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Probability, Convexity, and Harmonic Maps II: Smoothness via Probabilistic Gradient Inequalities.

by

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Probability, convexity, and harmonic maps II: Smoothness via probabilistic gradient inequalities.

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Abstract: A conformal change of metric is used to construct a coupling of two time-changed Riemannian Brownian motions, such that there is an upper bound on the probability of no coupling before either process leaves a small geodesic ball, with the bound being linear in the distance between the two starting points if these two points are close enough to the centre of the ball. This coupling is an alternative to that introduced by Cranston (1991), and is more geometric (and probabilistically less technical) than Cranston's coupling. It is used to provide a probabilistic approach to the regularity of harmonic maps, thus completing the probabilistic construction of solutions to small-image harmonic map Dirichlet problems described in Kendall (1990). The approach includes a mild generalization to the case of harmonic maps defined using non-Riemannian connections in the target manifold and smooth strictly elliptic second-order differential operators in the domain.

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1: Introduction.

A *harmonic map* $F : \mathbb{M} \to \mathbb{N}$ between Riemannian manifolds $\mathbb{M}$ and $\mathbb{N}$ is a smooth map which is a (local) extremal of the energy functional

$$\int \| \operatorname{grad} F\|^2 \, d\nu \tag{1.1}$$

Special cases include harmonic functions ($\mathbb{N} = \mathbb{R}$), geodesics ($\mathbb{M} = \mathbb{R}$) and a simple version of a class of maps used to model liquid crystals ($\mathbb{M} \subset \mathbb{R}^3$, $\mathbb{N} = S^2$). Harmonic maps have been much studied in recent years; see for example the surveys of Eells & Lemaire (1983, 1988).

It is now well-known that there is a rich and fruitful interplay between the theory of *harmonic functions* and the probability theory of Brownian motion and the stochastic calculus. It is therefore natural to ask whether this relationship extends in any way to the more general nonlinear setting of harmonic maps. Statistical expectations are not well-defined for manifold-valued random variables, and so it is not obvious whether there can be such an extension. Nevertheless one exists, starting with an observation (due to Bismut) to the effect that harmonic maps send Brownian motion into a random process which is a *Γ-martingale*; a manifold-valued process whose stochastic development is a local martingale. (See Kendall, 1987a, 1988, 1993 for surveys explaining these and other terms from stochastic differential geometry; a brief explanation is to be found at the start of Section 3.) This observation was initially used to provide probabilistic proofs of non-existence and uniqueness theorems for harmonic maps: see for example Kendall (1981, 1983), Goldberg & Mueller (1983), Elworthy & Kendall (1986).

More recently this approach has been used to derive existence and uniqueness for solutions to small-image Dirichlet harmonic map problems; see Kendall (1990), Picard (1991b). Strong points of the probabilistic approach include a close relationship with concepts such as convexity and also with linear potential theory (see for example the general Fatou radial limit theorem of Kendall, 1990 Theorem 6.4, also Kendall, 1991a, 1992a,b), and a ready means of generalization of the notion of harmonic map to include nonlinear elliptic systems arising by mild modifications of the Euler-Lagrange variational equations for (1.1) (see the discussion in Section 3 below). However in contrast to the analytic approach to harmonic maps (for example compare Aviles, Choi & Micallef, 1991 to Kendall, 1990), the probabilistic constructions show only that the solutions are continuous and *finely harmonic maps*; essentially this probabilistic or potential-theoretic definition means that they would be harmonic maps if they were differentiable up to second order.

The present paper gives a *probabilistic* proof of smoothness for continuous finely harmonic maps (including the mild modifications mentioned above) by using an idea of Cranston (1991); gradient inequalities (in Cranston's case for harmonic functions) can be obtained by careful construction of *couplings* for Riemannian Brownian motions (Lindvall, 1992 is a book expounding the general coupling method). Cranston showed that one can construct (on the same probability space) two examples X and Y of a Riemannian Brownian motion which are started from neighbouring points x and y, which are Brownian motions with respect to the same filtration, and which are such that the probability of their not *coupling* (meeting at some random time) before they leave a small neighbourhood $\operatorname{ball}(\mathbf{o}, R)$ is bounded above linearly in the distance $\operatorname{dist}_{\mathbb{M}}(x, y)$ if x, $y \in \operatorname{ball}(\mathbf{o}, R/2)$. Cranston's construction requires a somewhat technical analysis; Section 2 of the present paper shows how to avoid these technicalities by using time-changed Brownian motions referred to a conformal change of metric.

The calculations and constructions of Section 2 may well be of intrinsic interest to readers; for example they show how to deal effectively with the extra drift introduced by certain conformal changes of metric in dimensions other than 2. However the motivation here is to produce a coupling with a simpler probabilistic construction than that of Cranston, so as to allow easy modification to deal with the requirements of the harmonic map case. The probabilistic treatment of gradient estimates for harmonic maps is complicated by the fact that there may not exist a convex function comparable to the Riemannian distance $(x, y) \mapsto \operatorname{dist}_{\mathbb{N}}(x, y)$ on the target manifold. Section 3 shows how to modify the coupling construction to deal with this case, by the curious expedient of allowing a limited amount of independent randomness in the coupling construction. When chosen correctly, this does not destroy the linear upper bound on failure of coupling and yet makes the intervening Riemannian distance between the two processes into a submartingale random process. This permits the probabilistic derivation of a gradient inequality and hence (given an appeal to the regularity theory of *scalar linear* second-order elliptic partial differential equations) a probabilistic proof of regularity for finely harmonic maps.

Section 4 concludes the paper with a discussion of related work and some possible directions for future research.

Before turning to the main work of the paper we make a remark about our choice of notation for stochastic calculus. In stochastic differential geometry the symbol "d" tends to be somewhat overloaded (two kinds of stochastic differential, exterior derivative for forms, ...). Consequently we use suffices to indicate Itô differentials ("d_I") and Stratonovich differentials ("d_S"). For similar reasons we denote the bracket process or quadratic covariation of two continuous real semimartingales X, Y by "$[X, Y]$", reserving angle brackets for inner products of one kind or another — fortunately we have no need of a notation for Lie brackets in this paper. (Since we deal here only with continuous semimartingales we do not need to distinguish between the two kinds of bracket process).

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2: A modification of Cranston's coupling.

Suppose that $\text{ball}(\mathbf{o}, R)$ is an open geodesic ball of radius R, centred on a reference point $\mathbf{o}$ in a complete Riemannian manifold $\mathbb{M}$ of dimension d. Let g be the metric tensor of $\mathbb{M}$. Cranston (1991) showed how to construct Riemannian Brownian motions X and Y in $\text{ball}(\mathbf{o}, R)$, begun at points x and y in the open set $\text{ball}(\mathbf{o}, R/2)$ and adapted to the same filtration, in such a way as to ensure that the probability of not coupling is bounded above linearly in the distance between starting points:

$$\mathbb{P}[\,X, Y \text{ fail to couple before one of } X, Y \text{ exits } \text{ball}(\mathbf{o}, R)\,] \quad \leq \quad \text{constant} \times \text{dist}(x, y) \tag{2.1}$$

(In fact it suffices for X, Y to be Brownian motions with *bounded* drift.) The constant in the inequality (2.1) depends only on R, $\mathbf{o}$, and the geometry g of $\mathbb{M}$ (and the bounded drift of the Brownian motions if this is non-zero), and *not* on the choice of x and y in $\text{ball}(\mathbf{o}, R/2)$. Cranston's coupling construction is based on the coupling described in Kendall (1986a,b), and the main technical difficulty is to control the probability of exiting from $\text{ball}(\mathbf{o}, R)$ before coupling has taken place. This is overcome by a somewhat technical analysis of the joint behaviour of the three Riemannian distance processes $\text{dist}(X, Y)$, $\text{dist}(X, \mathbf{o})$, and $\text{dist}(Y, \mathbf{o})$.

Cranston applied his coupling construction to deduce a Lipschitz property for bounded harmonic functions $f : \text{ball}(\mathbf{o}, R) \to \mathbb{R}$; a martingale argument yields

$$|f(x) - f(y)| \quad \leq \quad \text{constant} \times \sup\left\{|f(z)| \;:\; z \in \text{ball}(\mathbf{o}, R)\right\} \times \text{dist}(x, y) \tag{2.2}$$

for all x, y in $\text{ball}(\mathbf{o}, R/2)$, for all such bounded harmonic functions f. The constant depends only on R and the geometry g of $\mathbb{M}$ (and again on the bounded drift of the Brownian motions if this is non-zero).

The derivation of the coupling inequality (2.1), and hence of the gradient inequality, would be considerably simplified if it were the case that X and Y never left the region $\text{ball}(\mathbf{o}, R)$. This can be arranged if X and Y are replaced by appropriately time-changed Brownian motions $\widetilde{X}$ and $\widetilde{Y}$, and if the coupling is constructed using the corresponding conformal Riemannian metric. Account must then be taken of the fact that $\widetilde{X}$ and $\widetilde{Y}$ are typically Brownian motions *with drift* when referred to the conformal metric, but if this can be overcome then the Lipschitz property (2.2) can be obtained from a coupling inequality for time-changed Brownian motions just as well as from the coupling inequality (2.1). (In particular it helps that the conformal metric and the original metric turn out to be quasi-isometric on $\text{ball}(\mathbf{o}, R/2)$.)

In this section we shall show how to fill in the details of this time-change/conformal isometry approach. It makes for a simpler probabilistic set-up than in Cranston's approach (though at the price of some general geometric calculations). The simpler probabilistic set-up will be helpful in Section 3, when we turn to gradient inequalities for harmonic maps, as it will be much easier to modify than Cranston's original argument would have been.

We begin by noting the effect that conformal change of metric has on Riemannian curvature.

Theorem 2.1: *Let $\text{ball}(\mathbf{o}, R) \subset \mathbb{M}$ be an open geodesic ball of centre $\mathbf{o}$ and radius R in a complete Riemannian manifold $\mathbb{M}$ of metric tensor g, as above. Define the function $\phi : \text{ball}(\mathbf{o}, R) \to (0, R^2]$ by*

$$\phi(x) \quad = \quad R^2 - \text{dist}(x, \mathbf{o})^2 \tag{2.3}$$

and suppose that $\tilde{g}$ is a metric which is conformal to g and is given by

$$\sum_i \sum_j \tilde{g}_{ij}\, dx^i dx^j \quad = \quad \sum_i \sum_j \frac{g_{ij}\, dx^i dx^j}{\phi^2(x)} \tag{2.4}$$

Then

(i) the Riemannian manifold $(\text{ball}(\mathbf{o}, R), \tilde{g})$ is (geodesically) complete;

(ii) its sectional curvatures are bounded below (and so it is stochastically complete; see Azencott, 1974, Yau, 1978, Dodziuk, 1983, Hsu, 1989). Moreover, its sectional curvatures are non-positive for all sufficiently small R.

Proof: This is purely a matter of calculation. We use the language of differential forms (the calculations can also be performed — albeit more clumsily — using covariant derivatives).

Part (i). Consider the distance of p from $\mathbf{o}$, for a boundary point $p \in \partial\,\mathrm{ball}(\mathbf{o}, R)$. We can write the new, conformal, metric in polar coordinates (based on $\mathbf{o}$ and using the old metric g) as

$$\frac{dr^2 + \sum_i \sum_j q_{ij}\, d\theta^i\, d\theta^j}{\left(R^2 - r^2\right)^2}$$

and hence

$$\widetilde{\mathrm{dist}}(p, \mathbf{o}) \quad = \quad \int_0^R \frac{dr}{(R^2 - r^2)} \quad = \quad \infty$$

where $\widetilde{\mathrm{dist}}$ is the distance function for the conformal metric $\tilde{g}$. It follows from the triangle inequality that

$$\widetilde{\mathrm{dist}}(p, x) \quad \geq \quad \widetilde{\mathrm{dist}}(p, \mathbf{o}) - \widetilde{\mathrm{dist}}(x, \mathbf{o}) \quad \geq \quad \infty$$

for any $x \in \mathrm{ball}(\mathbf{o}, R)$. Thus the boundary $\partial\,\mathrm{ball}(\mathbf{o}, R)$ lies at infinity in the conformal metric, and (geodesic) completeness follows.

Part (ii). Recall the procedure for computing the curvature matrix-valued 2-form. Let $\underline{Z}$ be an orthonormal frame field in the old metric g. If $\underline{\theta}$ is the coframe field for $\underline{Z}$ then the Levi-Civita connection form for g is the unique skew-symmetric matrix-valued 1-form $\underline{\underline{\omega}}$ solving

$$d\underline{\theta} + \underline{\underline{\omega}} \wedge \underline{\theta} \quad = \quad 0\,.$$

The curvature form for g is the skew-symmetric matrix-valued 2-form $\underline{\underline{\mathbf{\Omega}}}$ given by

$$\underline{\underline{\mathbf{\Omega}}} \quad = \quad d\underline{\underline{\omega}} + \underline{\underline{\omega}} \wedge \underline{\underline{\omega}}\,.$$

Now the corresponding orthonormal frame field in the conformal metric $\tilde{g}$ is $\underline{\widetilde{Z}} = \phi \underline{Z}$, with coframe field $\underline{\widetilde{\theta}} = \underline{\theta}/\phi$. It follows that the Levi-Civita connection form $\underline{\underline{\widetilde{\omega}}}$ for the conformal metric is $\underline{\underline{\widetilde{\omega}}} = \underline{\underline{\omega}} + \underline{\underline{\alpha}}$, where $\underline{\underline{\alpha}}$ is a skew-symmetric matrix-valued 1-form given by

$$\alpha_{ij} \quad = \quad 2\left(r r_j \widetilde{\theta}^i - r r_i \widetilde{\theta}^j\right)\,.$$

Here $r(x) = \mathrm{dist}(x, \mathbf{o})$ and the notation r_i *etcetera* is an abbreviation for the partial derivative of r determined by $\sum_i r_i \theta^i = dr$. Hence the curvature form $\underline{\underline{\widetilde{\mathbf{\Omega}}}}$ for the conformal metric is given by

$$\underline{\underline{\widetilde{\mathbf{\Omega}}}} \quad = \quad \underline{\underline{\mathbf{\Omega}}} \;+\; \underline{\underline{\alpha}} \wedge \underline{\underline{\omega}} + \underline{\underline{\omega}} \wedge \underline{\underline{\alpha}} \;+\; d\underline{\underline{\alpha}} + \underline{\underline{\alpha}} \wedge \underline{\underline{\alpha}}\,.$$

The sectional curvatures for the conformal metric are obtained by comparing $\underline{\underline{\widetilde{\mathbf{\Omega}}}}$ to the coframe field $\underline{\widetilde{\theta}} = \underline{\theta}/\phi$, or equivalently by comparing

$$\phi^2 \underline{\underline{\widetilde{\mathbf{\Omega}}}} \quad = \quad \phi^2 \underline{\underline{\mathbf{\Omega}}} \;+\; (\phi \underline{\underline{\alpha}}) \wedge (\phi \underline{\underline{\omega}}) + (\phi \underline{\underline{\omega}}) \wedge (\phi \underline{\underline{\alpha}}) \;+\; \phi^2 (d\underline{\underline{\alpha}} + \underline{\underline{\alpha}} \wedge \underline{\underline{\alpha}})$$

to $\underline{\theta}$. Now

$$\phi \alpha_{ij} \quad = \quad 2(R^2 - r^2)\left(r r_j \widetilde{\theta}^i - r r_i \widetilde{\theta}^j\right) \quad = \quad 2\left(r r_j \theta^i - r r_i \theta^j\right)$$

is bounded on $\mathrm{ball}(\mathbf{o}, R)$ when compared to the coframe field $\underline{\theta}$ and indeed is $O(R)$ as $R \to 0$, uniformly on $\mathrm{ball}(\mathbf{o}, R)$. Similarly $\phi^2 \underline{\underline{\mathbf{\Omega}}} = (R^2 - r^2)^2 \underline{\underline{\mathbf{\Omega}}}$ is bounded and $O(R^4)$, and $\phi \underline{\underline{\omega}}$ is bounded and $O(R^2)$, again as $R \to 0$ uniformly on $\mathrm{ball}(\mathbf{o}, R)$ and when compared to the coframe field $\underline{\theta}$.

These last two computations show that $\phi^2 \underline{\underline{\widetilde{\mathbf{\Omega}}}}$ is bounded on $\mathrm{ball}(\mathbf{o}, R)$ when compared to $\underline{\theta}$, and hence that the sectional curvatures of the Riemannian manifold $(\mathrm{ball}(\mathbf{o}, R), \tilde{g})$ are bounded.

We shall need to work only to $o(R^2)$ and so it suffices to consider $\phi^2(d\underline{\underline{\alpha}}+\underline{\underline{\alpha}}\wedge\underline{\underline{\alpha}})$ compared to $\underline{\theta}$ in order to assess the size of $\phi^2\widetilde{\underline{\underline{\Omega}}}$. Computation reveals (working throughout uniformly on ball$(\mathbf{o}, R)$ and comparing to the coframe $\underline{\theta}$) that:

$$\begin{aligned}\phi^2(d\underline{\underline{\alpha}}+\underline{\underline{\alpha}}\wedge\underline{\underline{\alpha}})_{ij} \quad &= \\ &- 4r^2\theta^i\wedge\theta^j + 2\phi\left(d(rr_j)\wedge\theta^i - d(rr_i)\wedge\theta^j\right) - 2r\phi\sum_k\left(r_j\omega_{ik}\wedge\theta^k - r_i\omega_{jk}\wedge\theta^k\right) .\end{aligned}$$

The last term is $O(R^3)$. To the middle term we apply the fact $d(rr_i) = [d(rr_i)]_{r=0} + o(1)$. This follows by some well-known asymptotics which we repeat here for the sake of completeness of exposition:

$$[\langle d(rr_i), Z_j\rangle]_{r=0} \quad = \quad \left[\langle d(Z_i\frac{r^2}{2}), Z_j\rangle\right]_{r=0} \quad = \quad \left[Z_jZ_i\frac{r^2}{2}\right]_{r=0} \quad = \quad \left[(Z_jZ_i - \nabla_{Z_j}Z_i)\frac{r^2}{2}\right]_{r=0}$$

since $\left[\nabla_{Z_j}Z_i(\frac{r^2}{2})\right]_{r=0} = 0$. Hence $\left[Z_jZ_i\frac{r^2}{2}\right]_{r=0}$ is tensorial in Z_i, Z_j, and so it suffices to consider $\left[(W^2)\,\frac{r^2}{2}\right]_{r=0}$ where W is the vectorfield of a unit speed geodesic γ through $\mathbf{o}$. But

$$\left[W^2\frac{r^2}{2}\right]_{r=0} \quad = \quad \left[\frac{d^2}{dt^2}\frac{\text{dist}(\gamma(t),\mathbf{o})^2}{2}\right]_{r=0} \quad = \quad 1\,.$$

Thus

$$d(rr_i) \quad = \quad d\left(Z_i\frac{r^2}{2}\right) \quad = \quad \theta^i + o(1)\,.$$

It follows,

$$\phi^2(d\underline{\underline{\alpha}}+\underline{\underline{\alpha}}\wedge\underline{\underline{\alpha}})_{ij} \quad = \quad -4r^2\theta^i\wedge\theta^j + 2(R^2-r^2)(\theta^j\wedge\theta^i - \theta^i\wedge\theta^j) + o(R^2) \quad = \quad -4R^2\theta^i\wedge\theta^j + o(R^2)\,.$$

Finally we deduce from this that

$$\phi^2\mathbf{\Omega}_{ij} \quad = \quad -4R^2\theta^i\wedge\theta^j + o(R^2) \tag{2.5}$$

which completes the proof of (ii) and hence of this theorem. □

Remark: It is immediate from the definition of $\tilde{g}$ that if $R < 1$ then the map

$$(\text{ball}(\mathbf{o}, R), g) \quad \longrightarrow \quad (\text{ball}(\mathbf{o}, R), \tilde{g})$$

is a metric contraction.

Remark: It is clear from the proof that as $R \to 0$ so the sectional curvatures of ball$(\mathbf{o}, R)$ under the conformal metric

$$\sum_i\sum_j\frac{g_{ij}\,dx^i dx^j}{(1-r^2/R^2)^2} \tag{2.6}$$

converge uniformly to -4.

Consider now a random process $\widetilde{X}$ on ball$(\mathbf{o}, R)$, which is time-changed Riemannian Brownian motion with respect to the old metric g, run at a varying rate using the conformality function ϕ so that

$$\widetilde{X}\left(\int\phi(\widetilde{X})^{-2}\,dt\right) \quad = \quad \widetilde{X}\left(\int\frac{dt}{(R^2-\text{dist}(\widetilde{X},\mathbf{o})^2)^2}\right)$$

is a Riemannian Brownian motion with respect to g. Then $\widetilde{X}$ will have infinitesimal generator $^1\!/_2\,\phi^2\Delta$ where Δ is the Laplace-Beltrami operator for g (so $\Delta f(x) = \left[\sum_i\frac{d^2}{dt^2}f\circ\gamma^i(t)\right]_{t=0}$ whenever $\gamma^1, \ldots, \gamma^d$ are orthonormal unit-speed geodesics issuing from x). Except in dimension $d = 2$ the generator $^1\!/_2\,\phi^2\Delta$ will *not* equal $^1\!/_2\,\widetilde{\Delta}$ (half the Laplace-Beltrami operator for the conformal metric $\tilde{g}$). Its description is given in the next theorem.

Theorem 2.2: *Using the above notation, and working in the setting of Theorem 2.1,*

$$
{}^1\!/\!_2\,\phi^2\Delta \quad = \quad {}^1\!/\!_2\,\widetilde{\Delta} + \tfrac{d-2}{2}\phi^2 \operatorname{grad}\log\phi \quad = \quad {}^1\!/\!_2\,\widetilde{\Delta} + \tfrac{d-2}{2}\widetilde{\operatorname{grad}}\log\phi \tag{2.7}
$$

where grad, $\widetilde{\text{grad}}$ *are gradients using metrics* g, $\tilde{g}$ *respectively. Moreover the norm of* $\frac{d-2}{2}\widetilde{\operatorname{grad}}\,\phi$ *under the metric* $\tilde{g}$ *is bounded uniformly over* $\operatorname{ball}(\mathbf{o}, R)$ *by* $(d-2)R$.

Proof: For all f, $h \in C_0^\infty(\operatorname{ball}(\mathbf{o}, R))$ we have

$$
\int h\,\Delta f\,d\nu \quad = \quad -\int \langle df, dh\rangle\,d\nu
$$

where the integration is over the domain $\operatorname{ball}(\mathbf{o}, R)$, ν is the volume measure for the Riemannian metric g, and $\langle\cdot,\cdot\rangle$ is the inner product for g. Now

$$
\begin{aligned}
d\nu &= \phi^d\,d\tilde{\nu}\,, \\
\langle df, dh\rangle &= \phi^{-2}\langle df, dh\rangle_{\tilde{g}}
\end{aligned}
$$

where $\tilde{\nu}$, $\langle\cdot,\cdot\rangle_{\tilde{g}}$ are the volume measure and inner product for the conformal Riemannian metric $\tilde{g}$. Hence

$$
\begin{aligned}
\int h\,\Delta f\,d\nu &= -\int \langle df, d(\phi^{d-2}h)\rangle_{\tilde{g}}\,d\tilde{\nu} + \int \langle df, d(\phi^{d-2})\rangle_{\tilde{g}}\,h\,d\tilde{\nu} \\
&= \int (h\phi^{-2}\,\widetilde{\Delta}\,f)\phi^d\,d\tilde{\nu} + (d-2)\int h\phi^{-2}\langle df, d\log\phi\rangle_{\tilde{g}}\,\phi^d\,d\tilde{\nu} \\
&= \int (h\phi^{-2}\,\widetilde{\Delta}\,f)\,d\nu + (d-2)\int h\phi^{-2}\langle f, \widetilde{\operatorname{grad}}\log\phi\rangle\,d\nu \\
&= \int (h\phi^{-2}\,\widetilde{\Delta}\,f)\,d\nu + (d-2)\int h\langle f, \operatorname{grad}\log\phi\rangle\,d\nu\,.
\end{aligned}
$$

The last two equations above are alternative derivations following from the definitions of the vectorfields $\widetilde{\operatorname{grad}}\log\phi = \phi^2\operatorname{grad}\log\phi$. The equations (2.7) are thereby established.

The norm of $\frac{d-2}{2}\widetilde{\operatorname{grad}}\log\phi$ is obtained by noting that $\|X\| = \phi\|X\|_{\tilde{g}}$ where $\|X\|$, $\|X\|_{\tilde{g}}$ are the norms of a vector X in the metrics g, $\tilde{g}$. Hence

$$
\begin{aligned}
\|\widetilde{\operatorname{grad}}\log\phi\|_{\tilde{g}} &= \phi^{-1}\|\widetilde{\operatorname{grad}}\log\phi\| \quad = \quad \phi^{-1}\|\phi^2\operatorname{grad}\log\phi\| \\
&= \|\operatorname{grad}\phi\| \quad = \quad \|2r\operatorname{grad}r\| \quad = \quad 2r \quad \leq \quad 2R
\end{aligned}
$$

This obtains the desired bound. □

Remark: From these calculations we see that

$$
\frac{d-2}{2}\widetilde{\operatorname{grad}}\log\phi(x) \quad = \quad -(d-2)r(x)\,\widetilde{\operatorname{grad}}_2\,\widetilde{\operatorname{dist}}(\mathbf{o}, x)\,.
$$

Thus the effect of this particular conformal change-of-metric is to convert a g-Brownian motion into a time-change of a $\tilde{g}$-Brownian motion with inward-pointing $\tilde{g}$-bounded drift; moreover the drift increases in $\tilde{g}$-length with the distance from the centre-point $\mathbf{o}$.

Remark: We choose to deal with the metric (2.4) rather than the metric (2.6) in order to make $\|\widetilde{\operatorname{grad}}\log\phi\|_{\tilde{g}}$ bounded on $\operatorname{ball}(\mathbf{o}, R)$ as $R \to 0$.

Remark: It may be of interest to note that these calculations were checked in special cases by using the `REDUCE` computer algebra package (Hearn, 1991) and in particular by using the contributed `REDUCE` program `EXCALC` (Schrüfer, 1991), which provides a syntax which is close to that of the theory of differential forms.

We can now state and prove the main result of this section.

Theorem 2.3: *Let* $\mathrm{ball}(\mathbf{o},R)\subset\mathbb{M}$ *be an open geodesic ball of centre* $\mathbf{o}$ *and radius* R *in a complete Riemannian manifold* $\mathbb{M}$ *of metric tensor* g*, as above. For all sufficiently small* R *(depending only on the geometry of* $\mathbb{M}$*) there is a constant* c_1 *depending only on* R *and the geometry* g *of* $\mathrm{ball}(\mathbf{o},R)$ *such that the following holds. For any two* x, $y\in\mathrm{ball}(\mathbf{o},R)$ *we can construct time-changed* g*-Riemannian Brownian motions* $\widetilde{X}$, $\widetilde{Y}$ *begun at* x, y*, which are time-changed* g*-Riemannian Brownian motions with respect to the same filtration, and are such that*

$$\mathbb{P}\left[\widetilde{X},\widetilde{Y}\text{ fail to couple before one of }\widetilde{X},\widetilde{Y}\text{ exits }\mathrm{ball}(\mathbf{o},R)\right]\quad\leq\quad c_1\times\widetilde{\mathrm{dist}}(x,y)\tag{2.8}$$

where $\widetilde{\mathrm{dist}}$ *is the Riemannian distance with respect to the conformal metric*

$$\sum_i\sum_j\tilde{g}_{ij}\,dx^i dx^j\quad=\quad\sum_i\sum_j\frac{g_{ij}\,dx^i dx^j}{\phi^2}\quad=\quad\sum_i\sum_j\frac{g_{ij}\,dx^i dx^j}{(R^2-\mathrm{dist}(x,\mathbf{o})^2)^2}\,.$$

We extend $\widetilde{X}$ *and* $\widetilde{Y}$ *beyond the coupling time by setting them equal after that time.*

Proof: Start with given x, $y\in\mathrm{ball}(\mathbf{o},R)$ and $\widetilde{X}$ constructed as in the discussion before Theorem 2.2, so that $\widetilde{X}(\int\phi(\widetilde{X})^{-2}dt)$ is a g-Riemannian Brownian motion. Theorems 2.1 and 2.2 now allow us to use the method of Kendall (1986b), Cranston (1991) to give a direct construction of a similar diffusion $\widetilde{Y}$ starting at y for which the probability of not coupling satisfies the required linear upper bound. The construction is as follows.

First set $\chi=\frac{d-2}{2}\widetilde{\mathrm{grad}}\log\phi$ to be the $\tilde{g}$-bounded vectorfield arising in equations (2.7). Then it is a consequence of Theorem 2.2 that $\widetilde{X}$ satisfies the following Stratonovich differential equations of stochastic development and parallel transport (see for example Kendall, 1987a, 1988, 1993):

$$\begin{aligned}d_S\widetilde{X}&=\widetilde{\Xi}\,d_S B+\chi(\widetilde{X})\,dt\\ d_S\widetilde{\Xi}&=\widetilde{H}_{\widetilde{\Xi}}\,d_S\widetilde{X}\end{aligned}\tag{2.9a}$$

Here $\widetilde{\Xi}$ takes values in the $\tilde{g}$-orthonormal frame bundle $\widetilde{O}(\mathrm{ball}(\mathbf{o},R))$ sitting above $\widetilde{X}$, while

$$\widetilde{H}_{\tilde{\xi}}\;:\;T_{\pi\tilde{\xi}}\,\mathrm{ball}(\mathbf{o},R)\quad\longrightarrow\quad T_{\tilde{\xi}}\,\widetilde{O}(\mathrm{ball}(\mathbf{o},R))$$

is the horizontal lift for the Levi-Civita connection for $\tilde{g}$. With $\widetilde{X}$ given as in the discussion before Theorem 2.2, the process B is ordinary Euclidean Brownian motion. Of course equation (2.9a) requires initial conditions: $\widetilde{X}_0=x$ and $\widetilde{\Xi}_0=\tilde{\xi}_0$ for some frame $\tilde{\xi}_0$ sitting above x.

The coupled process $\widetilde{Y}$ is defined by extending the stochastic differential system (2.9a). Suppose that R is chosen small enough to ensure that $\mathrm{ball}(\mathbf{o},R)$ is simply-connected and is of bounded non-positive curvature under the conformal metric $\tilde{g}$ (Theorem 2.1 assures us that this second property holds for all R which are sufficiently small!). Then there is a unique geodesic connecting any two points x, $y\in\mathrm{ball}(\mathbf{o},R)$ under the $\tilde{g}$ metric. Let

$$\widetilde{\mathfrak{M}}_{x,y}\;:\;T_x\,\mathrm{ball}(\mathbf{o},R)\quad\longrightarrow\quad T_y\,\mathrm{ball}(\mathbf{o},R)$$

be the "mirror-map under $\tilde{g}$": the $\tilde{g}$-isometry obtained by parallel transport along this geodesic followed by reflection in the plane normal to the tangent vector of the geodesic. The coupled process $\widetilde{Y}$ is defined as solving

$$\begin{aligned}d_S\widetilde{Y}&=\widetilde{\Psi}\,d_S\left([\widetilde{\Psi}^{-1}\widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}\widetilde{\Xi}]*B\right)+\chi(\widetilde{Y})\,dt\\ d_S\widetilde{\Psi}&=\widetilde{H}_{\widetilde{\Psi}}\,d_S\widetilde{Y}\end{aligned}\tag{2.9b}$$

Here $\widetilde{\Psi}$ takes values in $\widetilde{O}(\mathrm{ball}(\mathbf{o},R))$ sitting above $\widetilde{Y}$, while

$$[\widetilde{\Psi}^{-1}\widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}\widetilde{\Xi}]*B\quad=\quad\int[\widetilde{\Psi}^{-1}\widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}},\widetilde{\Xi}]\,d_I B$$

is the Itô integral of the Euclidean isometry $[\widetilde{\Psi}^{-1}\widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}\widetilde{\Xi}]$ with respect to the Euclidean Brownian motion B (being the Itô integral of an isometry-valued process, it remains a Euclidean Brownian motion). Equation (2.9b) also requires initial conditions: $\widetilde{Y}_0 = y$ and $\widetilde{\Psi}_0 = \widetilde{\psi}_0$ for some frame $\widetilde{\psi}_0$ sitting above y. Note that by Theorem 2.1 we know that $\operatorname{ball}(\mathbf{o}, R)$ is stochastically complete under $\tilde{g}$, and that $\widetilde{X}$, $\widetilde{Y}$ are $\tilde{g}$-Riemannian Brownian motions with bounded drift; consequently $\widetilde{X}$, $\widetilde{Y}$ stay in $\operatorname{ball}(\mathbf{o}, R)$ for all time. Hence the system (2.9a,b) defines $\widetilde{Y}$ up to the coupling time (at which $\widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}$ is no longer defined). The construction is continued after coupling by setting $\widetilde{Y} = \widetilde{X}$.

Following Kendall (1986a,b) and Cranston (1991), we can use a geometric form of the Itô formula and comparison arguments for Jacobi vectorfields to show that if the sectional curvatures under $\tilde{g}$ are bounded below by some constant $-\kappa^2$ depending on g and R (as is guaranteed by Theorem 2.1) then the distance between $\widetilde{X}$ and $\widetilde{Y}$ satisfies the following Itô stochastic differential equation up to the coupling time:

$$d_I\, \widetilde{\operatorname{dist}}(\widetilde{X},\widetilde{Y}) \quad = \quad 2\, d_I A \; + \; d\Lambda \, .$$

The process A is a real-valued Brownian motion begun at 0, while Λ is a scalar process of locally bounded variation satisfying the upper bound

$$\begin{aligned} \Lambda_t \quad &\leq \quad (d-1)\int_0^t \tfrac{1}{\kappa}\tanh\left(\tfrac{\kappa}{2}\,\widetilde{\operatorname{dist}}(\widetilde{X},\widetilde{Y})\right)\,ds \; + \; \int_0^t \left(\|\chi(\widetilde{X})\|_{\tilde{g}} + \|\chi(\widetilde{Y})\|_{\tilde{g}}\right)\,ds \\ &\leq \quad 2\tilde{c}\times t \end{aligned} \qquad (2.10)$$

(recall that Theorem 2.2 informs us that $\|\chi\|_{\tilde{g}} \leq (d-2)R$). In the above, κ and hence the constant $\tilde{c}$ are positive and depend only on R and on the original geometry g of $\operatorname{ball}(\mathbf{o}, R)$. (It is important here for ease of argument that $\tilde{g}$ makes $\operatorname{ball}(\mathbf{o}, R)$ into a simply-connected manifold of bounded non-positive sectional curvatures — though see the Remark at the end of this section.)

We have shown that until coupling the (conformal) distance process $\widetilde{\operatorname{dist}}(\widetilde{X},\widetilde{Y})$ is bounded above by a real Brownian motion of rate 4 and constant positive drift $2\tilde{c}$, begun at $\widetilde{\operatorname{dist}}(x,y)$. But this proves the theorem, since for a positive constant c_1 depending only on $\tilde{c}$ we have

$$\mathbb{P}\left[\,2A_t + 2\tilde{c}\,t + \widetilde{\operatorname{dist}}(x,y) > 0 \quad \text{for all} \quad t\,\right] \quad = \quad 1 - \exp\left(-c_1\,\widetilde{\operatorname{dist}}(x,y)\right) \quad \leq \quad c_1\,\widetilde{\operatorname{dist}}(x,y)\,.$$

But the event that $2A_t + 2\tilde{c}\,t + \widetilde{\operatorname{dist}}(x,y)$ fails ever to hit zero is implied by the event $\widetilde{\operatorname{dist}}(\widetilde{X},\widetilde{Y})$ also will not hit zero, and therefore has larger probability. This establishes the required result. □

Remark: In the theory of stochastic simulation, constructions such as that of $\widetilde{Y}$ from $\widetilde{X}$ are classified as using the method of *antithetic variates*. See Ripley (1987, Chapter 5).

Cranston (1991) establishes the gradient inequality even when the underlying Brownian motion has superimposed bounded drift, and the above approach allows for this also. This means that the results of Section 3 also cover, for example, the case of the Hermitian-harmonic maps of Jost & Yau (1993).

Corollary 2.4: *The conclusion of Theorem 2.3 still holds if we add a g-bounded drift vectorfield ζ to the underlying g-Riemannian Brownian motion (except that now the constant c_1 depends also on the supremum g-norm $\|\zeta\|_g$ of the bounded drift): we still obtain a bound of the form of (2.8) for similar time-changes of g-Riemannian Brownian motion with g-bounded drift.*

Proof: The supremum $\tilde{g}$-norm of $\phi^2 v$ is bounded above by $R^2\|\zeta\|_g$. Therefore the only modifications required to the proof of Theorem 2.3 are:

(i) equations (2.9a,b) are modified to include a drift term:

$$\begin{aligned} d_S\,\widetilde{X} \quad &= \quad \widetilde{\Xi}\,d_S\,B \; + \; \chi(\widetilde{X})\,dt \; + \; \zeta(\widetilde{X})dt \\ d_S\,\widetilde{\Xi} \quad &= \quad \widetilde{H}_{\widetilde{\Xi}}\,d_S\,\widetilde{X} \end{aligned} \qquad (2.9\text{a}')$$

$$\begin{aligned} d_S\,\widetilde{Y} \quad &= \quad \widetilde{\Psi}\,d_S\,\left([\widetilde{\Psi}^{-1}\widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}\widetilde{\Xi}] * B\right) \; + \; \chi(\widetilde{Y})\,dt \; + \; \zeta(\widetilde{Y})dt \\ d_S\,\widetilde{\Psi} \quad &= \quad \widetilde{H}_{\widetilde{\Psi}}\,d_S\,\widetilde{Y} \end{aligned} \qquad (2.9\text{b}')$$

(ii) the bound (2.10) is modified to include a bounded contribution from the drift term:

$$\begin{aligned}\Lambda_t &\leq (d-1)\int_0^t \tfrac{1}{\kappa}\tanh\left(\tfrac{\kappa}{2}\,\widetilde{\operatorname{dist}}(\widetilde{X},\widetilde{Y})\right)\,ds \;+\; \int_0^t \left(\|\chi(\widetilde{X})\|_{\tilde g}+\|\chi(\widetilde{Y})\|_{\tilde g}\right)\,ds \;+\; 2R^2\|\zeta\|_g\times t \\ &\leq 2\tilde c'\times t\end{aligned} \qquad (2.10')$$

Thus we obtain a linear upper bound on the probability of no coupling, as required. □

For the purpose of gradient bounds we need not suppose any smoothness for the drift vectorfield ζ, though we must then use a compactness argument to obtain weak solutions to the construction of equations $(2.9a, b)$. However the smoothness argument of Theorem 3.2 does require smoothness of ζ.

Note that any strongly elliptic diffusion with smooth coefficients can be represented as a Riemannian Brownian motion with added drift. (See the discussion in Ikeda & Watanabe, 1981, pages 273-274.) Consequently the coupling argument above applies to this general case.

The metrics g, $\tilde g$ are quasi-isometric on $\operatorname{ball}(\mathbf{o}, R/2)$. Hence we obtain the following:

Corollary 2.5: *In Theorem 2.3 and Corollary 2.4 we may replace the conformal metric distance* $\widetilde{\operatorname{dist}}$ *by the original metric distance* dist *so long as we restrict x, y to lie in* $\operatorname{ball}(\mathbf{o}, R/2)$*:*

$$\mathbb{P}\left[\widetilde{X},\widetilde{Y} \textit{ fail to couple before one of } \widetilde{X},\widetilde{Y} \textit{ exits } \operatorname{ball}(\mathbf{o},R)\right] \quad\leq\quad c_2\times\operatorname{dist}(x,y) \qquad (2.11)$$

for all x, $y \in$ $\operatorname{ball}(\mathbf{o}, R/2)$*. (The constant c_1 changes to c_2, but still depends only on R and the geometry g of* $\operatorname{ball}(\mathbf{o}, R)$ *and, in the case of nonzero bounded drift, the supremum g-norm $\|\zeta\|_g$ of the drift vector.)*

It will be useful in Section 3 to consider the consequence of using perfect correlation rather than the method of antithetic variates.

Corollary 2.6: *In the setting of Theorem 2.3 (and of Corollary 2.4, if the drift is smooth), suppose that R is small enough for the conformal metric $\tilde g$ to have non-positive curvatures. Then we can modify the coupling of $\widetilde{X}$, $\widetilde{Y}$ so that the mean distance between them grows at a fixed exponential rate: if $\hat X$, $\hat Y$ are the new coupled processes then*

$$\mathbb{E}\left[\widetilde{\operatorname{dist}}(\hat X_t,\hat Y_t)\right] \quad\leq\quad \exp\left(\tfrac{d-1}{2}t\right)\times\widetilde{\operatorname{dist}}(\hat X_0,\hat Y_0) \qquad (2.12)$$

in the case of Theorem 2.3 (in the case of smooth nonzero bounded drift replacing $\frac{d-1}{2}$ by some positive constant not depending on $\hat X_0$, $\hat Y_0$).

Proof: The modified construction simply replaces $\widetilde{\mathfrak{M}}_{x,y}$ by parallel transport along the intervening geodesic:

$$\widetilde{\mathfrak{T}}_{x,y} \,:\, T_x\operatorname{ball}(\mathbf{o},R) \quad\longrightarrow\quad T_y\operatorname{ball}(\mathbf{o},R)\,.$$

With $\hat X$, $\hat Y$ defined using the corresponding modifications of $(2.9a, b)$ the Itô analysis of $\widetilde{\operatorname{dist}}(\hat X,\hat Y)$ shows that it has *zero* martingale part, and its drift $\hat\Lambda$ is bounded above by

$$\begin{aligned}\hat\Lambda_t \quad\leq\quad & \\ & (d-1)\int_0^t \tfrac{1}{\kappa}\tanh\left(\tfrac{\kappa}{2}\,\widetilde{\operatorname{dist}}(\hat X,\hat Y)\right)\,ds \\ & + \int_0^t \left(\langle\chi(\hat X),\widetilde{\operatorname{grad}}_1\,\widetilde{\operatorname{dist}}(\hat X,\hat Y)\rangle_{\tilde g}+\langle\chi(\hat Y),\widetilde{\operatorname{grad}}_2\,\widetilde{\operatorname{dist}}(\hat X,\hat Y)\rangle_{\tilde g}\right)\,ds\,.\end{aligned}$$

(Add a term $\int_0^t \|D\zeta\|_g\times\widetilde{\operatorname{dist}}(\hat X,\hat Y)\,ds$ in the case of smooth nonzero bounded drift ζ.) Now we can compute

$$\chi(\hat X) \quad=\quad \frac{d-2}{2}\left(\widetilde{\operatorname{grad}}\log\phi\right)(\hat X) \quad=\quad -(d-2)r(\hat X)\,\widetilde{\operatorname{grad}}_2\,\widetilde{\operatorname{dist}}(\mathbf{o},\hat X)$$

(see the Remark at the end of the proof of Theorem 2.2). The $\tilde{g}$ metric has non-positive curvatures, so we can use comparison with Euclidean trigonometry (*via* Toponogov's theorem Cheeger & Ebin, 1975, Chapter 2) to show that the integrand of the second term

$$\begin{aligned}&\langle\chi(\hat{X}), \widetilde{\mathrm{grad}}_1\,\widetilde{\mathrm{dist}}(\hat{X},\hat{Y})\rangle_{\tilde{g}} + \langle\chi(\hat{Y}), \widetilde{\mathrm{grad}}_2\,\widetilde{\mathrm{dist}}(\hat{X},\hat{Y})\rangle_{\tilde{g}} \quad = \\ &-(d-2)\left(r(\hat{X})\langle\widetilde{\mathrm{grad}}_2\,\widetilde{\mathrm{dist}}(\mathbf{o},\hat{X}), \widetilde{\mathrm{grad}}_1\,\widetilde{\mathrm{dist}}(\hat{X},\hat{Y})\rangle_{\tilde{g}} + r(\hat{Y})\langle\widetilde{\mathrm{grad}}_2\,\widetilde{\mathrm{dist}}(\mathbf{o},\hat{Y}), \widetilde{\mathrm{grad}}_2\,\widetilde{\mathrm{dist}}(\hat{X},\hat{Y})\rangle_{\tilde{g}}\right)\end{aligned} \tag{2.13}$$

is non-positive.

To see this, consider the geodesic triangle with vertices at $\mathbf{o}$, $\hat{X}$, $\hat{Y}$. By Toponogov's theorem, the angles at $\hat{X}$, $\hat{Y}$ (of cosines $\langle\widetilde{\mathrm{grad}}_2\,\widetilde{\mathrm{dist}}(\mathbf{o},\hat{X}), \widetilde{\mathrm{grad}}_1\,\widetilde{\mathrm{dist}}(\hat{X},\hat{Y})\rangle_{\tilde{g}}$ and $\langle\widetilde{\mathrm{grad}}_2\,\widetilde{\mathrm{dist}}(\mathbf{o},\hat{Y}), \widetilde{\mathrm{grad}}_2\,\widetilde{\mathrm{dist}}(\hat{X},\hat{Y})\rangle_{\tilde{g}}$) must sum to less than π in radians. If both are no larger than $\pi/2$ then both terms of the sum in (2.13) must be non-positive.

On the other hand, suppose (without loss of generality) that the angle at $\hat{X}$ is greater than $\pi/2$ then, again by Toponogov's theorem, $\widetilde{\mathrm{dist}}(\mathbf{o},\hat{X}) < \widetilde{\mathrm{dist}}(\mathbf{o},\hat{Y})$. Hence $r(\hat{X}) < r(\hat{Y})$ and this together with the angle-sum inequality means again that the quantity in (2.13) must be negative.

The bound (2.12) follows directly, since $d_I\,\widetilde{\mathrm{dist}}(\hat{X},\hat{Y}) = d\hat{\Lambda}$ □

So the effect of the conformal change-of-metric is actually to bring coupled time-changed g-Brownian motions still closer together in the $\tilde{g}$-metric!

Remark: Further consideration of the methods of Kendall (1986b), Cranston (1991) shows that it is not necessary to restrict R to be "sufficiently small" in Theorem 2.3 and its corollaries. The only problem that may arise for large R is that the geodesic between x and y may no longer be unique, in which case the construction of the system $(2.9a, b)$ will break down. However it is shown in these two references how to apply the methods of Kendall (1987b) to extend the construction to deal with this kind of phenomenon (it should be noted that Cranston, 1991 gives the more elegant treatment), and its effect is essentially only to decrease the drift differential of $\widetilde{\mathrm{dist}}(\widetilde{X},\widetilde{Y})$ still further. This refinement is not required in the local considerations of Section 3 and so we do not discuss it further here.

3: Gradient inequalities for harmonic maps.

In Section 1 it was briefly indicated how probability may be applied to harmonic maps. We now describe this relationship in a little more detail.

We say that a *finely harmonic map* $F : \mathbb{M} \to \mathbb{N}$, from one Riemannian manifold to another, is a measurable map such that for quasi-almost all $x \in \mathbb{M}$ the random process $F(X)$ is a Γ-martingale whenever X is a Brownian motion on $\mathbb{M}$ begun at x. Recall that a Γ-martingale Y is a semimartingale whose stochastic development is a local martingale; equivalently $\phi(Y)$ is a submartingale in any domain for which the map $\phi : \mathbb{N} \to \mathbb{R}$ is convex. (Hence $\phi \circ F$ is subharmonic wherever F maps into a domain of convexity of ϕ, yielding a definition of finely harmonic maps in which probability does not appear.) In Kendall (1990) it was shown how to construct continuous finely harmonic maps as solutions to small-image Dirichlet problems.

If a finely harmonic map F is twice continuously differentiable then an application of the Itô formula shows that it must satisfy the nonlinear elliptic system of partial differential equations

$$ {}^{\mathbb{M}}\Delta F^\alpha + {}^{\mathbb{M}}g^{ij}\, {}^{\mathbb{N}}\Gamma^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j} = {}^{\mathbb{M}}g^{ij} \left\{ \frac{\partial F^\alpha}{\partial x^i \partial x^j} - {}^{\mathbb{M}}\Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k} + {}^{\mathbb{N}}\Gamma^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j} \right\} = 0 \qquad (3.1). $$

(Here and in the formula below we use the Einstein convention of summing over repeated indices.) These are actually the Euler-Lagrange variational equations for the energy functional (1.1). In such a case we say that F is a *harmonic map*. In this section we shall explore a way in which the coupling ideas of Section 2, together with standard regularity results for *scalar linear* elliptic partial differential equations, can be used to derive a probabilistic proof of regularity for continuous finely harmonic maps.

Of course there is an analytic theory of regularity for continuous *weakly* harmonic maps between Riemannian manifolds: see for example Hildebrandt (1985), also Aviles, Choi & Micallef (1991). However notice that the definitions of finely harmonic map and of harmonic map, as given above, make sense when the codomain $\mathbb{N}$ is furnished only with a connection which need not be compatible with any metric. Moreover we may replace the Brownian motion on $\mathbb{M}$ by a strictly elliptic diffusion with smooth coefficients, or equivalently replace the $\mathbb{M}$-Laplace-Beltrami part of equation (3.1)

$$ {}^{\mathbb{M}}\Delta F^\alpha = \frac{\partial F^\alpha}{\partial x^i \partial x^j} - {}^{\mathbb{M}}\Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k} $$

by the action on F^α of a strictly elliptic second-order differential operator L with smooth coefficients. As indicated in Section 2, any such L may be represented as $L = {}^{\mathbb{M}}\Delta + \zeta$, where ${}^{\mathbb{M}}\Delta$ is the Laplace-Beltrami operator derived from a suitable Riemannian metric on $\mathbb{M}$ and ζ is a smooth vectorfield on $\mathbb{M}$. From this we obtain the notions of *finely L-harmonic map* and *L-harmonic map*.) As noted in Kendall (1991a), the probabilistic method of Kendall (1990) applies to these generalizations essentially without change. We shall see that the probabilistic proof of regularity also generalizes easily to cover these extensions.

We begin by considering the simpler case of a Riemannian target $\mathbb{N}$ for which all the sectional curvatures are everywhere non-positive.

Theorem 3.1: *Suppose that $F : \mathbb{M} \to \mathbb{N}$ is a continuous finely L-harmonic map between complete Riemannian manifolds $\mathbb{M}$ and $\mathbb{N}$ ($\mathbb{M}$ perhaps with boundary), where $L = {}^{\mathbb{M}}\Delta + \zeta$ is the Laplace-Beltrami operator ${}^{\mathbb{M}}\Delta$ with superimposed locally bounded drift ζ. Suppose that $\mathbb{N}$ has non-positive sectional curvatures. Then F is locally Lipschitz in the interior of $\mathbb{M}$.*

Proof: We describe the proof only for the case of zero superimposed drift ζ; the changes required to prove the general case are straightforward, involving reference to Corollary 2.4 rather than Theorem 2.3.

First notice that here (and also in the general case) we are concerned with an essentially local situation, so that we need consider only the case of a continuous finely harmonic map

$$ F \;:\; \mathrm{ball}(\mathbf{o}, R) \longrightarrow U $$

where $\mathrm{ball}(\mathbf{o}, R)$ is an open geodesic ball contained in $\mathbb{M}$ and the codomain U is a bounded open subset of $\mathbb{N}$, and both $\mathrm{ball}(\mathbf{o}, R)$ and U may be taken as small as may be required. Moreover we need show the Lipschitz property of F only in the smaller $\mathrm{ball}(\mathbf{o}, R/2)$ (say).

The curvature assumption means that the distance function $\text{dist}_{\mathbb{N}}$ on $\mathbb{N}$ is convex as a function on the Riemannian product manifold $\mathbb{N}\times\mathbb{N}$. (In the terminology of Kendall, 1990, compact balls of $\mathbb{N}$ have convex geometry.) Suppose that $\widetilde{X}$, $\widetilde{Y}$ are time-changed Brownian motions begun at x, $y \in \text{ball}(\mathbf{o}, R/2)$ as in Theorem 2.3. Then the stochastic developments of $\widetilde{X}$, $\widetilde{Y}$ are time-changed Brownian motions and thus those of $F(\widetilde{X})$, $F(\widetilde{Y})$ are time-changed local martingales. But the local-martingale property is preserved under time-change, and so $F(\widetilde{X})$, $F(\widetilde{Y})$ are both Γ-martingales on $\mathbb{N}$. Consequently $(F(\widetilde{X}), F(\widetilde{Y}))$ is a Γ-martingale on $\mathbb{N}\times\mathbb{N}$ (using the product Riemannian metric) and so $\text{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))$ is a submartingale.

It follows from Theorem 2.3 that

$$\begin{aligned}\text{dist}_{\mathbb{N}}(F(x), F(y)) \quad &= \quad \text{dist}_{\mathbb{N}}(F(\widetilde{X}_0), F(\widetilde{Y}_0))\\ &\leq \quad \mathbb{E}\left[\text{dist}_{\mathbb{N}}(F(\widetilde{X}_\infty), F(\widetilde{Y}_\infty))\right] \quad \leq \quad c_2\,\text{diam}(U)\times\text{dist}_{\mathbb{M}}(x,y)\,.\end{aligned}$$

Here $\text{dist}_{\mathbb{M}}$ is the distance for $\mathbb{M}$, the constant c_2 is the constant which appears in Corollary 2.5, and the limits $F(\widetilde{X}_\infty)$, $F(\widetilde{Y}_\infty)$ are interpreted *via* Γ-martingale convergence of $F(\widetilde{X})$, $F(\widetilde{Y})$ (the Γ-martingales must converge because of a convexity argument; see for example Huang & Kendall, 1991, Theorem 4A). Thus F satisfies a Lipschitz condition on $\text{ball}(\mathbf{o}, R/2)$ as required. □

Remark: for this case the original Cranston coupling (using g-Brownian motions without time-changes) could have been used.

Once we know that F is a locally Lipschitz finely harmonic map then we can use probability to reduce the problem of regularity to the scalar linear case. Note that the local Lipschitz property is not a metric property and so applies to maps between smooth manifolds without metric structure.

Theorem 3.2: *Suppose that $F : \mathbb{M} \to \mathbb{N}$ is a locally Lipschitz finely L-harmonic map from a complete Riemannian manifold $\mathbb{M}$ (perhaps with boundary) to a manifold $\mathbb{N}$ furnished with a geodesically complete connection, where $L = {}^{\mathbb{M}}\Delta + \zeta$ is the Laplace-Beltrami operator ${}^{\mathbb{M}}\Delta$ with superimposed smooth drift ζ. Then F is a smooth map in the interior of $\mathbb{M}$.*

Proof: As before, we describe the proof only for the case of zero superimposed drift ζ, since the changes required to prove the general case are straightforward.

Again we can specialize to $F : \text{ball}(\mathbf{o}, R) \to U$, where R and U can be taken as small as required. We choose an arbitrary reference Riemannian metric for $\mathbb{N}$.

From the local Lipschitz property it follows that F has a weak first derivative DF in $\text{ball}(\mathbf{o}, R)$, which is bounded when we use the reference metric on $\mathbb{N}$ and the given Riemannian metric on $\mathbb{M}$ (see for example Ziemer, 1989, Theorem 2.1.4). We set $c_{lip} = \sup\{\|DF(x)\| : x \in \text{ball}(\mathbf{o}, R)\}$.

If X is Brownian motion on $\text{ball}(\mathbf{o}, R)$ (using the given metric on $\mathbb{M}$) with stochastic development B (a stopped Brownian motion) then $F(X)$ is a Γ-martingale on U with stochastic development N a local martingale. Since the Euclidean Brownian motion B generates the underlying filtration, we may use the martingale representation theorem (as discussed in Section 9 of Kendall, 1990) to deduce the existence of a predictable random process η mapping $T_X\mathbb{M}$ linearly to $T_{F(X)}\mathbb{N}$, and such that N can be expressed as the Itô integral

$$N \quad = \quad \left[\Psi^{-1}\,\eta\,\Xi\right] * B$$

where Ψ is the parallel transport in $O(\mathbb{N})$ for $F(X)$ and Ξ is the parallel transport in $O(\mathbb{M})$ for X. (In fact arguments based on the Markov property can be used to show that η is a measurable section of $T\mathbb{M}\otimes_F T\mathbb{N}$ over X.)

We will now show that

$$\eta \quad = \quad DF(X) \tag{3.2}$$

$\mathbb{P}$-almost-surely for almost all time until X exits $\text{ball}(\mathbf{o}, R)$.

For suppose that s is a smooth convex function on $\overline{U}$. Then $u = s\circ F$ is subharmonic and Lipschitz, with weak first derivative $(\text{grad}\, s)\circ DF$. If $\widetilde{X}$ is a time-changed Brownian motion as in Theorems 2.3, 3.1 above then regularity theory for its heat kernel shows us that

$$p^\epsilon(x) \quad = \quad \mathbb{E}\left[s\circ F(\widetilde{X}_\epsilon) \mid \widetilde{X}_0 = x\right]$$

defines a function p^ϵ which is smooth on $\mathrm{ball}(\mathbf{o},R)$ and which converges uniformly to $s\circ F$ on compact subsets of $\mathrm{ball}(\mathbf{o},R)$. Furthermore $p^\epsilon(\widetilde{X})$ is a submartingale (by a direct computation using iterated conditional expectations), and satisfies a Lipschitz estimate with respect to the $\tilde{g}$ metric on $\mathrm{ball}(\mathbf{o},R)$. To establish the Lipschitz estimate we use Corollary 2.6:

$$\begin{aligned} |p^\epsilon(x)-p^\epsilon(y)| &= \left|\mathbb{E}\left[s\circ F(\hat{X}_\epsilon) \;-\; s\circ F(\hat{Y}_\epsilon) \mid \hat{X}_0=x, \hat{Y}_0=y\right]\right| \\ &\le c_{lip}\|\operatorname{grad}(s)\|_\infty \exp\left(\frac{d-1}{2}\epsilon\right)\times \widetilde{\operatorname{dist}}(x,y)\,. \end{aligned}$$

On compact subsets of $\mathrm{ball}(\mathbf{o},R)$ we know that $\widetilde{\operatorname{dist}}$ is comparable with dist. Consequently Dp^ϵ is bounded on such compact subsets.

The convergence $p^\epsilon \to s\circ F$ holds uniformly on compact regions in $\mathrm{ball}(\mathbf{o},R)$, and Dp^ϵ is bounded on such regions, so we may use weak compactness methods to select a subsequence (which we will still denote by $\{p^\epsilon\}$) such that (using now the Brownian motion X rather than the time-changed Brownian motion $\widetilde{X}$)

$$\mathbb{E}\left[\int_0^T \langle Dp^\epsilon(X) - (\operatorname{grad} s)\circ DF(X), \underline{h}(X)\rangle\, dt \mid X_0=x\right] \quad\longrightarrow\quad 0 \qquad \text{as } \epsilon\to 0\,,$$

for any vectorfield $\underline{h}$ such that $\mathbb{E}\left[\int_0^T \|\underline{h}(X)\|^2\,dt \mid X_0=x\right] < \infty$, when x is in a given compact region of $\mathrm{ball}(\mathbf{o},R)$ and T is the exit time from this region. Over such stochastic intervals $[0,T]$ we may consider the Itô decomposition

$$d_I\, p^\epsilon(X) \quad=\quad [Dp^\epsilon(X)\,\Xi]\; d_I\, B \;+\; {}^1\!/_2 \operatorname{trace}\operatorname{Hess} p^\epsilon(dX,dX)$$

and deduce that the integral of the left-hand side converges to $s\circ F(X)$ in L^2, the integral of the first term on the right-hand side converges to $\int [(\operatorname{grad} s)\circ DF(X)\,\Xi]\; d_I\, B$ weakly in L^2, and thus the integral of the second term on the right-hand side must also converge weakly in L^2, say to V. The limit V is the weak limit of increasing processes (recall that p^ϵ is subharmonic), so itself is increasing, and we have therefore identified an Itô decomposition for $s\circ F$:

$$d_I\,(s\circ F(X)) \quad=\quad [(\operatorname{grad} s)\circ DF(X)\,\Xi]\; d_I\, B \;+\; dV\,.$$

However the Itô decomposition for $s\circ F(X)$ can be obtained directly: using the geometric form of the Itô formula and our formula for N the stochastic development of the Γ-martingale $F(X)$ we can show that

$$d_I\,(s\circ F(X)) \quad=\quad [(\operatorname{grad} s)\circ \eta\,\Xi]\; d_I\, B \;+\; {}^1\!/_2 \operatorname{trace}\operatorname{Hess} s(dX,dX)\,.$$

It follows,

$$\mathbb{E}\left[\int_0^T \|(\operatorname{grad} s)\circ\eta - (\operatorname{grad} s)\circ DF(X)\|^2\; dt \mid X_0=x\right] \quad=\quad 0\,.$$

For sufficiently small U the supply of smooth convex s is rich enough (use squares of coordinate functions) to allow us to deduce that $\eta = DF$ for almost all time, $\mathbb{P}$-almost-surely, as required.

Consider the Itô formula applied to $f(F(X))$ for a smooth function $f:\overline{U}\to\mathbb{R}$. Taking expectations we find that

$$\mathbb{E}[f(F(X_\sigma))] \quad=\quad f(F(X_0)) \;+\; {}^1\!/_2\; \mathbb{E}\left[\int_0^\sigma \operatorname{trace}\operatorname{Hess} f(DF(X),DF(X))\,dt\right]$$

for any stopping time σ. Hence we may deduce that $u=f\circ F$ is a weak solution of the scalar linear elliptic partial differential equation

$$\Delta u \quad=\quad [\operatorname{trace}\operatorname{Hess} f(DF,DF)] \qquad (3.3),$$

and the right-hand side is an L^∞ function on $\mathrm{ball}(\mathbf{o},R/2)$.

We can now apply the regularity theory of *scalar linear* elliptic partial differential equations. It follows from Theorem 8.34 and Corollary 8.35 of Gilbarg & Trudinger (1983) that equation (3.3) has a unique solution in $W^{1,2}$ which is also $C^{1,\alpha}$. Since $u = f \circ F$ is $W^{1,\infty} \subset W^{1,2}$ we deduce that u is $C^{1,\alpha}$. Since f is an arbitrary smooth function of compact support, it follows that F is also $C^{1,\alpha}$, and hence that DF and hence trace Hess $f(DF, DF)$ are C^{α}. The theory of scalar linear elliptic partial differential equations with C^{α} coefficients applies, so that we may deduce $u = f \circ F$ is C^2 (and indeed C^{∞} by repeating this procedure). It follows immediately that F is smooth (since f is an arbitrary smooth function), and so the proof is completed. □

If $\mathbb{N}$ is permitted to have some positive curvatures then the distance function $\mathrm{dist}_{\mathbb{N}}$ need no longer be convex. Even if the connection is not Riemannian, we can choose U to be small enough to have convex geometry: to support a function

$$\Phi \ : \ U \times U \quad \longrightarrow \quad [0,1]$$

which is convex and vanishes only on the diagonal (Emery & Meyer, 1989, Lemma (4.59), Kendall, 1990, 1991). However in general such functions behave like $\mathrm{dist}_{\mathbb{N}}$ raised to a power greater than 1. Consequently the obvious extension of the above methods can only supply a Hölder condition, rather than a full Lipschitz property. For example,

Corollary 3.3: *Suppose that $F : \mathbb{M} \to \mathbb{N}$ is a continuous finely L-harmonic map (for L, $\mathbb{M}$, $\mathbb{N}$ as in Theorem 3.2). Then F is locally Hölder($^1/_2$) in the interior of $\mathbb{M}$.*

Proof: Following Lemma (4.59) of Emery & Meyer (1989), choose R small enough that F restricted to the closed region $\overline{\mathrm{ball}(\mathbf{o}, R)}$ has codomain U of convex geometry: supporting a convex function

$$\Phi \ : \ U \times U \quad \longrightarrow \quad [0,1]$$

which is convex and vanishes only on the diagonal, and such that

$$\Phi(m,n) \quad \geq \quad \text{constant} \times \mathrm{dist}_{\mathbb{N}}(m,n)^2 \tag{3.4}$$

for m, $n \in U$. For example in the Riemannian case Φ can be chosen of the form

$$\Phi(m,n) \quad = \quad \left(\alpha^2 + \mathrm{dist}_{\mathbb{N}}(m,k)^2 + \mathrm{dist}_{\mathbb{N}}(n,k)^2\right) \mathrm{dist}_{\mathbb{N}}(m,n)^2 \tag{3.5}$$

for some reference point $k \in U$, and suitable α. (Emery gives a similar construction for manifolds with general connections.)

Using the coupling construction of Theorem 2.3 and Corollary 2.4 we see that for all x, y in $\mathrm{ball}(\mathbf{o}, R/2)$

$$\begin{aligned}
\mathrm{dist}_{\mathbb{N}}(F(x), F(y))^2 \quad &\leq \quad \text{constant} \times \Phi(F(x), F(y)) \quad \leq \quad \text{constant} \times \mathbb{E}\left[\Phi(F(\widetilde{X}_\infty), F(\widetilde{Y}_\infty))\right] \\
&\leq \quad \text{constant} \times \mathrm{diam}(U) \times \widetilde{\mathrm{dist}}(x,y) \quad \leq \quad \text{constant} \times \mathrm{diam}(U) \times \mathrm{dist}(x,y)
\end{aligned}$$

where the constant varies from formula to formula but does not depend on the choice of x, $y \in \mathrm{ball}(\mathbf{o}, R/2)$. (Note also that $F(\widetilde{X}_\infty)$, $F(\widetilde{Y}_\infty)$ are defined as Γ-martingale limits in U, using for example Huang & Kendall, 1991, Theorem 4A.) The local Hölder($^1/_2$) condition follows. □

However we see below that it is possible to recover regularity for continuous finely L-harmonic maps $F : \mathbb{M} \to \mathbb{N}$ where the codomain $\mathbb{N}$ is entirely general (and L is a strictly elliptic second-order differential operator with smooth coefficients), by varying the coupling and by using the Hölder($^1/_2$) result of Corollary 3.3.

Theorem 3.4: *Suppose that $F : \mathbb{M} \to \mathbb{N}$ is a continuous finely L-harmonic map, where L is a strictly elliptic second-order differential operator with smooth coefficients and $\mathbb{N}$ is a manifold with geodesically complete connection. Then F is smooth in the interior of $\mathbb{M}$ and in particular is a harmonic map.*

Proof: Give $\mathbb{M}$ the Riemannian metric such that $L = {}^{\mathbb{M}}\Delta + \zeta$, where ζ is a vectorfield (the intrinsic drift of L). Furnish $\mathbb{N}$ with a reference Riemannian metric.

Because of Theorem 3.2 it suffices to show that F is locally Lipschitz. As before, we can localize to the case

$$F \; : \; \overline{\text{ball}(\mathbf{o}, R)} \quad \longrightarrow \quad U$$

for U a small open subset of $\mathbb{N}$ and for R small. The method of proof of Theorem 3.1 applies directly so long as we can show the existence for each x, $y \in \text{ball}(\mathbf{o}, R/2)$ of a coupled pair $\widetilde{X}$, $\widetilde{Y}$ of time-changes of Riemannian Brownian motions with drift ζ in $\mathbb{M}$, beginning at x, y and such that

(i) there is a constant not depending on the choice of x and y such that

$$\mathbb{P}\left[\, \widetilde{X}, \widetilde{Y} \text{ fail to couple before either one exits from } \text{ball}(\mathbf{o}, R) \,\right] \quad \leq \quad \text{constant} \times \widetilde{\text{dist}}(x, y)\,;$$

(ii) the distance process $\text{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))$ is a submartingale up to the first time that $F(\widetilde{X}) = F(\widetilde{Y})$.

We begin by working on part (ii) of the requirements. First note that for sufficiently small U there is a positive constant c_3, depending only on the upper bound of sectional curvatures in U, and such that

$$\begin{aligned} &\text{Hess}\,\text{dist}_{\mathbb{N}}(m,n)((\underline{v}_1, \underline{v}_2), (\underline{v}_1, \underline{v}_2)) \quad \geq \\ &- c_3\, \text{dist}_{\mathbb{N}}(m,n) \left\{ \left(\|\underline{v}_1\|^2 - \langle \underline{v}_1, \text{grad}_1\, \text{dist}(m,n) \rangle^2 \right) \; + \; \left(\|\underline{v}_2\|^2 - \langle \underline{v}_2, \text{grad}_2\, \text{dist}(m,n) \rangle^2 \right) \right\} \end{aligned} \tag{3.6}$$

for $\underline{v}_1 \in T_m\mathbb{N}$, $\underline{v}_2 \in T_n\mathbb{N}$. (Of course this is working in the reference metric for $\mathbb{N}$. It uses standard estimates on Jacobi vectorfields, as found for example in Hildebrandt, 1985.)

There is also a positive constant c_4, depending only on the lower bound of sectional curvatures in U, such that

$$\begin{aligned} \text{Hess}\,\text{dist}_{\mathbb{N}}(m,n)((\underline{v}_1, 0), (\underline{v}_1, 0)) \quad &\geq \quad c_4 \times \frac{\|\underline{v}_1\|^2 - \langle \underline{v}_1, \text{grad}_1\, \text{dist}(m,n)\rangle^2}{\text{dist}_{\mathbb{N}}(m,n)} \\ \text{Hess}\,\text{dist}_{\mathbb{N}}(m,n)((0, \underline{v}_2), (0, \underline{v}_2)) \quad &\geq \quad c_4 \times \frac{\|\underline{v}_2\|^2 - \langle \underline{v}_2, \text{grad}_2\, \text{dist}(m,n)\rangle^2}{\text{dist}_{\mathbb{N}}(m,n)} \end{aligned} \tag{3.7}$$

for $\underline{v}_1 \in T_m\mathbb{N}$, $\underline{v}_2 \in T_n\mathbb{N}$. (Again, use standard estimates on Jacobi vectorfields from, for example, Hildebrandt, 1985.)

Fix $\alpha \in (0, 1/2)$ and choose $R < 1$ small enough so that U can be chosen small enough to ensure that

$$\frac{c_4}{c_5^{2\alpha}}\, \text{dist}_{\mathbb{N}}(m,n)^{2\alpha - 1} \quad > \quad c_3\, \text{dist}_{\mathbb{N}}(m,n) \; + \; \|\delta\|_\infty \tag{3.8}$$

for all m, $n \in U$, $m \neq n$, where c_5 is the constant for the Hölder($1/2$) property for F following from Corollary 3.3:

$$\text{dist}_{\mathbb{N}}(F(x), F(y)) \quad \leq \quad c_5\, \text{dist}(x,y)^{1/2} \tag{3.9}$$

for all x, $y \in \text{ball}(\mathbf{o}, 1)$ (say), and δ_ξ is a vector-valued bilinear form, bounded for all orthonormal frames $\xi \in O(U)$, representing the effect of the disagreement between specified and reference connections, so that if $F(\widetilde{X})$ is a Γ-martingale on $\mathbb{N}$ using the specified (non-Riemannian) connection for $\mathbb{N}$ then

$$\begin{aligned} d_S\, F(\widetilde{X}) \quad &= \quad \Xi^*\, d_S\, M \; + \; \delta_{\Xi^*}(d[M, M]) \\ d_S\, \Xi^* \quad &= \quad H_{\Xi^*}\, d_S\, F(\widetilde{X}) \end{aligned}$$

where Ξ^* is the parallel transport over $F(\widetilde{X})$ using the reference Riemannian metric, and M is a martingale (in fact $d_I\, M = [(\Xi^*)^{-1}\Theta^*]\, d_I\, M^*$ where Θ^* is the stochastic parallel transport over $F(\widetilde{X})$ using the specified, non-Riemannian connection and M^* is the corresponding stochastic development). The quantity δ_ξ can be represented in terms of the difference of traces of Hessians built up from the specified and reference connections.

We now modify the coupling construction $(2.9a,b)$ as follows. Suppose that A, B, C are independent d-dimensional Euclidean Brownian motions. Construct $\widetilde{X}$, $\widetilde{Y}$ starting at x, $y \in \operatorname{ball}(\mathbf{o}, R/2)$ with initial frames $\tilde{\xi}_0$, $\tilde{\psi}_0$ in $\widetilde{O}(\operatorname{ball}(\mathbf{o}, R))$ to solve the system

$$\begin{aligned} d_S \widetilde{X} &= \widetilde{\Xi}\, d_S W^{(\widetilde{X})} + \chi(\widetilde{X})\, dt + \zeta(\widetilde{X})\, dt \\ d_S \widetilde{\Xi} &= \widetilde{H}_{\widetilde{\Xi}}\, d_S \widetilde{X} \end{aligned} \qquad (3.10a)$$

$$\begin{aligned} d_S \widetilde{Y} &= \widetilde{\Psi}\, d_S W^{(\widetilde{Y})} + \chi(\widetilde{Y})\, dt + \zeta(\widetilde{Y})\, dt \\ d_S \widetilde{\Psi} &= \widetilde{H}_{\widetilde{\Psi}}\, d_S \widetilde{Y} \end{aligned} \qquad (3.10b)$$

$$\begin{aligned} d_I W^{(\widetilde{X})} &= \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha/2}\, d_I A + \sqrt{\left(1 - \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha}\right)}\; d_I B \\ d_I W^{(\widetilde{Y})} &= \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha/2}\, d_I C + \sqrt{\left(1 - \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha}\right)}\; \left[\widetilde{\Psi}^{-1}\, \widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}\, \widetilde{\Xi}\right]\; d_I B \end{aligned} \qquad (3.10c)$$

The point is that only $1 - \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha}$ of the quadratic variations of $\widetilde{X}$, $\widetilde{Y}$ are held in common *via* the "mirror-map" $\widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}$: the remainders derive from independent Brownian motions. As $\widetilde{X}$, $\widetilde{Y}$ draw together so the common quadratic variation increases; the exponent α is chosen so that the probability of not coupling is subject (as we will shortly see) to a bound which is linear in $\widetilde{\operatorname{dist}}(x, y)$. Nevertheless the independent variation supplies sufficient repulsion between $F(\widetilde{X})$, $F(\widetilde{Y})$ to overcome the nonconvexity of the distance function in $\mathbb{N}$, and also to overcome the disagreement between the specified connection for $\mathbb{N}$ and the Levi-Civita connection for its reference Riemannian metric.

An Itô analysis of $\operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))$ reveals the drift of its Itô differential to be

$$\begin{aligned} &\operatorname{trace}\left[\operatorname{Hess}\operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y})) \cdot ((\Xi^* \, dM, \Psi^* \, dN), (\Xi^* \, dM, \Psi^* \, dN))\right] \\ &\quad + \operatorname{grad}_1 \operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))\delta_{\Xi^*}(d[M, M]) + \operatorname{grad}_2 \operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))\delta_{\Psi^*}(d[N, N]) \end{aligned} \qquad (3.11)$$

where M, N are the (local) martingale parts of the stochastic developments of $F(\widetilde{X})$, $F(\widetilde{Y})$, and Ξ^*, Ψ^* are their stochastic parallel transports (using the reference metric for $\mathbb{N}$). But we may apply the martingale representation theorem to deduce the existence of matrix-valued processes $J^{\widetilde{X}}$, $J^{\widetilde{Y}}$ such that

$$\begin{aligned} d_I M &= J^{\widetilde{X}} \left[\operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha/2}\, d_I A + \sqrt{\left(1 - \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha}\right)}\; d_I B\right] \\ d_I N &= J^{\widetilde{Y}} \left[\widetilde{\Xi}^{-1}\, \widetilde{\mathfrak{M}}_{\widetilde{Y},\widetilde{X}}\, \widetilde{\Psi}\right] \left[\operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha/2}\, d_I C + \sqrt{\left(1 - \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha}\right)}\; \left[\widetilde{\Psi}^{-1}\, \widetilde{\mathfrak{M}}_{\widetilde{X},\widetilde{Y}}\, \widetilde{\Xi}\right]\; d_I B\right] \end{aligned} \qquad (3.12).$$

(This is shown for $J^{\widetilde{X}}$ by working in the filtration generated by $\widetilde{X}$ alone, for $J^{\widetilde{Y}}$ by working in the filtration generated by $\widetilde{Y}$ alone.)

From equations (3.11), (3.12) we deduce that the drift of the Itô differential of $\operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))$ is

$$\begin{aligned} &\operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha} \operatorname{trace}\left[\operatorname{Hess}_1 \operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y})) \cdot \left(\Xi^* J^{\widetilde{X}} dA, \Xi^* J^{\widetilde{X}} dA\right)\right] \\ &+ \left(1 - \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha}\right) \times \\ &\qquad \times \operatorname{trace}\left[\operatorname{Hess}\operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y})) \cdot \left((\Xi^* J^{\widetilde{X}} dB, \Psi^* J^{\widetilde{Y}} dB), (\Xi^* J^{\widetilde{X}} dB, \Psi^* J^{\widetilde{Y}} dB)\right)\right] \\ &+ \operatorname{dist}(\widetilde{X}, \widetilde{Y})^{\alpha} \operatorname{trace}\left[\operatorname{Hess}_2 \operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y})) \cdot \left(\Psi^* J^{\widetilde{Y}} dC, \Psi^* J^{\widetilde{Y}} dC\right)\right] \\ &\quad + \operatorname{grad}_1 \operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))\delta_{\Xi^*}(d[M, M]) + \operatorname{grad}_2 \operatorname{dist}_{\mathbb{N}}(F(\widetilde{X}), F(\widetilde{Y}))\delta_{\Psi^*}(d[N, N]) \end{aligned} \qquad (3.13)$$

Using the representation (3.12) of M, N in terms of A, B, C, and noting that A, B, C are all Brownian motions, the nonnegativity of this expression follows by employing the bounds (3.6), (3.7), which show that the quantity in (3.13) is bounded below by

$$\left(c_4 \frac{\mathrm{dist}(\widetilde{X},\widetilde{Y})^\alpha}{\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y}))} - \|\delta\|_\infty\right) dt \;\times\; \sum_{\underline{v}} \left\{ \|J^{\widetilde{X}}\underline{v}\|^2 + \|J^{\widetilde{Y}}\underline{v}\|^2 \right\}$$
$$- \quad c_3\,\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y}))\,dt \;\times\; \sum_{\underline{v}} \Big\{ \|J^{\widetilde{X}}\underline{v}\|^2 - \langle \Xi^* J^{\widetilde{X}}\underline{v}, \mathrm{grad}_1\,\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y}))\rangle^2$$
$$+ \|J^{\widetilde{Y}}\underline{v}\|^2 - \langle \Psi^* J^{\widetilde{Y}}\underline{v}, \mathrm{grad}_2\,\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y}))\rangle^2 \Big\}$$

where the sum is taken over an orthonormal basis of vectors $\{\underline{v}\}$. The sum is an absolutely continuous nonnegative random measure, and its density is bounded below because of the following:

$$c_4 \frac{\mathrm{dist}(\widetilde{X},\widetilde{Y})^\alpha}{\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y}))} \;-\; c_3\,\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y})) \;-\; \|\delta\|_\infty$$
$$\geq \quad \frac{c_4}{c_5^{2\alpha}} \mathrm{dist}(F(\widetilde{X}),F(\widetilde{Y}))^{2\alpha-1} \;-\; c_3\,\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y})) \;-\; \|\delta\|_\infty \quad\geq\quad 0$$

using the bounds (3.8), (3.9). Of course this analysis breaks down as soon as $F(\widetilde{X}) = F(\widetilde{Y})$. However we see that $\mathrm{dist}_{\mathbb{N}}(F(\widetilde{X}),F(\widetilde{Y}))$ is a submartingale up to the first time that $F(\widetilde{X}) = F(\widetilde{Y})$. This establishes part (ii) of our requirements.

Part (i) follows from an Itô analysis of the distance process $\widetilde{\mathrm{dist}}(\widetilde{X},\widetilde{Y})$. The drift of its stochastic differential is bounded above by

$$(d-1)\Big\{ \left(1 - \mathrm{dist}(\widetilde{X},\widetilde{Y})^\alpha\right) \kappa^{-1} \tanh\left(\frac{\kappa\,\widetilde{\mathrm{dist}}(\widetilde{X},\widetilde{Y})}{2}\right)$$
$$+ \tfrac{1}{2}\,\mathrm{dist}(\widetilde{X},\widetilde{Y})^\alpha \kappa \coth\left(\kappa\,\widetilde{\mathrm{dist}}(\widetilde{X},\widetilde{Y})\right) \Big\}\, dt + 2\|\chi\|_\infty\,dt + 2\|\zeta\|_\infty\,dt$$
$$\leq\; (d-1)\Big\{ \kappa^{-1} \tanh\left(\frac{\kappa\,\widetilde{\mathrm{dist}}(\widetilde{X},\widetilde{Y})}{2}\right)$$
$$+ \tfrac{1}{2}\,\widetilde{\mathrm{dist}}(\widetilde{X},\widetilde{Y})^\alpha \kappa \coth\left(\kappa\,\widetilde{\mathrm{dist}}(\widetilde{X},\widetilde{Y})\right) \Big\}\, dt + 2(\|\chi\|_\infty + \|\zeta\|_\infty)\,dt$$

(using the Remark about metric contraction following the proof of Theorem 2.1: recall that $R < 1$) while its martingale part is of quadratic variation bounded above by $4\,dt$. The probability of it failing to hit zero is bounded above by the probability $p(r_0)$ of ρ failing to hit zero before hitting the level 1, where ρ solves the stochastic differential equation

$$\begin{aligned} \rho_0 \quad &= \quad \widetilde{\mathrm{dist}}(\widetilde{X}_0,\widetilde{Y}_0) \quad = \quad r_0 \\ d_I\,\rho \quad &= \quad d_I\,W \;+\; \frac{c_6}{2\rho^{1-\alpha}}\,dt \end{aligned} \tag{3.14}$$

for W a (scalar) Brownian motion and for some positive c_6 not depending on x, y. But then the Itô formula may be applied to show

$$p'' \;+\; \frac{c_6}{r^{1-\alpha}} p' \quad = \quad 0\,; \quad p(0) = 0\,, \qquad p(1) = 1\,,$$

which can be solved to give

$$p(r) \quad = \quad \frac{\int_0^r \exp(-\alpha c_6 s^\alpha)\,ds}{\int_0^1 \exp(-\alpha c_6 s^\alpha)\,ds}\,.$$

So p is C^1 and moreover

$$p'(0) \quad = \quad \frac{1}{\int_0^1 \exp(-\alpha c_6 s^\alpha)\,ds} \quad > \quad 0$$

and so

$$p(r_0) \quad \leq \quad \text{constant} \times \widetilde{\mathrm{dist}}(x,y)\,.$$

This deals with Part (i) as required, and so the Theorem is proved. □

4: Conclusion.

The probabilistic approach to regularity described here covers the mild modifications of the Euler-Lagrange variational equations for (1.1) mentioned above; these modifications correspond to replacing the Riemannian Brownian motion by a strongly elliptic diffusion with smooth coefficients, and replacing the geometric datum of a Riemannian metric on the target manifold by that of a general connection. Hence these results are more than mere probabilistic variations on the analytic approach to harmonic maps (but see Wahl, 1981). A further possible generalization is hinted at by the recent work of Ben Arous, Cranston & Kendall (1993), which has derived coupling results holding for certain nilpotent diffusions. It seems at least possible that the probabilistic approach may therefore generalize to cover some nonlinear *hypoelliptic* partial differential systems. However it is not yet at all clear how the results of Ben Arous *et al.* can lead to gradient inequalities and hence to regularity results of the form described above. Note that the results of Picard (1991a, 1993) and also of Arnaudon (1993) (using the methods of Emery & Mokobodzki, 1991; see also Cohen, 1992) provide very general existence results for considerable probabilistic generalizations of harmonic maps (related to analytic work of Jost (1993) on generalizations of harmonic maps to metric spaces). However in general these maps need not be continuous; coupling theory (where it applies) should provide a convenient way of establishing continuity.

At the time of writing there are many other workers undertaking various probabilistic investigations into nonlinear partial differential equations. Of these, the nearest in spirit to the Γ-martingale approach to harmonic maps is provided by the work of Pardoux and Peng (see for example Peng, 1991). This constructs solutions to quasilinear partial differential systems by solving stochastic control reachability problems, and in this is similar in spirit to the approaches of Kendall (1990), Picard (1991b). However Pardoux and Peng require a global Lipschitz condition for their approach, and this is not available for harmonic maps because of the quadratic term

$$^{\mathbb{M}}g^{ij}\,{}^{\mathbb{N}}\Gamma^{\alpha}_{\beta\gamma}\frac{\partial F^{\beta}}{\partial x^{i}}\frac{\partial F^{\gamma}}{\partial x^{j}}$$

Perhaps one should seek for a synthesis of one of the Γ-martingale approaches and the Pardoux-Peng approach, combining (a) the Γ-martingale techniques for dealing with non-globally Lipschitz coefficients with (b) the greater generality (within the global Lipschitz context) of the Pardoux-Peng method.

In addition to the application described here, coupling methods have found recent use in other areas of stochastic analysis: linear partial differential equation theory (Kendall, 1989 established a partial domain monotonicity result for Neumann heat kernels), in stochastic differential geometry (for example Cranston, 1992), in spectral gap problems (see for example Chen, 1994), and in treatments of diffusions on fractals (see Barlow & Bass, 1993). It would be of great interest to determine to what extent coupling methods of the kind described here might be similarly applicable to gradient estimates for infinite-dimensional diffusions. (Of course the general coupling method is a well-established tool for general investigation of infinite dimensional random systems; see for example Liggett, 1985.) Preliminary calculations suggest that concepts such as Cameron-Martin space will arise in a very natural way.

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