for various values of stock volatility, η. We take $\delta = 0.0335$ giving values of $\beta > 1$.

We see that the position in the market consists of two terms. The first of these terms is the investment term. This is exactly the usual Merton (1971) strategy of holding a constant cash amount invested in the risky asset under exponential utility. The second term is the hedging term and thus represents the additional position the executive takes in the market in order to provide a partial hedge for his options. As we would expect, if $\rho > 0$, the hedge term is negative, so the market holdings are scaled down, and vice versa.

1.1. Impact of stock volatility

We can also consider how the exercise thresholds of the executive are altered by stock volatility, η. In a complete market, it is well known that the Black Scholes option value is increasing in stock volatility and therefore the threshold $\tilde{V}_{BS}$ is increasing with volatility. In Fig. 3 we plot the exercise thresholds for $n = 5$ options for various values of volatility. The graph shows increases in volatility η raise the level of each individual exercise threshold.[19] We observe that the impact of a change in volatility is more pronounced when fewer options remain unexercised. Recall, as each additional option is worth less to a risk averse agent, parameter changes (such as an increase in volatility) will have less impact on smaller prices and thresholds.

1.2. Restricted exercise

Although the executive's optimal behavior is to exercise his parcel of options one-at-a-time at a sequence of price thresholds, what if we restrict him from behaving optimally? We consider what his optimal behavior would be if he were forced to choose only one threshold at which to exercise all options. The single threshold is easily derived from Proposition 1 so we omit the details.[20] Fig. 4 illustrates the restricted exercise thresholds and compares to the unrestricted thresholds of Proposition 1. The two sets of markers in Fig. 4 represent the thresholds $\tilde{V}^i$ at which the ith remaining option is exercised (marked with $+$) and the restricted thresholds at which all options are exercised (marked with $*$). Once there is more than one option, each restricted threshold is higher than the corresponding unrestricted threshold at which just one option is exercised. That is, if the executive has to exercise all 10 options at once, he will do so at a higher stock price threshold than if he was just exercising his tenth *remaining* option. Each restricted threshold is somewhere between the first and last unrestricted thresholds so that the (single) restricted threshold is a (weighted) average of the unrestricted thresholds. As we would expect, the restricted threshold is always below the single Black Scholes threshold $\tilde{V}_{BS}$ which applies when the executive could perfectly hedge all risks. The difference between the Black Scholes threshold and the restricted threshold is simply reflecting the incompleteness of the market and the executive's risk aversion.

[19] In contrast, Henderson (2007) showed for the case of one option that the risk averse agent's threshold may increase or decrease with volatility. However, in her model, the CAPM equilibrium (3) was not required to hold since the model was in a real options context and the asset was not a traded stock. In that case, volatility had two effects – one via the convexity of the option payoff, and another via the reduction in expected return on the asset. The former causes the threshold to rise whilst the latter has the opposite effect. In our model, since the CAPM condition holds, the second of these effects does not occur and volatility always increases the thresholds.

[20] The single restricted exercise threshold $\tilde{V}_r^i$ solves $C_{i\gamma(1-\rho^2),\beta,K,0}(\tilde{V}_r^i) = 0$.

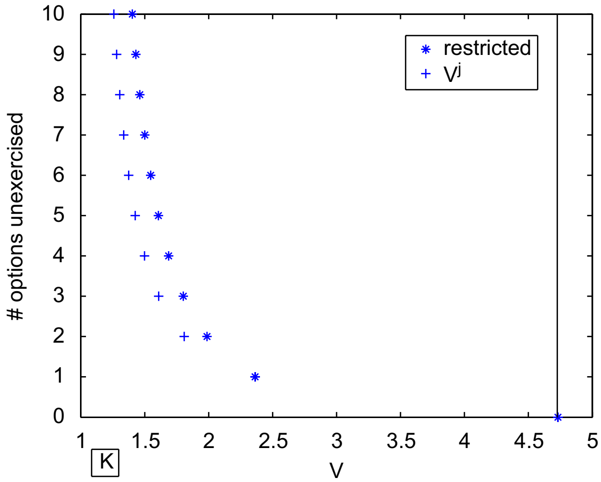

Fig. 4. *Restricted exercise*: A comparison of exercise thresholds $\tilde{V}^j;\ j=1,\ldots,n$ for the risk averse executive with portfolio choice (Proposition 1) with the restricted exercise thresholds. For a given number of options remaining, each restricted threshold is the single threshold at which all remaining options would be exercised, if forced to only exercise on one occasion. The threshold $\tilde{V}_{BS}=4.75$ is indicated with the solid line. Parameters are: $n=10$, $K=1$, $\rho=0.75$, $\gamma=1$, $\eta=0.5$, $\delta=0.0335$ giving $\beta=1.27$.

2. Optimal exercise and hedging policies with costly exercise

We now show that by including frictions in our model, we can recover the tendency of executives to exercise options on few occasions. We do this by including a fixed cost of exercise which is lost each time the executive exercises options, regardless of how many options are exercised.[21] One possible interpretation we have in mind is that the executive expends effort on making the decision of when to exercise which is costly to him. Another possible interpretation could be that for CEO's, exercising conveys a negative signal to the market. The executive will balance the exercise cost against the benefit of exercising optimally (as in Section 1).

Consider an executive who is granted n options and is subject to costly exercise. The per-exercise cost is represented by a constant c. Denote an exercise strategy of the n options by the vector of positive integers $q=(q_k,\ldots,q_1)$ where $\sum_{j=1}^k q_j=n$ and $q_j\geqslant 1; 1\leqslant j\leqslant k$. The size of the final block exercised is q_1, more generally, the size of the jth block exercised is q_{k-j+1} and the strategy q represents the sequence of such block sizes. There are $1\leqslant k\leqslant n$ exercise dates. We want to find the optimal of such q. In the case $c=0$, we are back in the setting of Proposition 1 where the optimal strategy was to take $k=n$ and $q_j=1,\ j=1,\ldots,n$. That is, the executive not subject to any exercise costs will exercise options one-at-a-time.

Each exercise strategy q will be associated with a set of exercise thresholds. If $k=1$, so all options are exercised at one time, the threshold $\tilde{V}_c^{q_1}$ is the level at which q_1 options are exercised. In this case $q_1=n$. If $k=2$ and the executive exercises q_2 options on one date, and the remaining q_1 options on a subsequent date (where $q_1+q_2=n$), then $\tilde{V}_c^{q_2,q_1}$ denotes the (first) threshold at which q_2 options are exercised, and $\tilde{V}_c^{q_1}$ denotes the second threshold, where the remaining q_1 options are exercised. In general, $q_j;\ j=1,\ldots,k$ options are exercised the first time $\tau_c^{q_j,q_{j-1},\ldots,q_1}$ that the stock price reaches threshold $\tilde{V}_c^{q_j,q_{j-1},\ldots,q_1}$. For example, if the executive begins with $n=3$ options, we may have $q=(1,1,1)$ with associated thresholds $\tilde{V}_c^{1,1,1}$ at which the first of three options is exercised, $\tilde{V}_c^{1,1}$ when the second is exercised and $\tilde{V}_c^{1}$where the final option is exercised. Note that options are only exercised if the stock price reaches the relevant threshold level. Other strategies are $q=(1,2),(2,1)$ or (3).

Before continuing further, we show how costly exercise impacts when the executive exercises multiple options on one date, so $k=1$. Assume there are l options to be exercised. The exercise cost c is paid when $q_1=l$ options are exercised at some time τ_c^l, and reduces the option payoff to $l(V_{\tau_c^l}-K)^+-c$. The options are only exercised if the payoff $l(V_{\tau_c^l}-K)^+$ exceeds c, so the effective payoff becomes $l(V_{\tau_c^l}-(K+c/l))^+$. In the presence of costs, the per-option strike is increased from K to $K+c/l$. This observation allows us to deduce the form of the exercise threshold.[22]

We now consider more than one exercise date, $k\geqslant 1$ and build solutions via a recursive formula based on the number of exercise dates. Let $p=(p_j,\ldots,p_1)$ where $\sum_{i=1}^j p_i<n$. Let $\tau_c^p\equiv\tau_c^{p_j,\ldots,p_1}$ be the first exercise time associated with the strategy p, at which p_j options are exercised. Let $\tilde{V}_c^p\equiv\tilde{V}_c^{p_j,\ldots,p_1}$ be the corresponding stock price threshold. Let $r=(l,p)$ where $l+\sum_{i=1}^j p_i\leqslant n$.[23] The following proposition gives the first exercise threshold τ_c^r associated with the strategy r, (at which l options are exercised), and the value to the executive of following this strategy.

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[21] This may be extended to the case where cost per exercise is made up of a fixed plus a proportional cost, $c+di$, for some constant d, where i is the number of options exercised. The effect of this form of cost could also be calculated in closed-form.

[22] The threshold $\tilde{V}_c^l$ solves $C_{l;\gamma(1-\rho^2),\beta,K+c/l,0}(\tilde{V}_c^l)=0$.

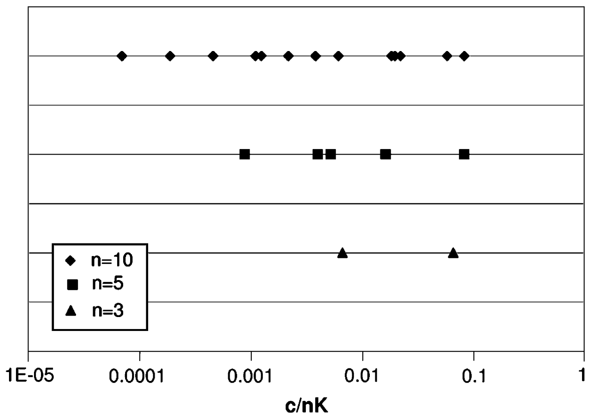

Fig. 5. *Costly exercise*: The plot displays optimal costly exercise strategies for $n = 3, 5$ and 10 options. The x-axis is in units of c/nK, indicating costs c as a proportion of total paid to exercise the n options, nK. The x-axis is in log scale. For example, $c/nK = 0.01$ indicates that costs c represent 1% of the total exercise amount nK. The markers indicate breakpoints between various optimal exercise strategies. The three sets of markers correspond to $n = 3$ (lowest), $n = 5$ and $n = 10$ (highest) options. For $n = 3$, the triangles indicate switches from strategies $q = (1, 1, 1)$ to $q = (2, 1)$ to $q = (3)$ as c/nK increases. For $n = 5$ options, the squares indicate the switches from strategies $q = (1, 1, 1, 1, 1), (2, 1, 1, 1), (2, 2, 1), (3, 1, 1), (4, 1)$ to (5), as c/nK increases. For $n = 10$ options, the diamonds indicate switching as c/nK increases through strategies: $q = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $(2, 1, 1, 1, 1, 1, 1, 1, 1)$, $(2, 2, 2, 1, 1, 1, 1)$, $(3, 2, 2, 1, 1, 1)$, $(3, 3, 2, 1, 1)$, $(4, 2, 2, 1, 1)$, $(4, 3, 2, 1)$, $(5, 3, 1, 1)$, $(6, 3, 1)$, $(7, 2, 1)$, $(7, 3)$, $(8, 2)$, $(9, 1)$, and (10). Parameters are: $K = 1$, $v = 1$, $\rho = 0.75$, $\gamma = 1, \eta = 0.5, \delta = 0.0125$ giving $\beta = 1.1$.

Proposition 6 (*Costly exercise*). *Suppose Ξ^p and $\tilde{V}_c^p$ are known. Then τ_c^r is characterized as the first passage time of V to constant threshold $\tilde{V}_c^r$ where $\tilde{V}_c^r \leqslant \tilde{V}_c^p$ satisfies*

$$C_{l\gamma(1-\rho^2),\beta,K+c/l,\Xi^p}(\tilde{V}_c^r) = 0$$

The constant Ξ^r (necessary for the next inductive step) is given by

$$\Xi^r = \Xi^{l,p} = \left(\frac{1}{\tilde{V}_c^r}\right)^{\beta}(1 - e^{-l\gamma(1-\rho^2)(\tilde{V}_c^r-(K+c/l))^+}(1 - \Xi^p(\tilde{V}_c^r)^\beta))$$

The value to the executive at time zero, given he has $l + \sum_{i=1}^{j} p_i$ options and exercises them according to pattern $r = (l, p_j, \ldots, p_1)$ is

$$G_c^r(x, v) = -\frac{1}{\gamma}\, e^{-\gamma x}\left[1 - (1 - e^{-l\gamma(1-\rho^2)(\tilde{V}_c^r-(K+c/l))^+}(1 - \Xi^p(\tilde{V}_c^r)^\beta))\left(\frac{v}{\tilde{V}_c^r}\right)^{\beta}\right]^{1/(1-\rho^2)}$$

We define the executive's optimal exercise strategy to be that $q = (q_k, \ldots, q_1)$ with $\sum_{i=1}^{k} q_i = n$ which maximizes value $G_c^q(x, v)$. We are interested in how this optimal strategy q varies with costs. We now illustrate the results of Proposition 6. We first consider an executive receiving three options. Possible strategies for the executive are to: exercise the three options simultaneously and pay c only once, exercise all three options separately and pay c each time, exercise one option first and then two later and pay c twice, or exercise two options on a single date and the final one later, again paying c twice. For each of these scenarios, the exercise cost c is only paid if the stock price threshold is attained.

We plot in Fig. 5 the cost breakpoints between various optimal strategies for $n = 3, 5$ and 10 options.[24] We want to express our results in a way that considers the relative size of c. We do this by considering c relative to the total amount paid to exercise the n options, which is nK. On the x-axis we give the ratio c/nK, in log scale. For example, at $c/nK = 0.01$, costs c represent 1% of the total amount paid.

For $n = 3$ options, we see that three of the four above possibilities occur. The two triangles on the graph give the values of c/nK where the optimal strategy changes. For very low costs, $c/nK < 0.0066$, the optimal exercise strategy is to exercise each option separately so $q = (1, 1, 1)$. Recall if there were no costs associated with exercise, the executive would optimally exercise the options individually and this remains true when costs are small. Between the two triangles on the graph, the strategy of exercising two options on a single date and then the third option at a later date, $q = (2, 1)$ is optimal. Finally, for large costs, where $c/nK > 0.063$, it is optimal to exercise all three options at once, $q = (3)$. Notice that it is never optimal to exercise one option followed by two options, so the strategy $q = (1, 2)$ does not occur.

[23] If $j = 0$ then p is the empty vector $\emptyset$ and $r = (l)$. In this case, for the purposes of the following proposition, $\Xi^\emptyset = 0$ and $\tilde{V}_c^\emptyset = \infty$.

[24] Of course grants of options are much bigger than this. As the number of options rises, enumerating all the possible exercise strategies becomes difficult. However 10 options are sufficient to demonstrate our conclusions.

For $n = 5$ options, the squares indicate the change points between different strategies. When $c/nK < 0.00088$, the options are exercised one-at-a-time. As c/nK increases, the optimal strategies are in turn $q = (2, 1, 1, 1)$, $(2, 2, 1)$, $(3, 1, 1)$, $(4, 1)$ and (5). When $n = 10$ options the change from one strategy to another is marked by the diamonds in Fig. 5 and the optimal strategies are given in the caption.

The results of Fig. 5 show that risk averse executives optimally exercise options in a small number of blocks once the exercise costs are large enough. We see that exercise costs need only be a small proportion of total outlay for block exercise to occur. For $n = 10$ options, costs of the order of 0.1% correspond to exercise strategies where 30% or 40% of options are exercised at one date, followed by smaller blocks. Costs around 1% correspond to a strategy of $(6, 3, 1)$ where only three exercise dates are chosen to exercise 10 options. So, the cost of effort does not have to be very large to have a major impact on the executive's exercise strategy.

The finding that executives should exercise in blocks is consistent with the empirical evidence of Huddart and Lang (1996) who find that the mean fraction of options of a single grant exercised by an employee at one time varied from 0.18 to 0.72. They also plot exercise frequency as a function of elapsed life and the percentage of the single option grant which is exercised, and note that '…the distribution across exercise percentages suggests employees typically exercise options in large blocks' (p. 21).

We can make a number of further observations from Figure 5. First, we find the largest block size increases as the cost c increases. This is intuitive since as the cost per exercise increases, the executive is tempted to exercise more options in a single block to save on costs. A related observation is that the number of blocks decreases with c. The executive exercises options on fewer occasions as costs rise.

It is to be expected that as costs rise, exercise takes place in fewer and larger blocks. However, less obvious is the finding that for a fixed value of c, if it is optimal to exercise across more than one date, the block size across the series of dates is always non-increasing. For example, for the situation with $n = 3$ options, it was never optimal to use the strategy $(1, 2)$. Indeed, for $n = 10$ options, only a small subset of possible exercise strategies actually appear as optimal ones. This pattern occurs in the model because of the features of costly exercise and risk aversion. The cost basically determines how many times the executive exercises, with a higher c leading to fewer exercise dates. When there are costs of exercise, the question becomes how the n options should be split into exercise quantities $(q_k, \ldots, q_1)$. Recall the exercise thresholds of Fig. 1, where in the absence of costs, options are exercised one-at-a-time. Since the thresholds in Fig. 1 are convex, if we split the sequence into k groups it is inevitable that there are more options in the first group and fewer in each subsequent group.[25] Whether block sizes decrease across exercise dates in practice is an empirically testable implication of the model.[26]

3. Implications for the cost to the company

In this section, we consider the impact of our findings on estimates of the cost of stock options to the company or equivalently, to shareholders. This is important since firms are required under SFAS 123R to recognize as an expense the fair value of their stock option grants using 'an appropriate valuation technique' which recognizes the timing of option exercise.

Since shareholders can diversify the risks of granting stock options, the cost to the company can be calculated as a standard tradeable option, conditional on the exercise behavior of executives, see Carpenter (1998). In the literature, this has been implemented in two main ways – via either a utility-based or reduced-form model. Carpenter (1998) and Bettis et al. (2005) use a utility-based binomial model with a single exercise threshold to calculate company cost. The main disadvantage of using a utility-based approach to model company cost is that risk aversion levels are unobserved and thus Carpenter (1998) and Bettis et al. (2005) calibrate their models to the data.

In contrast, the reduced-form approach does not model the executive's exercise policy from first principles, but rather models early exercise as an exogenous factor. Hull and White (2004) (see also Cvitanic et al., 2006)[27] assume all executives of a company exercise at a single constant exogenous barrier level, expressed as a percentage of the strike.

Under both of these approaches, the literature has assumed that all options are exercised at a single threshold. We comment here on the implications of our model for estimation of company cost under each of these approaches. First, since we find that exercising at multiple barriers is optimal, our model would suggest that an extension of the Hull and White model with a number of barriers should provide an improvement in their approximation of exercise behavior. Second, we ask in our utility-based framework, how much error does one make in estimating company cost by assuming a single exercise barrier rather than a sequence of barriers? The company cost estimate in our model is simply the value of an equivalent tradeable perpetual American option but with the exercise decision controlled by the executive. The exercise thresholds are exogenous inputs into the calculation of company cost. This way, the characteristics of executives (such as risk aversion, their hedging capabilities, the costs they face) impact on the cost to the company via their exercise decisions.

[25] We can use a similar explanation to argue why as c increases, the strategies $(7, 3), (8, 2), (9, 1)$ appear in this particular order.

[26] Vesting may alter this conclusion when there are multiple vest dates. Consider a situation where more options are about to vest and the stock price has already passed a number of thresholds during the vesting period. The executive will exercise a block of options on the vest date which may exceed the previous quantity exercised. However the hypothesis could be tested on options which have already vested.

[27] Other authors including Rubinstein (1995), Cuny and Jorion (1993) and Carpenter (1998) previously used exogenous exercise to obtain an ESO cost.

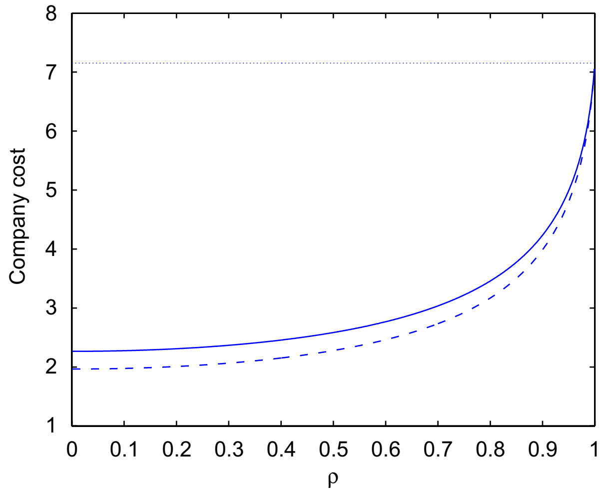

Fig. 6. *The cost to the company under different exercise strategies of the manager.* We take $c = 0$, $n = 10$ options, $v = K = 1$, $\gamma = 1$. The solid line represents the cost if the executive exercises according to the optimal strategy in Proposition 1 (one-at-a-time). The dashed line is the cost if the executive is restricted to exercise all options at one threshold. We take $\eta = 0.5$ and $\delta = 0.0125$ giving $\beta = 1.1$. The horizontal dotted line represents the company cost under the perpetual American Black Scholes model whereby the executive exercises all options at level $\tilde{V}_{BS}$.

Proposition 7. *Assuming a risk averse executive granted n options with strike K exercises according to the costly exercise model of Proposition* 6, *the cost to the company is given by*

$$\sum_{j=1}^{k} q_j(\tilde{V}^{q_j,\ldots,q_1} - K)\left(\frac{v}{\tilde{V}^{q_j,\ldots,q_1}}\right)^{\beta}$$

where $1 \leqslant k \leqslant n$ *is the number of exercise dates chosen by the executive, and* $\tilde{V}^{q_j,\ldots,q_1}$ *represents the threshold at which* q_j *options are exercised.*

In Fig. 6 we treat the case $c = 0$, that is, there are no costs to exercise.[28] The solid line in Fig. 6 represents the cost to the company of $n = 10$ options which are exercised optimally by the executive according to the 10 thresholds given in Proposition 1. We observe that company cost is increasing with correlation.[29] Since we observed earlier that correlation and risk aversion have opposite effects on thresholds, it is not surprising that company cost will be decreasing with risk aversion. The highest (dotted) line in Fig. 6 corresponds to the cost obtained under the Black Scholes assumption that the executive can trade the stock V, and thus exercises all options at the threshold $\tilde{V}_{BS}$. As anticipated, the company cost under the Black Scholes assumptions is significantly higher than the cost under our optimal exercise model for values of correlation lower than one. As correlation approaches one, the company cost under the optimal exercise model approaches that under the Black Scholes model.

We can now ask: what difference does it make to company cost if the executive is restricted to exercise at one rather than n thresholds? Fig. 6 also answers this question. The dashed line represents the cost to the company given the executive must exercise all options simultaneously. We see this results in a company cost which is lower than when the executive is free to exercise optimally. The difference between the two estimates is decreasing with correlation – when ρ is close to zero, the difference is around 15% for our parameters.[30] Our findings suggest utility-based models which have assumed a single threshold (Carpenter, 1998; Bettis et al., 2005) are likely to have underestimated company cost.

4. Robustness to model assumptions

4.1. Infinite maturity

We investigate the impact of the infinite maturity assumption on our results. To do this, we compute a binomial approximation of the model (see Grasselli, 2005) and compare the resulting exercise threshold to that obtained from our model. First we consider the simplest case where the executive only has one option. Both the exercise threshold from the

[28] Our conclusion that a single threshold underestimates company cost will hold also in the case $c > 0$ although we do not show this for space reasons.

[29] This arises since the company cost is an increasing function of the optimal thresholds of the executive, and higher correlation raised the optimal exercise thresholds, see Section 1.

[30] Similarly, the difference between the two estimates is increasing in risk aversion. For example, when $\gamma = 1$, the difference is about 10%.

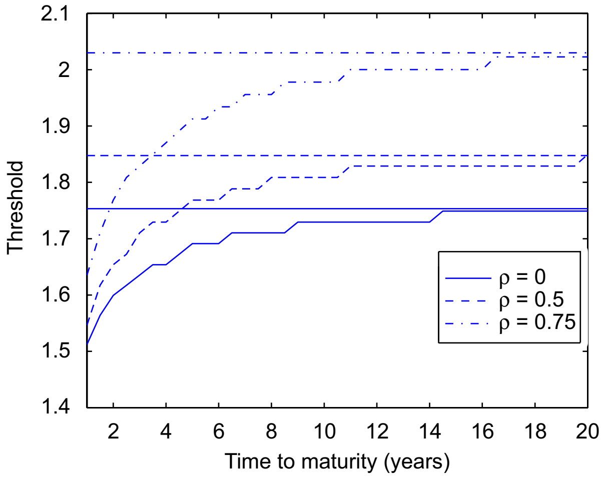

Fig. 7. *Convergence of exercise thresholds for $n = 1$ option.* There are three pairs of thresholds each corresponding to a different value of correlation. The thresholds are plotted for varying time-to-maturity. The horizontal lines correspond to thresholds calculated from the infinite maturity model of Proposition 1. The curved lines represent thresholds calculated from a binomial approximation. Risk aversion is fixed at $\gamma = 1$. Other parameters are: $\eta = 0.5$, $\delta = 0.07$, resulting in $\beta = 1.56$.

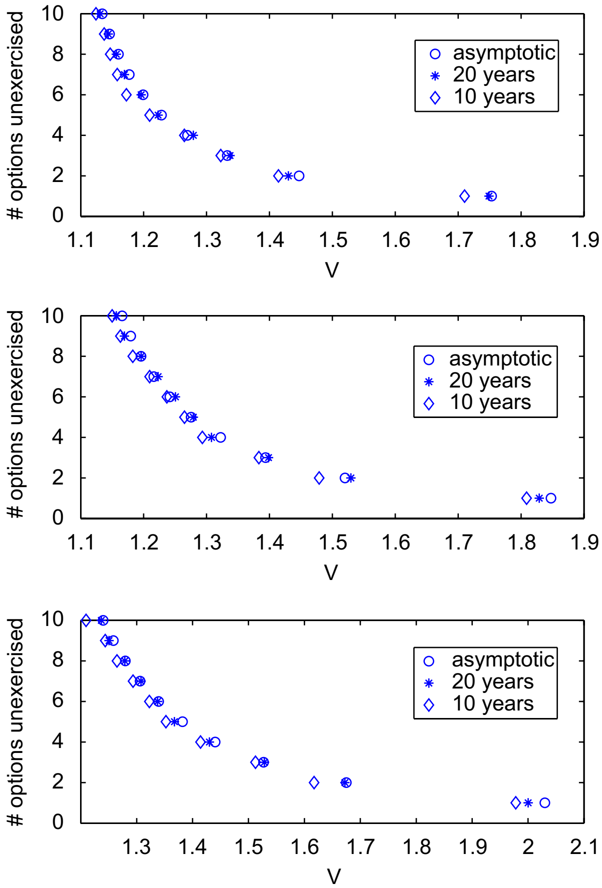

Fig. 8. *Convergence of exercise thresholds for $n = 10$ options.* The panels compare the exercise thresholds for 10 options calculated from the infinite maturity model (of Proposition 1) and under a binomial approximation with either a 10 or 20 year maturity. The three panels correspond to different values of correlation: the top panel takes $\rho = 0$, the middle panel takes $\rho = 0.5$ and the lower panel takes $\rho = 0.75$. Risk aversion is fixed at $\gamma = 1$. Other parameters are: $\eta = 0.5$, $\delta = 0.07$, resulting in $\beta = 1.56$.

binomial approximation and the threshold from the infinite maturity model of Proposition 1 are plotted in Fig. 7. As we expect, the binomial model gives a threshold which depends upon the time-to-maturity. As the time-to-maturity increases, the threshold converges to the (constant) infinite maturity threshold. We display up to 20 years to demonstrate this convergence. Many ESOs have maturity of 10 years. At the 10 year point on the figure, the binomial thresholds have converged to within a couple of percent of the infinite maturity threshold.

We can also consider the case of multiple options. In this case, the binomial approximation will give an exercise surface as a function of time-to-maturity and the stock price. Since it is difficult to depict clearly the comparison of the entire surface to the infinite maturity thresholds, we plot the binomial model thresholds for 10 and 20 year maturities in Fig. 8. The figure also plots the corresponding thresholds for the infinite maturity model of Proposition 1. We see the errors introduced by using the infinite maturity model decrease as the number of options increases. That is, for each additional option, the finite maturity threshold with 10 and 20 year maturity are closer to the infinite maturity threshold. Thus for a large grant of options, the thresholds at a 10 year maturity are going to be virtually indistinguishable from those of an infinite maturity option. Of course as the time-to-maturity reduces, this is no longer the case, but again, as the number of options increases, the convergence of the thresholds is faster, leading to smaller errors.

It is worth pointing out here that an issue better dealt with in the binomial framework (or a model which dealt with options of a finite maturity) is the treatment of multiple option grants on different dates.

4.2. Vesting restrictions

In the paper we have assumed that the options have no vesting period, which is not usually the case in practice. For example, a typical structure is where over the first four years, a quarter of the option grant becomes exerciseable each year. Clearly vesting is going to contribute to block exercises – if large parcels of options vest at certain times then large block exercises may follow. However, such block exercises motivated by vesting will occur immediately following the vest date. Huddart and Lang (1996) observe that although exercises that correspond to vest dates are common, they do not account for all the block-exercises in the data. For instance, there are situations where after the entire grant of options is fully vested, the remaining unexercised options appear to be exercised according to a block-like structure over their remaining life. Further, block exercise certainly occurs in proportions other than multiples of the vest parcel size of say 25%. These observations lead us to conclude that although vesting is certainly a contributor to block exercise, it cannot be the only reason block exercises occur.

Consider now the theoretical impact of vesting on exercise behavior. Under either risk-neutral or standard Black Scholes assumptions, the executive would exercise only in multiples of 25%, depending on whether the single stock price threshold had been reached when each 25% parcel of options vested. In our model with risk aversion (but no costs of exercise) and with no vesting, we find the executive exercises options individually. If vesting were also included in our model, multiple options would be potentially exercised together (constituting a block), but this would *only* occur on the vest dates when tranches of options became available for exercise. Clearly, without frictions such as costly exercise, neither vesting alone, nor risk aversion alone, nor vesting and risk aversion together can fully explain observed exercise patterns. Whilst vesting can provide a partial explanation for empirically observed block exercise behavior, it cannot provide the full story. By focusing on risk aversion and costly exercise in this paper, we show block exercise occurs *even in the absence of vesting*.

4.3. Employment termination

Executives may of course leave their firms, either voluntarily or non-voluntarily. Executives usually forfeit unvested options and are able to exercise vested options within a short time frame of leaving the firm, although the precise details across firms. The employee departure can be modeled by an exogenous exponentially distributed time with constant intensity, see Carr and Linetsky (2000) amongst others. Intuitively, we would expect that including employment termination in our model would reduce the exercise thresholds further relative to the perfect hedging case. That is, the executive would exercise options even earlier due to the additional termination risk faced. The impact of employment termination risk could therefore be approximated by assuming the executive had a higher risk aversion level. The addition of employment termination would not have an impact on the qualitative nature of the exercise structure we find.

5. Conclusions

Our emphasis in this paper is on providing a model which is consistent with well-known observed features in ESO exercise patterns such as early exercise, and the tendency of executives to exercise large quantities of options on a small number of occasions. We first show that risk averse executives who receive a grant of n identical options will optimally exercise them individually at an increasing sequence of stock price thresholds. All these thresholds are below the single Black Scholes threshold at which all n options are exercised. This implies that utility-based ESO models do not predict block exercise. Although our focus is on the determination of the executive's optimal exercise behavior given multiple options, this model also contributes to the literature on the executive's subjective or utility-based valuation of options.[31]

The introduction of a friction such as costly exercise into our utility-based model generates exercise patterns consistent with observed behavior. Costs need only be very small (compared to the total outlay to exercise) in order for block exercise to occur. Another reason for block exercise which could have been modeled might be liquidity needs of the executive. The costly exercise model also gave the new, empirically testable prediction that executives should begin by exercising large blocks of options, but the block sizes should become smaller over time.

Our framework also allows us to explore the estimation of the cost of options to the company. We show that restricting the executive to exercise all options at one time leads to an underestimate of company cost. This suggests that existing models which (implicitly or explicitly) assume a single exercise date result in an estimate of costs on the low side. Related to this is the issue of testing models on exercise data. Carpenter (1998) and Bettis et al. (2005) have tested utility-based models on exercise data but assume that the executive exercises all options on only one occasion. As we demonstrate, this assumption is not consistent with optimal behavior under a utility-based model and in fact, we show executives optimally exercise options over a number of dates.

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Appendix A

This appendix contains proofs of the results in the main text.

A.1. Horizon-unbiased utilities: a derivation of (6)

For simplicity consider the case of $n = 1$ option. The option is exercised at some time τ of the executive's choosing. First consider the executive's problem under the restriction that $\tau < T$ for some finite horizon T. That is, assume the option has a finite maturity T. Assume the executive's preferences are described by a utility function $U(x)$ for wealth at T.

Temporarily ignoring the option, consider solving the optimal investment problem

$$\sup_{(\theta_s)_{t<s\leqslant T}} \mathbb{E}[U(\hat{X}_T)|\hat{X}_t = x]$$

where $\hat{X}_s = x + \int_t^s \theta_u \, \mathrm{d}M/M$ denotes non-option wealth at time $t<s<T$. That is, $\hat{X}_s$ is wealth generated purely from the investment in the market asset. Denote by

$$\tilde{U}(t,x) = \sup_{(\theta_s)_{t\leqslant s\leqslant T}} \mathbb{E}[U(\hat{X}_T)|\mathscr{F}_t] \tag{A.1}$$

the solution to this investment problem at $t<T$. This is just the indirect utility or value function. We now want to choose a time $\tau \leqslant T$ at which to exercise the option to receive $(V_\tau - K)^+$. Conditioning shows that

$$\sup_{\tau\leqslant T} \sup_{(\theta_s)_{s\leqslant T}} \mathbb{E}U(\hat{X}_T + (V_\tau - K)^+) = \sup_{\tau\leqslant T} \sup_{(\theta_s)_{s\leqslant \tau}} \mathbb{E}\tilde{U}(\tau, \hat{X}_\tau + (V_\tau - K)^+)$$

It is equivalent to either solve the stopping/exercise problem over the finite horizon T (with utility U), or to solve the stopping/exercise problem up to the (exercise) time τ but using $\tilde{U}(t,x)$ in (A.1) to evaluate wealth at τ. In this sense, the real problem (in which the horizon is T and the executive has preferences described by utility function $U(x)$) is translated into one in which it is *as if* the optimization ends at τ and the induced function $\tilde{U}$ depends on market parameters through λ.

[31] There are many papers which value options to the executive under a utility-based model, see Footnote 3. However, only a few of these incorporate a traded market asset in which the executive can invest optimally. Detemple and Sundaresan (1999) present a trinomial model, Ingersoll (2006) gives a model to value a marginal quantity of options and Kahl et al. (2003) value stock but not options. These models all rely on numerical methods. Henderson (2005) gives values in closed-form, but only for European style options. For reasons of space, we do not provide details of the option valuation to the executive. However, this can easily be done in closed-form, given the exercise thresholds, along with sensitivities to various model parameters.

In the case of exponential utility (with an extra constant A), $U(x) = -\frac{A}{\gamma}\,\mathrm{e}^{-\gamma x}$, standard calculations show that $\tilde{U}(t,x) = -(A\,\mathrm{e}^{-\lambda^2 T/2})(1/\gamma)\,\mathrm{e}^{\lambda^2 t/2-\gamma x}$ and then from the above,

$$A \sup_{\tau\leqslant T}\sup_{(\theta_s)_{s\leqslant T}} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{-\gamma(\hat{X}_T+(V_\tau-K)^+)}\right] = A\,\mathrm{e}^{-\lambda^2 T/2}\sup_{\tau\leqslant T}\sup_{(\theta_s)_{s\leqslant\tau}} \mathbb{E}\left[-\frac{1}{\gamma}\,\mathrm{e}^{\lambda^2\tau/2-\gamma(\hat{X}_\tau+(V_\tau-K)^+)}\right]$$

Clearly, it is equivalent to solve the problem on the right-hand side without the pre-factor $A\,\mathrm{e}^{-\lambda^2 T/2}$. The relevant problem to solve becomes

$$\sup_{\tau\leqslant T}\sup_{(\theta_s)_{s\leqslant\tau}} \mathbb{E}\left[-\frac{1}{\gamma}\,\mathrm{e}^{\lambda^2\tau/2-\gamma(\hat{X}_\tau+(V_\tau-K)^+)}\right] \tag{A.2}$$

and (dropping the pre-factor) we now call

$$\tilde{U}(t,x) = -\frac{1}{\gamma}\,\mathrm{e}^{\lambda^2 t/2}\,\mathrm{e}^{-\gamma x} \tag{A.3}$$

the horizon-unbiased exponential utility. This is exactly as given in (6).

In the paper, we consider the infinite horizon problem. In this case the natural generalization is to remove the restriction $\tau\leqslant T$ in (A.2) and to consider

$$\sup_{\tau}\sup_{(\theta_s)_{s\leqslant\tau}} \mathbb{E}[-\mathrm{e}^{\lambda^2\tau/2-\gamma(\hat{X}_\tau+(V_\tau-K)^+)}] = \sup_{\tau}\sup_{(\theta_s)_{s\leqslant\tau}} \mathbb{E}[\tilde{U}(\tau,X_\tau)]$$

where $X_\tau = \hat{X}_\tau + (V_\tau - K)^+$. This is exactly the formulation in (5) in the special case of $n = 1$ option. When there are n options, the relevant time is the time when the last option is exercised and all remaining wealth is invested optimally after that.

We now give an alternative interpretation of $\tilde{U}$ in (A.3). This is based on the idea that it is the choice which ensures an executive facing an optimal investment problem (without options) over horizon τ (where τ is a stopping time, $\tau<\infty$) is indifferent to the choice of horizon. We show $\tilde{U}$ is such that the solution of the investment problem

$$\sup_{(\theta_u)_{t\leqslant u\leqslant\tau}} \mathbb{E}[\tilde{U}(\tau,\hat{X}_\tau)|\hat{X}_t = x] \tag{A.4}$$

does not depend on the horizon τ. The intuition is that under this formulation, the executive with options to exercise does not already have an in-built preference for early or late exercise arising from the underlying investment problem.

To show $\tilde{U}$ has this interpretation, we show that $\tilde{U}(t,\hat{X}_t)$ is a super-martingale in general, and a martingale for the optimal θ (recall $\tilde{U}\leqslant 0$). Once we have these properties we have that

$$\tilde{U}(t,x) = \sup_{(\theta_u)_{t\leqslant u\leqslant\tau}} \mathbb{E}[\tilde{U}(\tau,\hat{X}_\tau)|\hat{X}_t = x]$$

and we can write

$$\tilde{U}(t,x) = \sup_{\tau}\sup_{(\theta_u)_{t\leqslant u\leqslant\tau}} \mathbb{E}[\tilde{U}(\tau,\hat{X}_\tau)|\hat{X}_t = x]$$

since $\tilde{U}(t,x)$ does not depend on the horizon τ.

We now show these properties. Applying Itô's formula to $\tilde{U}(t,\hat{X}_t)$ and integrating gives

$$\tilde{U}(\tau,\hat{X}_\tau) = \tilde{U}(t,\hat{X}_t) + \int_t^\tau \frac{\tilde{U}(s,\hat{X}_s)}{2}[\lambda-\gamma\theta_s\sigma]^2\,\mathrm{d}s - \int_t^\tau \gamma\theta_s\sigma\tilde{U}(s,\hat{X}_s)\,\mathrm{d}B_s$$

It follows that $\mathbb{E}\tilde{U}(\tau,\hat{X}_\tau)\leqslant\tilde{U}(t,\hat{X}_t)$ for any θ, and using the optimal strategy solving the investment problem (A.4), $\theta^*_s = \lambda/\gamma\sigma$, we have

$$\sup_{(\theta_u)_{t\leqslant u\leqslant\tau}} \mathbb{E}[\tilde{U}(\tau,\hat{X}_\tau)] = \tilde{U}(t,\hat{X}_t)$$

Hence $\tilde{U}(t,\hat{X}_t)$ is a super-martingale in general and a martingale for the optimal θ.

Proof of Proposition 1. The executive's optimization problem at an intermediate time t, and with $i\leqslant n$ options remaining unexercised is to find

$$H^i(t,x,v) = \sup_{t\leqslant\tau^i\leqslant\cdots\leqslant\tau^1}\sup_{(\theta_s)_{t\leqslant s<\tau^1}} \mathbb{E}_t[\tilde{U}(\tau^1,X_{\tau^1})|X_t = x, V_t = v]$$

where $\tilde{U}(t,x)$ is given in (6). Recall also that wealth X_{τ^1} includes the payoffs of all options exercised at times prior to and including date τ^1. First define the time-independent function $G^i(x,v)$ via

$$G^i(x,v) = \sup_{t\leqslant\tau^i\leqslant\cdots\leqslant\tau^1}\sup_{(\theta_s)_{t\leqslant s<\tau^1}} \mathbb{E}_t\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2(\tau_1-t)-\gamma X_{\tau^1}}\middle| X_t = x, V_t = v\right] \tag{A.5}$$

Then

$$H^i(t,x,v) = \sup_{t\leqslant\tau^i\leqslant\cdots\leqslant\tau^1} \sup_{(\theta_s)_{t\leqslant s<\tau^1}} \mathbb{E}_t\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2\tau_1-\gamma X_{\tau^1}}\middle|X_t=x,V_t=v\right]$$
$$= \mathrm{e}^{(1/2)\lambda^2 t} \sup_{t\leqslant\tau^i\leqslant\cdots\leqslant\tau^1(\theta_s)} \sup_{t\leqslant s<\tau^1} \mathbb{E}_t\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2(\tau_1-t)-\gamma X_{\tau^1}}\middle|X_t=x,V_t=v\right]$$
$$= \mathrm{e}^{(1/2)\lambda^2 t}G^i(x,v) \tag{A.6}$$

and the problem reduces to finding $G^i(x,v)$ for $i\leqslant n$. Since $H^i(0,x,v)=G^i(x,v)$, the value today to the executive of the i remaining options is just $G^i(x,v)$ where $G^i(x,v)$ is a time-independent function.

We now proceed by backwards induction. We first solve the case where there is only one option remaining and obtain threshold $\tilde{V}^1$ (Part A). The problem with only one option is closely related to that studied in Henderson (2007). The general form is proved by induction. We propose the form of the solution when $(i-1)$ options remain to be exercised, and then use this to solve for the case of i remaining unexercised options (Part B).

Part A: When there is only one option remaining unexercised, the value function $H^1(t,x,v)$ is given by

$$H^1(t,x,v) = \sup_{t\leqslant\tau^1}\sup_{\theta^1}\mathbb{E}_t[\tilde{U}(\tau^1,X_{\tau^1})|X_t=x,V_t=v]$$

where θ^1 denotes holdings in M between times t and τ^1. By (A.6) and (A.5), $H^1(t,x,v)=\mathrm{e}^{(1/2)\lambda^2 t}G^1(x,v)$ and

$$G^1(x,v) = \sup_{t\leqslant\tau^1}\sup_{\theta^1}\mathbb{E}_t\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2(\tau_1-t)-\gamma X_{\tau^1}}\middle|X_t=x,V_t=v\right]$$

By time homogeneity, we propose the exercise time as the first passage time of V to a constant threshold $\tilde{V}^1$ such that $\tau^1=\inf\{t:V_t\geqslant\tilde{V}^1\}$. For now we assume V follows (1) with expected return ν and dividend yield δ. Later we specialize to the equilibrium choice $\nu=\mu\rho\eta/\sigma$ in (3).

In the continuation region, $H^1(t,x,v)=\mathrm{e}^{(1/2)\lambda^2 t}G^1(x,v)$ is a martingale under the optimal strategy θ^1, and the Bellman equation is given by

$$\tfrac{1}{2}\lambda^2G^1+(\nu-\delta)vG^1_v+\tfrac{1}{2}\eta^2v^2G^1_{vv}+\sup_{\theta^1}\left\{\theta^1\mu G^1_x+\tfrac{1}{2}\sigma^2(\theta^1)^2G^1_{xx}+\theta^1\sigma\rho\eta vG^1_{xv}\right\}=0 \tag{A.7}$$

Optimizing over strategies gives

$$\theta^1 = -\frac{\lambda G^1_x}{\sigma G^1_{xx}}-\frac{G^1_{xv}\rho\eta v}{\sigma G^1_{xx}} \tag{A.8}$$

and substituting back into the Bellman equation (A.7) gives

$$0=\frac{1}{2}\lambda^2G^1+(\nu-\delta)vG^1_v+\frac{1}{2}\eta^2v^2G^1_{vv}-\frac{1}{2}\frac{(\lambda G^1_x+\rho\eta vG^1_{xv})^2}{G^1_{xx}} \tag{A.9}$$

with boundary condition $G^1(x,0)=-\frac{1}{\gamma}\,\mathrm{e}^{-\gamma x}$. Value-matching at the exercise threshold gives

$$G^1(x,\tilde{V}^1)=-\frac{1}{\gamma}\,\mathrm{e}^{-\gamma x}\,\mathrm{e}^{-\gamma(\tilde{V}^1-K)^+}$$

and finally, smooth-pasting gives $G^1_v(x,\tilde{V}^1)=\mathrm{e}^{-\gamma x}\,\mathrm{e}^{-\gamma(\tilde{V}^1-K)^+}$.

The problem is to solve (A.9) subject to these three conditions. We first factor out wealth x by proposing $G^1(x,v)=-\frac{1}{\gamma}\,\mathrm{e}^{-\gamma x}J^1(v)$ and setting $J^1(v)=(I^1)^{1/(1-\rho^2)}$, giving

$$0=vI^1_v(\nu-\delta-\lambda\rho\eta)+\tfrac{1}{2}\eta^2v^2I^1_{vv}$$

with corresponding transformed conditions on I^1. Proposing a solution of the form $I^1(v)=Lv^\psi$ for some constant L results in the quadratic

$$\psi(\psi-1)\eta^2/2+\psi(\nu-\delta-\lambda\rho\eta)=0$$

and we denote the non-zero root of the quadratic by $\beta=1-2(\nu-\delta-\lambda\rho\eta)/\eta^2$. The general solution is $I^1(v)=Lv^\beta+B$. Using $I^1(0)=1$ gives $B=1$. If $\beta\leqslant 0$ then smooth-pasting fails and $L=0$. In this case, the manager waits indefinitely. However, if $\beta>0$, value-matching gives an expression for L and

$$I^1(v)=1-(1-\mathrm{e}^{-\gamma(1-\rho^2)(\tilde{V}^1-K)^+})\left(\frac{v}{\tilde{V}^1}\right)^\beta$$

Smooth-pasting gives the exercise threshold $\tilde{V}^1$ solves

$$\tilde{V}^1 - K = \frac{1}{\gamma(1-\rho^2)}\ln\left[1 + \frac{\gamma(1-\rho^2)}{\beta}\tilde{V}^1\right].$$

Part B: We will use the notation $\theta^i \equiv (\theta_s)_{\tau^{i+1} \leqslant s < \tau^i}$, where $\tau^{n+1} = 0$, to denote the holdings between exercise dates. Consider the problem when i options remain to be exercised. Observe that

$$\begin{aligned}
H^i(t,x,v) &= \sup_{\tau^i \leqslant \cdots \leqslant \tau^1} \sup_{(\theta_s)_{t \leqslant s < \tau^1}} \mathbb{E}_t\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2\tau_1 - \gamma X_{\tau^1}}\middle| X_t = x, V_t = v\right] \\
&= \sup_{\tau^i \leqslant \cdots \leqslant \tau^1} \sup_{\theta^i,\ldots,\theta^1} \mathbb{E}_t\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2\tau_1 - \gamma X_{\tau^1}}\middle| X_t = x, V_t = v\right] \\
&= \sup_{\tau^i \leqslant \cdots \leqslant \tau^1} \sup_{\theta^i,\ldots,\theta^1} \mathbb{E}_t\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2\tau_1 - \gamma(X_t + \int_t^{\tau^1}\theta \mathrm{d}M/M + \sum_{j=1}^{i}(V_{\tau^j}-K)^+)}\middle| X_t = x, V_t = v\right] \\
&= \sup_{\tau^i}\sup_{\theta^i} \mathbb{E}_t\left[\sup_{\tau^{i-1} \leqslant \cdots \leqslant \tau^1}\ \sup_{\theta^{i-1},\ldots,\theta^1} \mathbb{E}_{\tau^i}\left[-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2\tau_1 - \gamma(X_{\tau^i} + \int_{\tau^i}^{\tau^1}\theta \mathrm{d}M/M + \sum_{j=1}^{i-1}(V_{\tau^j}-K)^+)}\middle| X_{\tau^i}, V_{\tau^i}\right]\middle| X_t = x, V_t = v\right] \\
&= \sup_{\tau^i}\sup_{\theta^i} \mathbb{E}_t[H^{i-1}(\tau^i, X_{\tau^i}, V_{\tau^i}) | X_t = x, V_t = v]
\end{aligned}$$

Hence using (A.6) we have

$$G^i(x,v) = \sup_{\tau^i}\sup_{\theta^i} \mathbb{E}[\mathrm{e}^{(1/2)\lambda^2\tau^i} G^{i-1}(X_{\tau^i}, V_{\tau^i}) | X_0 = x, V_0 = v] \tag{A.10}$$

We now propose the form of the exercise thresholds $\tilde{V}^j$ and time-independent functions $G^j(x,v)$ for all $j \leqslant i-1$. Then we can use (A.10) to solve for the value function $G^i(x,v)$ when i options remain. Suppose that for all $j \leqslant i-1$, $\tau^j = \inf\{t : V_t \geqslant \tilde{V}^j\}$

$$G^j(x,v) = -\frac{1}{\gamma}\,\mathrm{e}^{-\gamma x}\left[1 - (1 - \mathrm{e}^{-\gamma(1-\rho^2)(\tilde{V}^j - K)^+}(1 - A^{j-1}(\tilde{V}^j)^\beta))\left(\frac{v}{\tilde{V}^j}\right)^\beta\right]^{1/(1-\rho^2)}, \quad v \leqslant \tilde{V}_j \tag{A.11}$$

where A^j and $\tilde{V}^j$ are given in the statement of the proposition. Now we substitute (A.11) into (A.10) to solve for the function $G^i(x,v)$ when there are i options remaining. This gives

$$\begin{aligned}
G^i(x,v) &= \sup_{\tau^i}\sup_{\theta^i} \mathbb{E}\left\{-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2\tau_i}\mathrm{e}^{-\gamma X_{\tau^i}}\left[1 - (1 - \mathrm{e}^{-\gamma(1-\rho^2)(\tilde{V}^{i-1}-K)^+}(1 - A^{i-2}(\tilde{V}^{i-1})^\beta))\left(\frac{V_{\tau^i}}{\tilde{V}^{i-1}}\right)^\beta\right]^{1/(1-\rho^2)}\right\} \\
&= \sup_{\tau^i}\sup_{\theta^i} \mathbb{E}\left\{-\frac{1}{\gamma}\,\mathrm{e}^{(1/2)\lambda^2\tau_i}\mathrm{e}^{-\gamma(X_{\tau^i_-} + (V_{\tau^i}-K)^+)}[1 - A^{i-1}(V_{\tau^i})^\beta]^{1/(1-\rho^2)}\right\}.
\end{aligned}$$

Again, by time homogeneity, we know the optimal τ^i is of the form $\tau^i = \inf\{t : V_t \geqslant \hat{V}^i\}$ for some constant $\hat{V}^i$. We now show $\hat{V}^i = \tilde{V}^i$ and that $G^i(x,v)$ is as stated in the proposition.

The ith remaining option is exercised if

$$G^i(x,v) = -\frac{1}{\gamma}\,\mathrm{e}^{-\gamma(x+(v-K)^+)}[1 - A^{i-1}v^\beta]^{1/(1-\rho^2)}$$

In the continuation region, $H^i(t,x,v) = \mathrm{e}^{(1/2)\lambda^2 t}G^i(x,v)$ is a martingale under the optimal strategy θ^i and the Bellman equation is

$$\frac{1}{2}\lambda^2 G^i + (v-\delta)vG^i_v + \frac{1}{2}\eta^2 v^2 G^i_{vv} + \sup_{\theta^i}\left\{\theta^i \mu G^i_x + \frac{1}{2}\sigma^2(\theta^i)^2 G^i_{xx} + \theta^i\sigma\rho\eta v G^i_{xv}\right\} = 0$$

Optimizing over strategies gives

$$\theta^i = -\frac{\lambda G^i_x}{\sigma G^i_{xx}} - \frac{G^i_{xv}\rho\eta v}{\sigma G^i_{xx}} \tag{A.12}$$

Substituting back into the Bellman equation gives

$$0 = \frac{1}{2}\lambda^2 G^i + (v-\delta)vG^i_v + \frac{1}{2}\eta^2 v^2 G^i_{vv} - \frac{1}{2}\frac{(\lambda G^i_x + \rho\eta v G^i_{xv})^2}{G^i_{xx}} \tag{A.13}$$

with boundary, value-matching and smooth-pasting conditions

$$G^i(x,0) = -\frac{1}{\gamma}\,\mathrm{e}^{-\gamma x} \tag{A.14}$$

$$G^i(x,\hat{V}^i) = -\frac{1}{\gamma}\,\mathrm{e}^{-\gamma x}\mathrm{e}^{-\gamma(\hat{V}^i-K)^+}[1-\Lambda^{i-1}(\hat{V}^i)^\beta]^{1/(1-\rho^2)} \tag{A.15}$$

$$G^i_v(x,\hat{V}^i) = \frac{1}{-\gamma}\mathrm{e}^{-\gamma x}\left[\gamma(1-\Lambda^{i-1}(\hat{V}^i)^\beta)^{1/(1-\rho^2)}\mathrm{e}^{-\gamma(\hat{V}^i-K)^+}\,\frac{\beta\Lambda^{i-1}(\hat{V}^i)^{\beta-1}}{1-\rho^2}\mathrm{e}^{-\gamma(\hat{V}^i-K)^+}[1-\Lambda^{i-1}(\hat{V}^i)^\beta]^{1/(1-\rho^2)-1}\right] \tag{A.16}$$

Notice the Bellman equation (A.13) is identical to (A.9) in Part A, but we have a different set of conditions to satisfy. The same approach is used as in Part A, proposing solution $G^i(x,v) = -\frac{1}{\gamma}\,\mathrm{e}^{-\gamma x}J^i(v)$ and $J^i(v) = (I^i)^{1/(1-\rho^2)}$. Again, we are led to a quadratic with non-zero root β and using (A.14) gives solution $J^i(v) = Lv^\beta + 1$.

Again, the smooth-pasting condition (A.16) requires $\beta > 0$ for a solution. Provided $\beta > 0$, (A.15) gives an expression for the constant L and

$$G^i(x,v) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}\left[1-(1-\mathrm{e}^{-\gamma(1-\rho^2)(\hat{V}^i-K)^+}(1-\Lambda^{i-1}(\hat{V}^i)^\beta))\left(\frac{v}{\hat{V}^i}\right)^\beta\right]^{1/(1-\rho^2)}.$$

The smooth-pasting condition (A.16) now gives $\hat{V}^i$ solves

$$\hat{V}^i - K = \frac{1}{\gamma(1-\rho^2)}\ln\left(1+\frac{\gamma(1-\rho^2)}{\beta}(1-\Lambda^{i-1}(\hat{V}^i)^\beta)\hat{V}^i\right)$$

and hence $\hat{V}^i = \tilde{V}^i$ and $G^i(x,v)$ is as stated in the proposition. Now finally consider the equilibrium choice in (3), $v = \mu\rho\eta/\sigma$. We obtain $\beta = 1 + 2\delta/\eta^2 > 1$ and so smooth-pasting holds. The solutions are as given in the proposition. □

Proof of Proposition 2. We prove thresholds $\tilde{V}^j$ are decreasing in j, for $j = 1,\ldots,n$. Let $a = \gamma(1-\rho^2) > 0$ to simplify the notation of the proof. From Proposition 1 we have that the thresholds $\tilde{V}^j$, $j = 1,\ldots,n$ solve

$$\tilde{V}^j - K = \frac{1}{a}\ln\left(1+\frac{a}{\beta}(1-\Lambda^{j-1}(\tilde{V}^j)^\beta)\tilde{V}^j\right) \tag{A.17}$$

where

$$\Lambda^0 = 0,\quad \Lambda^j = \frac{1}{(\tilde{V}^j)^\beta}[1-\mathrm{e}^{-a(\tilde{V}^j-K)^+}(1-\Lambda^{j-1}(\tilde{V}^j)^\beta)],\quad j = 1,\ldots,n-1 \tag{A.18}$$

Define $A_j = \Lambda^{j-1}(\tilde{V}^j)^\beta$, $j = 2,\ldots,n$ so that $A_1 = 0$. Define also $E_j = 1/(1-A_j)$, $j = 1,\ldots,n$ so that $E_1 = 1$. Since our interest is in solutions for which $\tilde{V}^j > K$, the R.H.S. of (A.17) is positive, and hence $A_j < 1$, $E_j > 1$ for all $j = 2,\ldots,n$. Using these definitions, we rewrite (A.17) as

$$\tilde{V}^j - K = \frac{1}{a}\ln\left(1+\frac{a}{\beta}\frac{\tilde{V}^j}{E_j}\right) \tag{A.19}$$

From (A.19) we see for $j = 1,\ldots,n$, $\tilde{V}^j$ are decreasing in j if and only if E_j are increasing in j. (The function $f(x) = (1/a)\ln(1+(a/\beta)(\tilde{V}/x))$ is decreasing in x for fixed threshold level $\tilde{V}$). From (A.18) and (A.17) we obtain

$$\left(1-\frac{1}{E_j}\right)(\tilde{V}^{j-1})^\beta = (\tilde{V}^j)^\beta\left(1-\frac{1}{E_{j-1}+\frac{a}{\beta}\tilde{V}^{j-1}}\right) \tag{A.20}$$

Now suppose $\tilde{V}^j \geqslant \tilde{V}^{j-1}$. Then from (A.20),

$$1-\frac{1}{E_j} \geqslant 1-\frac{1}{E_{j-1}+\frac{a}{\beta}\tilde{V}^{j-1}}$$

and $E_j \geqslant E_{j-1} + \frac{a}{\beta}\tilde{V}^{j-1}$ so $E_j > E_{j-1}$. But from earlier, this implies $\tilde{V}^j < \tilde{V}^{j-1}$ which is a contradiction. Therefore $\tilde{V}^j < \tilde{V}^{j-1}$. □

Proof of Corollary 3. We take the limit as $\gamma \to 0$ in the equations of Proposition 1 for the special case where $\lambda = 0.\rho = 0$ implying $v = 0$ and thus $\beta_0 = \beta$. □

Proof of Corollary 4. It is straightforward to take the limit as $\rho \to 1$ in the equations of Proposition 1. □

Proof of Proposition 6. When $k = 1$, by definition $p = \emptyset$, and letting $q_1 = l$, it is immediate to verify that Proposition 6 gives the solution in Footnote 26 which is easily obtained via Proposition 1 with modified strike, $K + c/l$. When $k > 1$, we work backwards as in Proposition 1. □

Proof of Proposition 7. The cost to the company of n options is given by the complete market or tradeable option value, conditional on the executive exercising optimally. Conditional on q_j options being exercised at threshold $\tilde{V}^{q_j,\ldots,q_1}$, the company cost is given by

$$\sum_{j=1}^{k} q_j(\tilde{V}^{q_j,\ldots,q_1} - K)\mathbb{P}^{\mathbb{Q}}(\tilde{\tau}_c^{q_j,\ldots,q_1} < \infty)$$

where $\tilde{\tau}_c^{q_j,\ldots,q_1} = \inf\{t : V_t \geqslant \tilde{V}^{q_j,\ldots,q_1}\}$ is the first passage time of V to $\tilde{V}^{q_j,\ldots,q_1}$ (under risk-neutral probabilities $\mathbb{P}^{\mathbb{Q}}$) and $1 \leqslant k \leqslant n$ is the number of exercise dates. Under risk-neutral probabilities, the stock price follows $\mathrm{d}V/V = -\delta\,\mathrm{d}t + \eta\,\mathrm{d}W^{\mathbb{Q}}$, where $W^{\mathbb{Q}}$ is a Brownian motion. Then by standard techniques,

$$\mathbb{P}^{\mathbb{Q}}(\tilde{\tau}_c^{q_j,\ldots,q_1} < \infty) = \left(\frac{v}{\tilde{V}^{q_j,\ldots,q_1}}\right)^{\phi}$$

where ϕ solves the quadratic $\phi(\phi-1)\eta^2/2 - \delta\phi = 0$, with roots $\phi = 0$ and $1 + 2\delta/\eta^2 = \beta$. Taking the non-zero root β gives the result. □

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