ON THE REGULARITY OF SOLUTIONS TO THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITH POLYNOMIALLY GROWING COLLISION KERNEL

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Abstract. The paper is devoted to the propagation of smoothness (more precisely L_∞-moments of the derivatives) of the solutions to the spatially homogeneous Boltzmann equation with polynomially growing collision kernels.

1. Introduction

The paper is devoted to the spatially homogeneous d-dimensional Boltzmann equation

$$\frac{\partial f}{\partial t} = Q(f,f), \tag{1.1}$$

where $t \geq 0$, $v \in \mathbf{R}^d$, $d \geq 3$, and the collision operator Q is given by the standard formula

$$Q(f,g)(v) = \int_{\mathbf{R}^d} dw \int_{S^{d-1}_{w-v,+}} dn\, B(|v-w|,\theta)[f(v')g(w') - f(v)g(w)]. \tag{1.2}$$

Here n denotes the unit vector in the direction $v'-v$, $S^{d-1}_{w-v,+} = \{n \in S^{d-1} : (n, w-v) \geq 0\}$, dn denotes the Lebesgue measure on S^{d-1}, θ is the (necessarily acute) angle between $w-v$ and n (or $v'-v$),

$$\begin{cases} v' &= v + (w-v,n)n \\ w' &= w - (w-v,n)n \end{cases} \Leftrightarrow \begin{cases} v &= v' + (w'-v',n)n \\ w &= w' - (w'-v',n)n \end{cases} \tag{1.3}$$

and the collision kernel $B(|z|,\theta)$ is a given measurable non-negative function on $\mathbf{R}_+ \times [0,\pi/2]$ of polynomial growth, i.e.

$$\int_{S^{d-1}_{w-v,+}} B(|z|,\theta)\, dn \leq C(1+|z|^\beta) \tag{1.4}$$

with some constants $C > 0$ and $\beta \geq 0$ (and with some additional assumptions discussed below).

Key words and phrases. Boltzmann equation with polynomially growing collision kernel, L^∞-bounds, propagation of smoothness.

The aim of the paper is to show that if initial conditions are smooth with derivative having a power decay as $|v| \to \infty$, then the same holds for the energy preserving solutions. In passing we give explicit L^∞- bounds for the solutions and prove a seemingly new existence and uniqueness result for $\beta > 2$. Some of the results obtained can be extended to more general kinetic equations discussed in [Ko1], [Ko2]. The property of propagation of smoothness is important both theoretically, as it forbids a spontaneous creation of shocks, and practically, say, for error estimates of interpolation schemes used for numerical calculations. For the latter, the explicit bounds for derivatives are, of course, relevant. Our estimates are uniform in time for all positive β. For $\beta = 0$ (maxwellian gas) our estimates for L_∞-moments depend on time, but on the other hand they are much simpler (they are monotone in the corresponding norms of initial data without a "pathological" blow up, as the mass goes to zero, like in case of positive β) and they can be obtained without any reference to the corresponding integral moments.

Our paper is close in spirit to the recent work [MV] devoted to the propagation of the integral moments of the derivatives of the solutions to the Boltzmann equation in case $\beta \in (0, 2)$, but it is quite different by its results and technique used. The paper [MV] is based on the regularity of the gain term in the sense discovered by Lions in [Li] (and further developed in [W],[Lu1],[BD]) and deals with integral moments of the derivatives. Our method of deducing the L_∞-bounds for the derivatives from the integral ones is an extension of the methods from [Ca],[Ar2],[Gu1] on the analysis of the solutions of the Boltzmann equation to the analysis of the derivatives of these solutions.

The paper is organized as follows. Further in this introductory section we recall basic representations of the collision kernel which we need for our analysis. The details of the deduction of these representations are widely presented in the literature, see e.g. [Gu1], [Gu2] for $d = 3$. For the general background on the Boltzmann equation we refer to the monographs [Ce], [CIP], [Ma], see also some recent results and references in [MW] and [Vi]. Section 2 is devoted to the analysis of maxwellian gas, i.e. to the case $\beta = 0$ in (1.4). This case deserves a special treatment, because (i) it requires special methods, (ii) is not included in usual treatments of L_∞-bounds (see [Gu1], [Gu2]), (iii) is often required as an intermediate approximation to the case of growing kernels. Section 3 deals with L^∞-bounds of the solutions in case of arbitrary $\beta > 0$. The novelty here is two-folds. Firstly we extend the results of [Ca], [Ar2], [Gu1] (devoted to the case $d = 3$ and $\beta \in (0, 1]$) to arbitrary d and $\beta > 0$, and secondly we give explicit dependence of the bounds on the initial functions (not just a vague statement that they depend on a lower bound of the mass). This progress is achieved by simplification and modification of the methods from [Ca], [Ar2], [Gu1] in several directions. However, the main objective of this section is to extend the whole technique in a way suitable for estimating derivatives of the solutions. In Section 4 our main results on the propagation of L_∞-bounds for the derivatives are obtained in case of arbitrary $\beta > 0$. As we mentioned above, the propagation of the corresponding integral moments was analysed in [MV]. Of course, the knowledge of integral moments for higher derivatives can be used to get uniform bounds for lower derivatives, but our method allows to obtain uniform bounds to derivatives of arbitrary given order without any references to higher derivatives. Moreover, we also analyse the smoothness with respect to the initial data, which could be instructive for the analyis of the full (spatially non-trivial) Boltzmann equation. In Appendix, some auxiliary results are collected.

Recall first that (1.3) describes the general transformation of the pairs of vectors that preserve momentum and energy, i.e. for arbitrary $v, w \in \mathbf{R}^d$ and $n \in S^{d-1}_{w-v,+}$ the vectors v', w' given by (1.3) satisfy

$$v' + w' = v + w, \quad |v'|^2 + |w'|^2 = |v|^2 + |w|^2, \tag{1.5}$$

and vice versa any pair v', w' satisfying (1.5) is given by (1.3) with some (uniquely defined) $n \in S^{d-1}_{w-v,+}$. In the kinetic theory of gases a non-negative measurable function f on $\mathbf{R}^d$ describes the density of a gas and hence the state of a system, the integrals $\int f(v)\,dv$ and $\int |v|^2 f(v)\,dv$ describe the total mass and the energy of the state f and are denoted by $M(f)$ and $E(f)$ respectively, the integral $\int v f(v)\,dv$ denotes the total momentum of the state f, and the vectors v, w (respectively v', w') are interpreted as the velocities of two particles just before (respectively, just after) a collision.

Extending B to the angles $\theta \in [\pi/2, \pi]$ by $B(|z|, \theta) = B(|z|, \pi - \theta)$ yields

$$Q(f,g)(v) = \frac{1}{2}\int_{\mathbf{R}^d} dw \int_{S^{d-1}} dnB(|v-w|,\theta)[f(v')g(w') - f(v)g(w)]. \tag{1.6}$$

On the other hand, writing

$$n = \frac{w-v}{|w-v|}\cos\theta + m\sin\theta, \qquad dn = \sin^{d-2}\theta\, d\theta\, dm \tag{1.7}$$

with $m \in S^{d-2}$ and dm being the Lebesgue measure on S^{d-2} yields

$$Q(f,g)(v) = \int_{\mathbf{R}^d} dw \int_0^{\pi/2} d\theta \int_{S^{d-2}} dm\, \sin^{d-2}\theta B(|v-w|,\theta)[f(v')g(w') - f(v)g(w)]. \tag{1.8}$$

Clearly one can write

$$Q(f,f) = G(f,f) - fL(f), \tag{1.9}$$

where L is the linear operator

$$Lf(v) = \int_{\mathbf{R}^d} dw \int_{S^{d-1}_{w-v,+}} B(|v-w|,\theta) f(w)\, dn \tag{1.10}$$

and $G(f,f)$ is called the gain term and is the quadratic form of the bilinear functional

$$G(f,g)(v) = \int_{\mathbf{R}^d} dw \int_{S^{d-1}_{w-v,+}} B(|v-w|,\theta) f(v')g(w')\, dn. \tag{1.11}$$

Multiplying $Q(f,g)$ by an appropriate test function ψ, integrating, and changing the variables of integration in the gain term yields the following weak form of the collision operator

$$\int \psi(v)Q(f,g)(v)\, dv = \frac{1}{2}\int_{\mathbf{R}^{2d}} dvdw \int_{S^{d-1}} dnB(|v-w|,\theta)(\psi(v') - \psi(v))f(v)g(w).$$

By symmetry, one may as well write w' and w as the arguments for ψ in this integral. Consequently,

$$\int \psi(v)\frac{1}{2}(Q(f,g) + Q(g,f))(v)\, dv$$

$$= \frac{1}{8}\int_{\mathbf{R}^{2d}} dvdw \int_{S^{d-1}} dnB(|v-w|,\theta)(\psi(v') + \psi(w') - \psi(v) - \psi(w))(f(v)g(w) + g(v)f(w)). \tag{1.12}$$

Changing the variables of integration $\theta \mapsto \pi/2 - \theta$, $m \mapsto -m$ (and hence $v' \mapsto w'$, $w' \mapsto v'$) in (1.11) implies that if $B(|z|, \theta)\, dn$ is invariant under this transformation, or, more explicitly, if

$$\sin^{d-2}\theta B(|z|,\theta) = \sin^{d-2}(\pi/2 - \theta)B(|z|, \pi/2 - \theta) \tag{1.13}$$

for all $|z|$ and θ, then the bilinear form G is symmetric, i.e. $G(f,g) = G(g,f)$ for all $f, g \in L_1(\mathbf{R}^d)$.

Next, noting that there is a one-to-one correspondence between the pairs $v'w'$ satisfying (1.5) and the vectors $\omega \in S^{d-1}$ such that

$$v' = \frac{v+w}{2} + \frac{|v-w|}{2}\omega, \quad w' = \frac{v+w}{2} - \frac{|v-w|}{2}\omega, \quad \omega \in S^{d-1}, \tag{1.14}$$

one gets the following centered representation of the collision operator (1.2):

$$Q(f,g)(v)=\int_{\mathbf{R}^d} dw \int_{S^{d-1}} S(|v-w|,\theta)[f(v')g(w')-f(v)g(w)]\,d\omega \tag{1.15}$$

with

$$B(|v-w|,\theta)=2^{d-1}\cos^{d-2}\theta S(|v-w|,\theta). \tag{1.16}$$

Equivalently, denoting by $S^{|v-w|/2}_{(v+w)/2}$ the sphere in $\mathbf{R}^d$ centered at $(v+w)/2$ and with radius $|v-w|/2$, and by $d\sigma$ the Lebesgue measure on this sphere, one can write

$$Q(f,g)(v)=\int_{\mathbf{R}^d} dw \int_{S^{|v-w|/2}_{(v+w)/2}} d\sigma(v')S(|v-w|,\theta)(\frac{2}{|v-w|})^{d-1}[f(v')g(w')-f(v)g(w)].$$

In particular, multiplying by a test function $\psi(v)$, integrating and changing the variables of integration $(v,w,v')\in \mathbf{R}^d\times\mathbf{R}^d\times S^{|v-w|/2}_{(v+w)/2}$ to $(v',w',v)\in\mathbf{R}^d\times\mathbf{R}^d\times S^{|v'-w'|/2}_{(v'+w')/2}$ yields the important alternative weak representation for the gain term

$$\int \psi(v)G(f,g)(v)\,dv=\int_{\mathbf{R}^{2d}} dvdw \int_{S^{|v-w|/2}_{(v+w)/2}} d\sigma(v')S(|v-w|,\theta)(\frac{2}{|v-w|})^{d-1}\psi(v')f(v)g(w). \tag{1.17}$$

In L_∞-theory of the Boltzmann equation the crucial role is played by the Carleman representation of the collision operator

$$Q(f,g)(v)=\int_{\mathbf{R}^d} dv' \int dE_{v,v'}(w')\frac{B(|v-w|,\theta)}{|v'-v|^{d-1}}[f(v')g(w')-f(v)g(w)] \tag{1.18}$$

with $\theta=\arctan(|w'-v|/|v'-v|)$, where $E_{v,z}$ denotes the $(d-1)$-dimensional plane in $\mathbf{R}^d$ that passes through v and is perpendicular to $z-v$ and $dE_{v,z}$ denotes the Lebesgue measure on this plane. This representation is obtained from (1.2) by changing the variables of integration w,n to $v'\in\mathbf{R}^d$, $w'\in E_{v,v'}$.

We shall need also a modification of Carleman's representation proposed in [Gu1]. Namely, assuming for simplicity the symmetry condition (1.13), which in terms of function S given by (1.16) reads as

$$\forall\,\theta \quad S(|v|,\theta)=S(|v|,\pi/2-\theta), \tag{1.19}$$

decomposing the integral over n in (1.11) into the sum of two integrals over the sets with $0\le\theta<\pi/4$ and $\pi/4\le\theta\le\pi/2$ and changing the variable of integration $n=(v'-v)/|v'-v|$ to the new $n=(w'-v)/|w'-v|$ (which means changing θ to $\pi/2-\theta$ m to $-m$, and v',w' to w',v') in the second integral (and using the symmetry condition (1.13)) yields

$$G(f,g)(v)=\int_{\mathbf{R}^d} dw \int_{S^{d-1}_{w-v,+}\cap\{\theta\in[0,\pi/4]\}} dnB(|v-w|,\theta)f(v')g(w')$$

$$+\int_{\mathbf{R}^d} dw \int_{S^{d-1}_{w-v,+}\cap\{\theta\in[0,\pi/4]\}} dnB(|v-w|,\theta)f(w')g(v').$$

Making in these integrals the same Carleman's transformation as above leads to the representation

$$G(f,g)(v)=\int_{\mathbf{R}^d} f(v')|v-v'|^{-(d-1)}\,dv' \int_{E_{v,v',\pi/4}} B(|v'-w'|,\theta)g(w')\,dE_{v,v'}(w')$$

$$+\int_{\mathbf{R}^d} g(v')|v-v'|^{-(d-1)}\,dv' \int_{E_{v,v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{v,v'}(w') \tag{1.20}$$

of the gain term, which we shall call the Carleman-Gustafsson representation. Here $E_{v,v',\lambda}$ is the ball in $E_{v,v'}$ with radius $|v'-v|\tan\lambda$ and centre at v and where θ is the acute angle with $\tan\theta = |w'-v|/|v'-v|$.

Another useful transformation of the gain term (also due to Carleman [Ca]) is obtained by the following trick. For an arbitrary real λ, let χ_λ denote the indicator of the half-line $[\lambda,\infty)$. For a given v and a function f one can introduce $f^{v,ext}$, $f^{v,int}$ by

$$f = f^{v,ext} + f^{v,int}, \quad f^{v,ext}(w) = \chi_{|v|/\sqrt{2}}(w) f(w). \tag{1.21}$$

A key observation is that $G(f^{v,int}, g^{v,int})(v) = 0$ for all f, g, which holds because, for any w and any pair v', w' satisfying (1.5) (and hence forming two opposite points on the sphere with poles v and w), either $|w'| \geq |v|/\sqrt{2}$ or $|v'| \geq |v|/\sqrt{2}$. This enables us to write $G(f,g)(v) = G(f, g^{v,ext}) + G(f^{v,ext}, g^{v,int})$, which together with (1.18) implies

$$G(f,g)(v) = \int_{\mathbf{R}^d} f(v')|v-v'|^{-(d-1)} dv' \int_{E_{v,v'}} B(|v'-w'|,\theta) g^{v,ext}(w') dE_{v,v'}(w'),$$

$$+\int_{\mathbf{R}^d} g^{v,int}(v')|v-v'|^{-(d-1)} dv' \int_{E_{v,v'}} B(|v'-w'|,\theta) f^{v,ext}(w') dE_{v,v'}(w'), \tag{1.22}$$

which we shall call the reduced Carleman representation. The same modification of the Carleman-Gustafsson representation (1.20) leads to the following reduced Carleman-Gustafsson representation (that we shall need only for $f = g$):

$$G(f,f)(v) = \int_{\mathbf{R}^d} \frac{(f+f^{v,int})(v')}{|v-v'|^{(d-1)}}\,dv' \int_{E_{v,v',\pi/4}} B(|v'-w'|,\theta) f^{v,ext}(w')\,dE_{v,v'}(w')$$

$$+\int_{\mathbf{R}^d} \frac{f^{v,ext}(v')}{|v-v'|^{(d-1)}}\,dv' \int_{E_{v,v',\pi/4}} B(|v'-w'|,\theta)(f+f^{v,int})(w') dE_{v,v'}(w') \tag{1.23}$$

To conclude the introduction, we note that the basic norms used for the analysis of the Boltzmann equation are defined for vector-valued function $g = \{g_i\}_{i=1}^n$ on $\mathbf{R}^d$ as

$$\|g\|_{\infty,r} = \sup[\max_i |g_i(v)|(1+|v|^r)], \quad \|g\|_{1,s} = \int \sum_{i=1}^n |g_i(v)|(1+|v|^k)\,dv.$$

The corresponding Banach spaces are denoted respectively $L_{\infty,r}$ and $L_{1,s}$, the notation L_1 being reserved for space $L_{1,0}$.

By $C(a,b,...)$ we shall denote various constants depending on parameters $a, b, ...$ By χ_a we denote the indicator function of $[a,\infty)$, i.e. $\chi_a(x) = 1$ (respectively 0) for $x \geq 1$ (respectively otherwise).

2. Maxwellian gas

We shall consider here the Maxevellian gas with a symmetric kernel, i.e. we shall assume that the function $S(|v|,\theta)$ given by (1.6) satisfies the following condition

$$2^{d-1} S(|v|,\theta) \leq c_0 < \infty, \quad S(|v|,\theta) = S(|v|,\pi/2-\theta). \tag{2.1}$$

A rather comprehensive discussion of the known properties of the Boltzmann equation for Maxwelian molecules can be found in [Bo]. We shall need here only the well known (see e.g. [Ar1]) that under the above condition, for arbitrary non-negative $f_0 \in L_1$ there exists a unique non-negative L_1-solution f_t of the Boltzmann equation that preserves the mass, i.e. $M(f_t) = M(f_0)$ for all $t \geq 0$. Moreover, if $E(f_0) < \infty$, then $E(f_t) = E(f_0)$ for all t, and for other integral norms the estimate

$$\|f_t\|_{1,s} \leq \|f_0\|_{1,s} \exp\{C(s,d)c_0 t M(f_0)\} \tag{2.2}$$

holds (it is easy to get more precise time independent estimates, but we do not need them, as our L_∞-estimates for maxwellian gas are time dependent anyway). On the other hand, for any two such solutions f_t and g_t

$$\|f_t - g_t\|_{1,0} \leq \|f_0 - g_0\|_{1,0} \exp\{c_0 t A(d-2)(M(f_0) + M(g_0))\}. \tag{2.3}$$

We start with $L_{\infty,s}$-bounds of the solutions f_t.

Proposition 2.1. *If $f_0 \in L_{1,0} \cap L_{\infty,0}$, then $f_t \in L_{1,0} \cap L_{\infty,0}$ and $\|f_t\|_{\infty,0} \leq \exp\{2c_0 A(d-2)M_0 t\}\|f_0\|_{\infty,0}$ with $M_0 = M(f_0)$ for all $t > 0$.*

Proof. (i) Suppose first that $\|f_t\|_{\infty,0} < \infty$ for all t, and let us obtain the required bounds. From the Carleman-Gustafsson representation

$$G(f_t, f_t) \leq 2c_0 \int_{\mathbf{R}^d} f_t(v')|v-v'|^{-(d-1)} \|f_t\|_{\infty,0} A(d-2)|v-v'|^{d-1}\, dv' \leq 2c_0 M_0 A(d-2)\|f_t\|_{\infty,0}.$$

Hence

$$\|f_t\|_{\infty,0} \leq \|f_0\|_{\infty,0} + \int_0^t 2c_0 A(d-2)M_0 \|f_s\|_{\infty,0}\, ds,$$

which implies the statement of the theorem by Gronwall's lemma.

(ii) To justify this bound let us approximate the collision kernel $B(|v-w|,\theta)$ by a family of cutoff kernels $B_n(v,w,\theta) = B(|v-w|,\theta)\phi_n(R(v,w))$. Here ϕ_n is a family of infinitely smooth functions on $\mathbf{R}^+$ with uniformly bounded derivatives and with range $[0,1]$, and such that $\phi_n(x)$ vanishes (respectively equals one) for $x \geq n$ (respectively $x \leq n-1$), and $R(v,w) = \max\{\|z\| : z \in S_{v,w}\}$, where $S_{v,w}$ denotes the $(d-1)$-dimensional sphere with v and w being its north and south poles. As by definition $R(v,w) = R(v',w')$, one has $\int Q_n(f,f)(v)\, dv = 0$ for any $f \in L_{1,0}$, where Q_n denotes the corresponding collision operator. Consequently, the L_1-theory of the solutions of the Boltzmann equation with the collision operator Q_n is the same as for the collision operator Q giving a unique mass preserving non-negative solution for any non-negative initial function with a finite mass. On the other hand, the evolution defined by Q_n preserves the values of an arbitrary initial function in the points v with $|v| \geq n$. As $Q_n(f,f)$ depends on f locally Lipshitz continuous in $L_{\infty,0}$, there exists a unique local solution f^n in $L_{\infty,0}$ of the Cauchy problem for the corresponding Boltzmann equation for any initial $f_0 \in L_{\infty,0} \cap L_{1,0}$, which obviously preserves the $L_{1,0}$-norm and hence coincides with the unique L_1-solutions. Due to the bounds on $\|f_t\|_{\infty,0}$ obtained above (which are the same for solutions f^n), this solution can not explode in finite times, and hence coincides with the L_1-solution for all times and has the required bounds for $\|f_t^n\|_{\infty,0}$. It remains to observe that the solutions f_t^n converge in L_1-sense to the solutions f_t. This implies almost sure pointwise convergence and hence the preservation of the common (essential) upper bound.

Theorem 2.2. *Let $f_0 \in L_{1,s} \cap L_{\infty,k}$ and either $k < \min(d-1,s)$, or $\min(k,s) > 1$. Then $f_t \in L_{1,s} \cap L_{\infty,k}$ for all t and the corresponding norms are bounded uniformly for $t \leq T$ with an arbitrary T. In particular, if either $k < \min(d-1, sd/(d+s))$ or $k > 1$ and $s \geq d/(d-1)$, then*

$$\|f_t\|_{\infty,k} \leq \|f_0\|_{\infty,k} \exp\{C(k,s,d)c_0 t \sup_{\tau \leq t}(\|f_\tau\|_{\infty,0} + \|f_\tau\|_{1,s})\},$$

which is bounded by (2.2) and Proposition 2.1.

Proof. (i) We shall only obtain the required uniform bounds for the solutions assuming that all norms $\|f_t\|_{\infty,k}$ are finite. A justification is precisely the same as in the previous theorem, because again the L_1-convergence of approximations implies the preservation of a common upper bound.

(ii) Suppose first that $k < d-1$ and $k \leq sd/(d+s)$. Let us divide the integral in Gustafsson-Carleman into the sum $I_1+I_2+I_3$ of three integrals decomposing the domain of integration into three parts

$$D_1 = \{v' : |v'| < |v|/2\}, \quad D_2 = \{v' : |v'| \geq |v|/2, |v'-v| \geq 1\}, \quad D_3 = \{v' : |v'| \geq |v|/2, |v'-v| < 1\}.$$

In D_1 one has $|w'| \geq |v|/2$, hence $f(w') \leq 2^k \|f\|_{\infty,k}/(1+|v|^k)$ and

$$\int_{E_{v,v',\pi/4}} f(w') dE_{v,v'}(w') \leq 2^k A(d-2)\|f\|_{\infty,k}|v'-v|^{d-1}(1+|v|^k)^{-1};$$

in D_2

$$\int_{E_{v,v',\pi/4}} f(w') dE_{v,v'}(w') \leq A(d-2)\|f\|_{\infty,k}\int_0^{|v-v'|}\frac{r^{d-2}}{1+r^k}dr \leq \frac{A(d-2)}{d-1-k}\|f\|_{\infty,k}|v'-v|^{d-1-k}$$

(here the condition $k < d-1$ was used); and in D_3

$$\int_{E_{v,v',\pi/4}} f(w') dE_{v,v'}(w') \leq A(d-2)\|f\|_{\infty,0}|v-v'|^{d-1}.$$

Hence

$$G(f,f) \leq A(d-2)c_0\|f\|_{\infty,k}[2^k\frac{M(f)}{1+|v|^k} + \frac{1}{d-1-k}\int_{|v'|\geq|v|/2}|v-v'|^{-k}f(v')dv']$$

$$+A(d-2)c_0\|f\|_{\infty,0}\int_{|v'|\geq|v|/2, |v'-v|\leq 1} f(v')dv'.$$

Using Proposition A2 with $r=0$, $\lambda = k$ yields the estimate

$$\int_{|v'|\geq|v|/2}|v-v'|^{-k}f(v')dv' \leq C(k,s,d)(\|f\|_{1,s}+\|f\|_{\infty,0})(1+|v|)^{-(s-ks/d)}$$

$$\leq C(k,s,d)(\|f\|_{1,s}+\|f\|_{\infty,0})(1+|v|)^{-k}$$

for $k \leq sd/(s+d)$. As clearly

$$\int_{|v'|\geq|v|/2, |v'-v|\leq 1} f(v')dv' \leq A(d-2)C(k)\|f\|_{\infty,k}(1+|v|^k)^{-1},$$

it follows that $G(f,f) \leq C(k,s,d)c_0\|f\|_{\infty,k}(\|f\|_{1,s}+\|f\|_{\infty,0})(1+|v|^k)^{-1}$. Consequently, Gronwall's lemma implies the required estimate for $\|f_t\|_{\infty,k}$.

(iii) Consider now the general case with $k < d-1$. Clearly it is sufficient to prove the statement for $s \leq d-1$. This assumption implies that $sd/(s+d) < d-1$. Then by (i) we conclude that $\|f_t\|_{\infty,k_1}$ is bounded for $k_1 = sd/(s+d)$. Repeating the previous arguments we can now use Proposition A2 with $r = k_1$, which proves that $\|f_t\|_{\infty,k_1}$ are bounded whenever $s + k(k_1-s)/d \geq k$, in particular, for

$$k_2 = \frac{sd(d+s)}{d^2+sd+s^2} = \frac{sd(d^2-s^2)}{d^3-s^3}.$$

By induction we show that $\|f_t\|_{\infty,k_1}$ are bounded for all

$$k_n = \frac{sd(d^n - s^n)}{d^{n+1} - s^{n+1}}, \quad n = 1, 2, ...,$$

and hence for all $k < s$, as $\lim_{n\to\infty} k_n = s$.

(iv) Assume $k > 1$. Clearly

$$\int_{E_{v,v'}} f^{v,ext}(w') \cos^{d-2}\theta \, dE_{v,v'}(w') \le |v - v'|^{d-2} \int_{E_{v,v'}} f^{v,ext}(w') \frac{dE_{v,v'}(w')}{|w' - v|^{d-2}}.$$

We represent this integral as the sum $I_1 + I_2$ of two integrals, where I_1 is taken over the domain $D = \{w' : |w'| \ge 2|v|\}$. To estimate I_1 we use the polar coordinate in $E_{v,v'}$ centered at the nearest to the origin point of $E_{v,v'}$. As $|w' - v| \ge |v|$, and hence $|w' - v| \ge r/2$ in D, this gives

$$I_1 \le A(d-2)2^{d-2}|v - v'|^{d-2}\|f\|_{\infty,k} \int_{\{r: r^2+\rho^2 \ge (|v|/2)^2\}} \frac{dr}{1 + (\rho^2 + r^2)^{k/2}},$$

where ρ is the distance from $E_{v,v'}$ to the origin. Hence $I_1 \le C(d)\|f\|_{\infty,k}|v - v'|^{d-2}(1+|v|)^{1-k}$ whenever $k > 1$. To estimate I_2 we use polar coordinate centered at v, which gives

$$I_2 \le C(d)|v - v'|^{d-2}\|f\|_{\infty,k}(1 + |v|)^{-k} \int_0^{3|v|} dr,$$

and hence again the same estimate as for I_1. Hence the modified Carleman representation implies

$$G(f, f)(v) \le C(d, k)c_0\|f\|_{\infty,k}(1 + |v|)^{1-k} \int f(v')|v' - v|^{-1} dv'.$$

Suppose first $s \ge d/(d-1)$. Then $s - s/d \ge 1$ and applying Proposition A2 with $\lambda = 1$, $r = 0$ yields

$$G(f, f)(v) \le C(d, s, k)c_0\|f\|_{\infty,k}(\|f\|_{1,s} + \|f\|_{\infty,0})(1 + |v|)^{-k},$$

which implies the required bound for $\|f_t\|_{\infty,k}$ by Gronwall's lemma.

Suppose now $s < d/(d-1)$. If $k < s$, the required boundedness is already shown in (iii) (as then $k < \min(s, d-1)$). Assume $k \ge s$. Again by the previous results of (ii), (iii), we know that the norms $\|f_t\|_{\infty,r}$ are bounded for all $r < s$, and we can use Proposition A2 with $\lambda = 1$ and any such r when estimating the upper bound for G obtained above. This yields

$$G(f, f)(v) \le C(d, s, k)c_0\|f\|_{\infty,k}(\|f\|_{1,s} + \|f\|_{\infty,r})(1 + |v|)^{-k},$$

whenever $s + \min(1, (r - s)/d) \ge 1$. This holds for some $r < s$ whenever $s > 1$. The proof is then again completed by Gronwall's lemma.

Corollary. *Suppose* $f_0 \in L_{\infty,k}$ *with* $k > d + 1$. *Then* $f_t \in L_{\infty,k}$ *for all* t. *In particular, if* $k > d^2/(d-1)$, *then for an arbitrary* $\epsilon > 0$ *such that* $k - d - \epsilon > d/(d-1)$

$$\|f_t\|_{\infty,k} \le \|f_0\|_{\infty,k} \exp\{C(k, \epsilon, d)c_0 t \sup_{\tau \le t}(\|f_\tau\|_{\infty,0} + \|f_\tau\|_{1,k-d-\epsilon})\},$$

and consequently $\|f_t\|_{\infty,k} \le \|f_0\|_{\infty,k}(1+O(t\|f_0\|_{\infty,k}))$ *uniformly for* $t \le T$, $\|f_0\|_{\infty,k} \le A$ *with arbitrary* $T > 0$, $A > 0$.

Proof. This follows from Theorem 2.2 and a simple observation that $f \in L_{\infty,k}$ implies $f \in L_{1,s}$ for any $s < k - d$.

The next two theorems are devoted to the smooth dependence of the solutions on the initial values.

Theorem 2.3. *Let $f_0 = f_0(x,.)$ be a bounded family of non-negative functions from $L_{1,0}$ that depends smoothly on a real parameter $x \in \mathbf{R}$ in the sense that*

(i) $\mathbf{M}_0 = \sup_s \|f_0(x,.)\|_{1,0} < \infty$,

(ii) $\sup_x \|f_0(x,.)\|_{\infty,0} < \infty$,

(iii) for each $v \in \mathbf{R}^d$ the derivative $\nabla_x f_0(x,v)$ exists for almost all x,

(iv) both $\sup_{x,v} |\nabla_x f_0(x,v)|$ and $\sup_x \|\nabla_x f_0(x,.)\|_{1,0}$ are bounded.

Then for all $t > 0$ and $v \in \mathbf{R}^d$, the derivative $\nabla_x f_t(x,v)$ exists for almost all x and

$$\sup_{x,v} |\nabla_x f_t(x,v)| \le \exp\{2c_0 \mathbf{M}_0 t\} \sup_{x,v} |\nabla_x f_0(x,v)|$$
$$+C(d)c_0 t \exp\{c_0 C(d)\mathbf{M}_0 t\} \sup_x \|f_0(x,.)\|_{\infty,0} \sup_x \|\nabla_x f_0(x,.)\|_{1,0},$$
$$\sup_x \|\nabla_x f_t(x,.)\|_{1,0} \le \exp\{2c_0 A(d-2)\mathbf{M}_0 t\} \sup_x \|\nabla_x f_0(x,.)\|_{1,0}.$$

Proof. From the Carleman-Gustafsson representation we find that for any two non-negative f and g

$$|G(f,f) - G(g,g)| \le 2c_0[\max(\|f\|_{\infty,0}, \|g\|_{\infty,0})\|f-g\|_{1,0} + \max(M(f), M(g))\|f-g\|_{\infty,0}].$$

Suppose now that f_t and g_t are two solutions to the Boltzmann equation. As

$$\dot f - \dot g + (f-g)Lf + g(Lf - Lg) = G(f,f) - G(g,g),$$

the above inequality and Gronwall's lemma imply

$$\|f_t - g_t\|_{\infty,0} \le \exp\{2c_0 \max(M(f_0), M(g_0))t\}\|f_0 - g_0\|_{\infty,0}$$
$$+C(d)c_0 t \exp\{2c_0 \max(M(f_0), M(g_0))t\} \max(\sup_{s\le t} \|f_t\|_{\infty,0}, \sup_{s\le t} \|g_t\|_{\infty,0}) \sup_{s\le t} \|f_s - g_s\|_{1,0}.$$

Using this inequality, Theorem 2.1 and (2.3) yield

$$\|f_t(x_1,.) - f_t(x_2,.)\|_{\infty,0} \le \exp\{2c_0\mathbf{M}_0 t\}\|f_0(x_1,.) - f_0(x_2,.)\|_{\infty,0}$$
$$+C(d)c_0 t \exp\{c_0 C(d)\mathbf{M}_0 t\} \sup_x \|f_0(x,.)\|_{\infty,0}\|f_0(x_1,.) - f_0(x_2,.)\|_{1,0}.$$

Dividing by $|x_1 - x_2|$ we get that

$$\sup_{x_1,x_2,v} \frac{|f_t(x_1,v) - f_t(x_2,v)|}{|x_1 - x_2|} \le \exp\{2c_0\mathbf{M}_0 t\} \sup_{x,v} |\nabla_x f_0(x,v)|$$
$$+C(d)c_0 t \exp\{c_0 C(d)\mathbf{M}_0 t\} \sup_x \|f_0(x,.)\|_{\infty,0} \sup_x \|\nabla_x f_0(x,.)\|_{1,0}.$$

Hence $\nabla_x f(x,v)$ exists almost everywhere and has the required uniform bound. From the integral bound and Lebesgue dominated convergence one obtains the required bound for $\|\nabla_x f(x,.)\|_{1,0}$.

Theorem 2.4. *Let $f_0 = f_0(x,.)$ be a bounded family of functions from $L_{1,0}$ depending on a parameter $x \in \mathbf{R}^n$ in such a way that all partial derivatives of f_0 with respect to x up to order k exist (almost everywhere for all v) and have uniformly (with respect to x) bounded norms in $L_{\infty,0}$ and $L_{1,0}$. Then the same holds for all f_t with uniform bounds as $t \in [0,T]$ for arbitrary T. In particular,*

$$\|\nabla_{x_i}\nabla_{x_j} f_t(x,.)\|_{1,0} \le \exp\{c_0 C(d)\mathbf{M}_0 t\}\|\nabla_{x_i}\nabla_{x_j} f_0(x,.)\|_{1,0} + C_1,$$
$$\|\nabla_{x_i}\nabla_{x_j} f_t(x,.)\|_{\infty,0} \le \exp\{c_0 C(d)\mathbf{M}_0 t\}\|\nabla_{x_i}\nabla_{x_j} f_0(x,.)\|_{\infty,0} + C_2,$$

where C_1, C_2 depend on the bounds of f_0 and its first order derivatives with respect to x.

Proof. This is obtained by induction using the same arguments as in the proof of Theorem 1. First one obtains integral bounds for differences, then point-wise bounds for the corresponding derivatives, and then integral bounds for these derivatives.

We turn now to the question of the propagation of smoothness. For an arbitrary $\omega \in [0,1]$ and a function f on $\mathbf{R}^d$, let us define the Hölder modulus of continuity as

$$\Omega^\omega(f) = \limsup_{\delta\to 0} \Omega_\delta(f)\delta^{-\omega}, \quad \Omega_\delta(f) = \sup_{v,w:|v-w|=\delta} |f(v) - f(w)|.$$

Let us say that f belongs to the Hölder class H^ω, if $\Omega^\omega(f)$ is finite.

Theorem 2.5. *Let $f_0 \in L_{1,0} \cap L_{\infty,0}$, and let f_t be the unique mass preserving solution of the corresponding Boltzmann equation. Let S be differentiable with respect to the first variable with $S_1 < \infty$ being the uniform upper bound for the magnitude of this derivative.*

(i) If $f_0 \in H^\omega$ for some $\omega \in [0,1]$, then the same holds for all f_t. Moreover,

$$\Omega^\omega(f_t) \leq \exp\{c_0 C(d) M(f_0) t\} \Omega^\omega(f_0)$$

for $\omega < 1$ and

$$\Omega^1(f_t) \leq \exp\{c_0 C(d) M(f_0) t\}[\Omega^1(f_0) + tC(d)M(f_0) S_1 \sup_{\tau \leq t} \|f_\tau\|_{\infty,0}].$$

(ii) If f_0 is uniformly continuous, then the same holds for all f_t, and f_t solves the corresponding Boltzmann equation not only in L_1-sense, but also in $C(\mathbf{R}^d)$-sense.

(iii) If $\nabla f_0(v)$ exists for almost all v and $\|\nabla f_0\|_{\infty,0} = \Omega^1(f_0) < \infty$, then f_t are almost everywhere differentiable for all t, and $\|\nabla f_t\|_{\infty,0} = \Omega^1(f_t) < \infty$ has the bound from (i).

Proof. (i) Let v_1, v_2 be two arbitrary points with $|v_1 - v_2| = \delta$. From the Gustafsson-Carleman representation

$$G(f,f)(v_1) - G(f,f)(v_2) = 2\int_{\mathbf{R}^d} f(v')|v_1 - v'|^{-(d-1)} dv' \int_{E_{v_1,v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{v_1,v'}(w')$$
$$- 2\int_{\mathbf{R}^d} f(v')|v_2 - v'|^{-(d-1)} dv' \int_{E_{v_2,v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{v_2,v'}(w')$$

(recall that $B(|v|,\theta) = 2^{d-1} S(|v|,\theta)\cos^{d-2}\theta$). Let us represent this expression as the sum $I_1 + I_2$, where I_1 (respectively I_2) stands for the integration over the domain $D = \{v' : |v_1 - v'| \geq |v_2 - v'|\}$ (respectively the complement of D). We shall now estimate I_1. For $v' \in D$ let

$$\tilde{v} = \tilde{v}(v_1, v_2, v') = v' + \frac{|v_2 - v'|}{|v_1 - v'|}(v_1 - v'),$$

i.e. $\tilde{v}$ is placed on the interval connecting v' and v_1 in such a way that $|\tilde{v} - v'| = |v_2 - v'|$. Then we can write

$$I_1 = 2\int_D f(v')|v_1 - v'|^{-(d-1)} dv' \int_{E_{v_1,v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{v_1,v'}(w')$$
$$- 2\int_D f(v')|\tilde{v} - v'|^{-(d-1)} dv' \int_{E_{v_1,v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{\tilde{v},v'}(w') + 2\int_D f(v')|v_2 - v'|^{-(d-1)} dv'$$
$$\times \left(\int_{E_{\tilde{v},v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{\tilde{v},v'}(w') - \int_{E_{v_2,v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{v_2,v'}(w') \right).$$

The balls $E_{\tilde{v},v',\pi/4}$ and $E_{v_2,v',\pi/4}$ have both radius $|v_2 - v'|$ and are connected by the obvious rotation (that does not change the orthogonal compliment to the two-dimensional plane passing through the points v', v_1, v_2) preserving the Lebesgue measure. The largest distance between the corresponding points of $E_{\tilde{v},v',\pi/4}$ and $E_{v_2,v',\pi/4}$ is $\sqrt{2}|\tilde{v} - v_2| \leq 2|v_1 - v_2|$. Hence for any two such points u, w there exists $\tilde{u}$ such that $|u - \tilde{u}| = |w - \tilde{u}| = \delta$ and consequently $|f(u) - f(w)| \leq 2\Omega_\delta(f)$. Therefore

$$\left| \int_{E_{\tilde{v},v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{\tilde{v},v'}(w') - \int_{E_{v_2,v',\pi/4}} B(|v'-w'|,\theta) f(w') dE_{v_2,v'}(w') \right|$$
$$\leq 4c_0 \frac{A(d-2)}{d-1} |v_2 - v'|^{d-1} \Omega_\delta(f),$$

and consequently the last term in the expression for I_1 is bounded by$C(d)c_0M(f)\Omega_\delta(f)$. On the other hand, the first two terms of the expression for I_1 can be written as

$$\int_D f(v')dv' \int_0^1 r^{d-2}dr \int_{S^{d-2}} dn[f(v_1+r|v_1-v'|n)B(|v_1-v'|\sqrt{1+r^2}, \arctan r)$$

$$-f(\tilde{v}+r|\tilde{v}-v'|n)\tilde{S}(|\tilde{v}-v'|\sqrt{1+r^2}, \arctan r)].$$

Again the largest distance between the points $v_1+r|v_1-v'|n$ and $\tilde{v}+r|\tilde{v}-v'|n$ does not exceed $\sqrt{2}\delta$, and it follows that the last expression does not exceed in magnitude

$$\int_D f(v')dv' \int_0^1 r^{d-2}dr \int_{S^{d-2}} dn(2c_0\Omega_\delta(f)+2^d\|f\|_{\infty,0}S_1\delta) \le C(d)M(f)(c_0\Omega_\delta(f)+\|f\|_{\infty,0}S_1\delta).$$

We conclude (taking into account that I_2 can be estimate in the same way as I_1) that

$$\Omega_\delta(G(f,f)) \le C(d)M(f)(c_0\Omega_\delta(f)+\|f\|_{\infty,0}S_1\delta).$$

Now clearly $|Lf(v_1)-Lf(v_2)| \le C(d)S_1M(f)|v_1-v_2|$. Hence $(d/dt)\Omega_\delta(f_t) \le C(d)M(f)(c_0\Omega_\delta(f_t)+\|f_t\|_{\infty,0}S_1\delta)$. (Of course, it would be more correct to write this inequality in the integral form.) Consequently, by Gronwall's lemma

$$\Omega_\delta(f_t) \le \exp\{c_0C(d)M(f_0)t\}\Omega_\delta(f_0)+c_0^{-1}(\exp\{c_0C(d)M(f_0)t\}-1)S_1\sup_{\tau\le t}\|f_\tau\|_{\infty,0}\delta.$$

Dividing by δ^ω and passing to lim sup as $\delta\to 0$ one gets the required estimates for $\Omega^\omega(f_t)$ and concludes that $f_t\in H^\omega$ for all $t>0$.

(ii) Follows from (i) with $\omega=0$ and $\Omega^0(f_0)=0$, and an observation from the proof of Proposition 2.1 that uniform bounds for $\|f_t\|_{\infty,0}$ imply uniform bounds for $\|G(f_t,f_t)\|_{\infty,0}$.

(iii) Follows from (i) with $\omega=1$, and from the well known fact that uniform Lipshitz continuity implies the differentiability almost everywhere.

Theorem 2.6. *Suppose $f_0\in L_{1,0}\cap L_{\infty,0}$, f_0 is differentiable and $\nabla f_0\in L_{\infty,0}$. Let S' (respectively B') denotes the partial derivative of S (respectively B) with respect to the first variable. Let S' exists everywhere and $|S'|$ is uniformly bounded by a number $S_1<\infty$.*

(i) If ∇f_0 is continuous, then the same holds for all ∇f_t, and ∇f_t is a $C(\mathbf{R}^d)$-solution of the differentiated Boltzmann equation that can be written in the form

$$\frac{d}{dt}(\nabla f,m)(v)+(\nabla f(v),m)Lf(v)+f(v)(\nabla Lf(v),m)$$

$$=2\int_{\mathbf{R}^d} f(v')|v-v'|^{-(d-1)}dv' \int_{E_{v_1,v',\pi/4}} B(|v'-w'|,\theta)dE_{v_1,v'}(w')$$

$$\times\left((\nabla f(w'),u(w';v,v',m))\frac{\sin\psi}{|v'-v|}-(\nabla f(w'),w'-v')\frac{\cos\psi}{|v'-v|}\right)$$

$$-2\int_{\mathbf{R}^d} f(v')|v-v'|^{-(d-1)}dv' \int_{E_{v_1,v',\pi/4}} B'(|v'-w'|,\theta)f(w')\frac{\cos\psi}{|v'-v|}|w'-v'|dE_{v_1,v'}(w'),$$

where $\psi=\psi(m,v'-v)\in[0,\pi]$ denotes the angle between $v'-v$ and m, $u(w';v,v',m)$ is the vector lying in the two-dimensional plane $P(v,v',m)$ passing through the points $v,v',v+m$ that has the magnitude of the projection $\tilde{u}$ of $w'-v'$ on this plane, is perpendicular to $\tilde{u}$ and has an acute angle with m.

(ii) If $f_0\in L_{\infty,k}\cap L_{1,s}$ and $\nabla f_0\in L_{\infty,k}$ with $k<\min(d-1,s)$, then $\nabla f_t\in L_{\infty,k}$ for all t and

$$\|\nabla f_t\|_{\infty,k} \le \exp\{C(k,s,d)t\sup_{\tau\le t}(\|f_\tau\|_{\infty,k}+\|f_\tau\|_{1,s})\}$$

$$\times[\|\nabla f_0\|_{\infty,k} + C(k,s,d)S_1 \sup_{\tau\le t}((\|f_\tau\|_{\infty,0} + \|f_\tau\|_{1,s})\|f_\tau\|_{\infty,k})].$$

(iii) If $f_0 \in L_{\infty,k} \cap L_{1,s}$ and $\nabla f_0 \in L_{\infty,k}$ with $\min(k,s) > 2$, then $\nabla f_t \in L_{\infty,k}$ for all t and it solves the differentiated Boltzmann equation in $L_{\infty,k}$-sense.

(iv) If $\nabla f_0 \in L_1$, then the same holds for all ∇f_t, and ∇f_t is a L_1-solution of the differentiated Boltzmann equation that can be written in the form

$$\frac{d}{dt}(\nabla f, m) = Q((\nabla f, m), f) + Q(f, (\nabla f, m)). \tag{2.4}$$

Remark. The last formula offers another approach to the analysis of the upper bounds $\|\nabla f_t\|_{\infty,k}$, which on the one hand does not require any assumptions on the derivatives of S or B, but on the other hand allows to give bounds on $\|\nabla f_t\|_{\infty,r}$ only when the corresponding integral bounds are available. We shall follow this approach in Section 4 when analyzing the growing collision kernels.

Proof. (i) Let $v_1 = v + \delta m$, η be the angle between $v - v'$ and $v_1 - v'$. Let R_η denote the rotation in $\mathbf{R}^d$ that rotates $P(v, v', m)$ around the point v' on the angle η (so that the direction $v - v'$ turns to the direction $v_1 - v'$) and does not move the orthogonal complement of $P(v, v', m)$. At last, let H be a scaling transformation of $\mathbf{R}^d$ defined as

$$H : w \mapsto v' + (w - v')\frac{|v_1 - v'|}{|v - v'|}.$$

From the calculations of the increments of $G(f, f)$ in the proof of Theorem 2.5 it follows that

$$G(f,f)(v_1) - G(f,f)(v) = \int f(v')\, dv' \int_0^1 r^{d-2}\, dr \int_{S^{d-1}} dn$$

$$\times[f(\tilde{w}')\tilde{S}(|w' - v'|\frac{|v_1 - v'|}{|v - v'|}, \arctan r) - f(w')\tilde{S}(|w' - v'|, \arctan r)],$$

where $w' = v + r|v - v'|n$, $\tilde{w}' = R_\eta H w' = H R_\eta w'$, and where the unit sphere S^{d-2} lies in the hyperplane passing through v perpendicular to $v - v'$.

By the cosine-rule

$$|v' - v_1| = \sqrt{|v' - v|^2 + \delta^2 - 2\delta|v' - v|\cos\psi} = |v - v'|(1 - \delta\frac{\cos\psi}{|v - v'|} + O(\delta^2)),$$

and by the sine-rule

$$\sin\eta = \delta\sin\psi/|v - v'|.$$

Hence

$$\tilde{S}(|w' - v'|\frac{|v_1 - v'|}{|v - v'|}, \arctan r) - \tilde{S}(|w' - v'|, \arctan r)$$

$$= -\delta\cos\psi\frac{|w' - v'|}{|v' - v|}S'(|w' - v'|, \arctan r) + o(\delta),$$

$$\tilde{w}' - w' = -\delta(w' - v')\frac{\cos\psi}{|v' - v|} + \delta u(w'; v, v', m)\frac{\sin\psi}{|v - v'|} + O(\delta^2).$$

These formulas, the (proved above) uniform boundedness of ∇f_t and the dominated convergence theorem imply that $\nabla G(f_t, f_t)(v), m)$ equals the r.h.s. of the differentiated Boltzmann equation as required.

Also the calculations of the previous theorem show that this non-homogeneous linear (with respect to ∇f) equation depends continuously on ∇f in $L_{\infty,0}$ and hence has the unique solution. As the space of continuous function $C(\mathbf{R}^d)$ is closed in $L_{\infty,0}$, the solution belongs to this space whenever the initial condition does.

(ii) One can either use the differentiated Boltzmann equation written above or work with increments. Let us choose the latter approach. Let

$$\Omega_\delta(f;v) = \sup_{w:|w-v|=\delta} |f(w) - f(v)|.$$

By Theorem 2.2, $f_t \in L_{\infty,k}$ for all t, and hence the function $\Omega_\delta(f_t;.)$ also belongs to $L_{\infty,k}$. In particular,

$$\Omega_\delta(f_t;w) \le 2^k \delta(1+|v|^k)^{-1}\|\Omega_\delta(f_t;.)\|$$

whenever $|v-w| \le \delta$. Consequently, following the arguments of the proof of the previous theorem one finds that for arbitrary v

$$\Omega_\delta(G(f,f);v) \le C(d)\int f(v')|v-v'|^{-(d-1)}dv' \int_{E_{v,v',\pi/4}} \Omega_\delta(f;w')dw'$$

$$+C(d)S_1\delta \int f(v')|v-v'|^{-(d-1)}dv' \int_{E_{v,v',\pi/4}} f(w')dw'.$$

Estimating the r.h.s. by the same method as in the proof of Theorem 2.2 (where one chooses $r=\lambda=k$ when applying Proposition A2 and takes into account that $s+k(k-s)/d \ge k$ whenever $\min(s,d) \ge k$) yields

$$\Omega_\delta(G(f,f);v) \le C(k,s,d)\|\Omega_\delta(f;.)\|_{\infty,k}(\|f\|_{1,s}+\|f\|_{\infty,k})(1+|v|^k)^{-1}$$

$$+C(k,s,d)\delta S_1\|f\|_{\infty,k}(\|f\|_{1,s}+\|f\|_{\infty,0})(1+|v|^k)^{-1}.$$

Consequently, Gronwall's lemma implies

$$\|\Omega_\delta(f_t;.)\|_{\infty,k} \le \exp\{C(k,s,d)t\sup_{\tau\le t}(\|f_\tau\|_{\infty,k}+\|f_\tau\|_{1,s})\}$$

$$\times[\|\Omega_\delta(f_0;.)\|_{\infty,k} + \delta C(k,s,d)S_1 \sup_{\tau\le t}((\|f_\tau\|_{\infty,0}+\|f_\tau\|_{1,s})\|f_\tau\|_{\infty,k})].$$

Dividing by δ and passing to the limit yields the required estimate for ∇f_t.

(iii) We shall consider now the differentiated Boltzmann equation on ∇f (with given f_t) as a linear equation in $L_{\infty,k}$. To prove the statement, it is clearly enough to show that the r.h.s. of this equation depends continuously on ∇f in the norm of $L_{\infty,k}$. To see this, we shall rewrite the r.h.s. $\nabla G(f,f)(v),m)$ of this equation using the modified Carleman representation. Using this representation, following the same arguments as when deducing the equation in (i), and taking into account the condition $k>2$ together with the dominated convergence theorem to justify the limiting procedure under the integration one finds that

$$|(\nabla G(f,f)(v),m)| \le C(d)|G(f,f)(v)|$$

$$+C(d)\int_{\mathbf{R}^d} f(v')|v-v'|^{-(d-1)}dv' \int_{E_{v,v'}} S_1 f(w')\frac{|w'-v'|}{|v'-v|}\cos^{d-2}\theta \, dE_{v,v'}(w')$$

$$+c_0C(d)\int_{\mathbf{R}^d} f(v')|v-v'|^{-(d-1)}dv' \int_{E_{v,v'}} \cos^{d-2}\theta \max_i |(\nabla_i f)^{v,ext}(w')|\frac{|w'-v'|}{|v'-v|}\, dE_{v,v'}(w').$$

Dealing with the last term as in the proof of Theorem 2.2 one estimates it by

$$c_0C(d)\int f(v')|v-v'|^{-2}\, dv' \|\nabla f\|_{\infty,k}(1+|v|)^{2-k}.$$

To estimate this by

$$c_0 C(d,s,k)\|\nabla f\|_{\infty,k}(\|f\|_{1,s}+\|f\|_{\infty,k})(1+|v|)^{-k}$$

using Proposition A2, one needs $s+2(k-s)/d \geq 2$, which holds for $\min(s,k)>2$ (and of course, $2<d$). The obtained estimate shows that $\nabla G(f,f)$ depends continuously on ∇f in $L_{\infty,k}$-norm, which implies the existence of the unique $L_{\infty,k}$-bounded solution of the differentiated Boltzmann equation.

(iv) Let $v_1 = v+m$ with some unit vector m. Writing

$$Q(f,g)(v_1)-Q(f,g)(v) = \frac{1}{2}\int_{\mathbf{R}^d} dw_1 \int_{S^{d-1}} dn\, B(|v_1-w_1|,\theta_1)[f(v_1')g(w_1')-f(v_1)g(w_1)]$$
$$-\frac{1}{2}\int_{\mathbf{R}^d} dw \int_{S^{d-1}} dn\, B(|v-w|,\theta)[f(v')g(w')-f(v)g(w)],$$

where θ (respectively θ_1) is the angle between n and $v'-v$ (respectively $v_1'-v_1$), and changing the variable of integration in the first term by $w_1=w+m$ yields

$$Q(f,g)(v+m)-Q(f,g)(v) = \frac{1}{2}\int_{\mathbf{R}^d} dw \int_{S^{d-1}} dn\, B(|v-w|,\theta)$$
$$\times[(f(v'+m)g(w'+m)-f(v+m)g(w+m))-(f(v')g(w')-f(v)g(w))],$$

which implies the required form of the differentiated Boltzmann equation. Clearly, this equation is a non-homogeneous linear equation with respect to ∇f that depends continuously on ∇f in $L_{1,0}$. Hence it has the unique solution in this space for any initial $\nabla f_0 \in L_{1,0}$. As this solution can be obtained by the same approximation procedure as when working in $L_{\infty,0}$, the solutions in L_1 and $L_{\infty,0}$ coincide.

By induction, one can get the following result on higher derivatives.

Theorem 2.7. *Suppose $S(|v|,\theta)$ is infinitely smooth with bounded derivatives with respect to the first variable, and $f_0 \in L_1 \cap L_{\infty,0}$. If f_0 has uniformly bounded derivatives up to order l with arbitrary l, the same holds for all f_t. Moreover, if all these derivatives belong to $L_{\infty,k}$ and $f_0 \in L_{1,s}\cap L_{\infty,k}$ with either $k<\min(s,d-1)$ or $\min(k,s,d)>l$, the same holds for the derivatives of all f_t.*

3. $L_{\infty,r}$-BOUNDS OF THE SOLUTIONS IN CASE $\beta > 0$

Generalizing the form of the collision kernels of the cutoff hard potentials (where $\beta\in(0,1)$), we shall assume from now on that

$$S(|v|,\theta) = |v|^\beta h(\theta) \tag{3.1}$$

with $\beta>0$ and h being a non-negative measurable function such that

$$\forall\ \theta\ 2^{d-1}h(\theta)\leq c_0<\infty, \quad 2\int_0^{\pi/2}\sin^{d-2}(2\theta)h(\theta)\,d\theta = c_1>0. \tag{3.2}$$

For large β we shall occasionally assume the following additional condition (though this can be essentially relaxed):

$$\forall\ \theta\ 2^{d-1}\max(1,\cos^{d-2-\beta}\theta)h(\theta)\leq c_0<\infty, \quad \text{if} \quad \beta>d-2. \tag{3.3}$$

Under conditions (1.16), (3.1), the weak representation (1.12) of the collision operator can be written as

$$\int \psi(v)Q(f,f)(v)\,dv$$
$$= \int_{\mathbf{R}^{2d}}|v-w|^\beta dvdw \int_0^{\frac{\pi}{2}}\sin^{d-2}(2\theta)h(\theta)\,d\theta\int_{S^{d-2}}(\psi(v')+\psi(w')-\psi(v)-\psi(w))f(v)f(w)\,dm. \tag{3.4}$$

Following the long development of the theory, it was shown in [MW] that for arbitrary $\beta>0$ and a non-negative $f_0\in L_{1,2}$ there exists a solution f_t of the Boltzmann equation that preserves the mass and the energy, i.e. $M(f_0)=M(f_t)$ and $E(f_0)=E(f_t)$ for all t, and moreover, $\|f_t\|_{1,s}$ is bounded uniformly in times whenever $f_0\in L_{1,s}$, where $s>0$ is arbitrary. We shall always denote by f_t such a solution. It is also shown in [MW] that such solution is unique if $\beta\leq 2$. We shall give a uniqueness result for arbitrary β in the next section.

The aim of this Section is to prove the following result.

Theorem 3.1. *Assume (1.16), (1.19), (3.1), (3.2) hold and* $\beta > 0$. *Let* $s \geq 2$, $r \geq 0$, $f_0 \in L_{1,s} \cap L_{\infty,r}$ *and is non-negative. Let*

$$F = F(s, f_0) = \frac{c_0}{c_1 \sigma} \sup_{t \geq 0} \|f_t\|_{1,s} \frac{\|f_0\|_{1,2}}{M_0^2}, \tag{3.5}$$

where $\sigma = \sigma(\beta, d, h_0, M_0, E_0)$ *is given by (A4) with* $h = h_0 = \int f_0(v) \ln f_0(v)\, dv$. *If* $\beta < d - 1$, *let* $m = 1, 2, \ldots$ *be chosen in such a way that* $(m-1)\beta < d - 1 - \beta \leq m\beta$, *and if* $\beta < 1$, *let* $n = 1, 2, \ldots$ *be chosen in such a way that* $(n-1)\beta < 1 - \beta \leq n\beta$.

(i) Suppose $r \leq s$ *and* $\beta \leq s$. *Then* $\|f_t\|_{\infty,r}$ *are uniformly bounded for all times. Moreover, if* $1 \leq \beta < d - 1$, *then*

$$\sup_{t \geq 0} \|f_t\|_{\infty,r} \leq C(d, s, r, \beta) \frac{c_0 \|f_0\|_{1,2}}{c_1 \sigma M_0^2} [\sup_{\tau \geq 0} \|f_\tau\|_{1,\beta-1} + \|f_0\|_{\infty,r}](F^{m+1} M_0 + F^m \|f_0\|_{\infty,r}).$$

If $\beta < 1$ $(\beta \geq d-1)$, *then the same estimate holds with* $(F^n M_0 + F^{n-1}\|f_0\|_{\infty,r})$ *instead of the value* $[\sup_{\tau \geq 0} \|f_\tau\|_{1,\beta-1} + \|f_0\|_{\infty,r}]$ *(with* $F[\sup_{\tau \geq 0} \|f_\tau\|_{1,\beta-(d-1)} + \|f_0\|_{\infty,r}]$ *instead of the value* $(F^{m+1} M_0 + F^m \|f_0\|_{\infty,r}$, *respectively).*

(ii) Suppose $\max(d-1, s) < r \leq d+s$, $\beta \leq s+1$, *and (3.3) holds. Then again* $\|f_t\|_{\infty,r}$ *are uniformly bounded for all times and the same estimates as in (i) hold.*

(iii) Suppose $r > s$, $r > d - 1 + 2\max(1, 2^{\beta-1}) c_0/c_1$, $\beta \leq s+1$, *and (3.3) holds. Then*

$$\sup_{t \geq 0} \|f_t\|_{\infty,r} \leq C(d, s, r, \beta, c_0, c_1) \rho^{r-s} \frac{\|f_0\|_{1,2}}{\sigma M_0^2} \sup_{\tau} \|f_\tau\|_{1,\beta-1} (F^m M_0 + F^{m-1} \|f_0\|_{\infty,r})$$

for $1 \leq \beta < d-1$, *where* ρ *is the maximum of the r.h.s. of (3.24), (3.25), and (3.29) below. The same holds with* $(F^n M_0 + F^{n-1}\|f_0\|_{\infty,r})$ *instead of* $\sup_\tau \|f_\tau\|_{1,\beta-1}$, *and with* $\sup_\tau \|f_\tau\|_{1,\beta-(d-1)}$ *instead of* $(F^m M_0 + F^{m-1}\|f_0\|_{\infty,r})$, *respectively for* $\beta \in (0,1)$ *and* $\beta \geq d-1$.

Remarks.

1. The results of this theorem are essentially known for $\beta \in (0, 1]$ and $d = 3$ and are dealt with in several papers, see the statement (iii) (and the basic method) for $\beta = 1$ in the seminal book [Ca], further developments in [MTc], [Gl], [Ar2], and the most general exposition in [Gu1]. However, the explicit bounds were seemingly never given even for this case. Our proof is obtained by simplifying and extending the estimates from [Gu1] in several directions. It is also adapted to further extensions in the next section.
2. All formulae are simplified for $s = 2$ as $F(2, f_0) = c_0 \|f_0\|_{1,2}^2 / (\sigma c_1 M_0^2)$.

Before proving the theorem, we obtain several lemmas that give upper bounds for the convolutions $\star$ of the solutions with the power functions $P_\lambda(v) = |v|^\lambda$, for the Radon transform of these solutions, and for the gain term. We assume everywhere that (1.19), (3.1), (3.2) hold with some $\beta > 0$, and that the initial function f_0 is non-negative. To shorten the formulas we shall write M_0 and E_0 for $M(f_0)$ and $E(f_0)$ respectively.

Lemma 3.1. *Let* $0 \leq \gamma < d - 1$, $s \geq 2$, $r \in [0, s+d]$, $a \geq 1/2$. *Let* $n = 0, 1, 2, \ldots$ *be such that* $(n-1)\beta < \gamma \leq n\beta$, *and suppose* $n\beta - \gamma \leq s$. *Let* $f_0 \in L_{1,s} \cap L_{\infty,r}$. *Then*

$$\sup_{t \geq 0} \|f_t \star P_{-\gamma}\|_{\infty,0} \leq C(d, \gamma, \beta) \left[\left(\frac{c_0}{c_1 \sigma}\right)^n \sup_{\tau \geq 0} \|f_\tau\|_{1,n\beta-\gamma} + \chi_1(n) \left(\frac{c_0}{c_1 \sigma}\right)^{n-1} \|f_0\|_{\infty,0} \right]. \tag{3.6}$$

Moreover, for all t

$$(f_t \star P_{-\gamma})(v) \leq C(d, \gamma, \beta, s, r)(1 + |v|)^{-\min(\gamma, s+\gamma(r-s)/d)} [F^n M_0 + \chi_1(n) F^{n-1} \|f_0\|_{\infty,r}], \tag{3.7}$$

$$\int \chi_{a|v|}(|z|) f_t(z)|v-z|^{-\gamma}\, dz \le \frac{C(d,\gamma,\beta,s,r)}{(1+|v|)^{s+\gamma(r-s)/d}}[F^n M_0 + \chi_1(n) F^{n-1}\|f_0\|_{\infty,r}], \tag{3.8}$$

where $\sigma = \sigma(\beta, d, h_0, M_0, E_0)$ and $F = F(s, f_0)$ are the same as in Theorem 3.1.

Proof. First observe that $f_0 \in L_{s,1} \cap L_{\infty,r}$ with any $s > 0$, $r \ge 0$ implies that $f_0 \ln f_0 \in L_1(\mathbf{R}^d)$, and consequently h_0 is well defined and $\int f_t(v) \ln f_t(v)\, dv \le \int f_0(v) \ln f_0(v)\, dv = h_0$ for all t. Next, if $\gamma = 0$, (3.6) and (3.7) follow from the conservation of mass.

Suppose now $0 < \gamma \le \beta$. For an arbitrary $j > 0$, multiplying the Boltzmann equation by $\chi_1(j|v-z|)|v-z|^{-\gamma}$ and integrating yields

$$\frac{d}{dt}\int f_t(z)\chi_1(j|v-z|)|v-z|^{-\gamma}\, dz + \int f_t(z) L f_t(z) \chi_1(j|v-z|)|v-z|^{-\gamma}\, dz$$

$$= \int G(f_t, f_t)(z)\chi_1(j|v-z|)|v-z|^{-\gamma}\, dz \le C(d,\gamma) c_0 M_0 \|f_t\|_{1,\beta-\gamma},$$

where for the last inequality we used Proposition A4 (i) and the conservation of mass. As the entropy is not increasing it follows from Proposition A2 and its Corollary, and also from (1.10), (1.16), (3.1), (3.2), (3.4) that

$$L f_t(z) \ge A(d-2) c_1 \sigma M(f_t) = A(d-2) c_1 \sigma M_0 \tag{3.9}$$

for all t with $A(d)$ being as always the area of d-dimensional unit sphere. Consequently from Proposition A2 with $r = s = 0$ it follows that

$$\int f_t(z)\chi_1(j|v-z|)|v-z|^{-\gamma}\, dz \le \max\left(\frac{C(d,\gamma)c_0}{c_1\sigma} \sup_\tau \|f_\tau\|_{1,\beta-\gamma}, \int f_0(z)|v-z|^{-\gamma}\, dz\right)$$

$$\le C(d,\gamma)\left(\frac{C(d,\gamma)c_0}{c_1\sigma}\sup_\tau \|f_\tau\|_{1,\beta-\gamma} + \|f_0\|_{\infty,0}\right),$$

where we used the fact that both $1/\sigma$ and c_0/c_1 are bounded from below. Consequently, passing to the limit $j \to \infty$ proves (3.6) in this case.

For $\beta < \gamma \le 2\beta$ in the same way

$$\frac{d}{dt}\int f_t(z)\chi_1(j|v-z|)|v-z|^{-\gamma}\, dz + \int f_t(z) L f_t(z) \chi_1(j|v-z|)|v-z|^{-\gamma}\, dz$$

$$\le C(d,\gamma) c_0 M_0 \|f_t \star P_{-(\gamma-\beta)}\|_{\infty,0},$$

which is already shown to be bounded by (3.6) with $\gamma - \beta$ instead of β and with $n = 1$. Hence again by (3.9) and Proposition A3 (and because $1/\sigma$ and c_0/c_1 are bounded from below) one proves (3.6) for $\beta < \gamma \le 2\beta$. Simple induction argument yields the proof of (3.6) for all $\gamma < d-1$.

Next, as clearly

$$\int_{\{|z| \le |v|/\sqrt{2}\}} f_t(z)|v-z|^{-\gamma}\, dz \le 2^\gamma M(f_t)|v|^{-\gamma} = 2^\gamma M_0 |v|^{-\gamma},$$

it follows from (3.6) that the integral on the left-hand side does not exceed the value

$$C(d,\gamma,\beta)\left[\left(\frac{c_0}{c_1\sigma}\right)^n \sup_\tau \|f_\tau\|_{1,n\beta-\gamma} + \chi_1(n)\left(\frac{c_0}{c_1\sigma}\right)^{n-1}\|f_0\|_{\infty,0}\right](1+|v|)^{-\gamma}$$

for any t. Consequently, (3.7) follows from (3.8), because $n\beta - \gamma \le s$ and F is bounded from below (up to a constant) by $c_0 E_0 / c_1 \sigma$.

To prove (3.8) first observe that it is evident for $\gamma = 0$. Suppose $0 < \gamma \le \beta$. As in the proof of (3.6) above, multiplying the Boltzmann equation by $\chi_{a|v|}(z)|v-z|^{-\gamma}$, integrating and using Proposition A4 (i) yields

$$\frac{d}{dt}\int f_t(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz + \int f_t(z)Lf_t(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz$$

$$= \int G(f_t,f_t)(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz \le C(d,\gamma)c_0\|f_t\|_{1,s}M_0(1+|v|)^{-(s+\gamma-\beta)}.$$

Notice that here we did not need to multiply by $\chi_1(j|v-z|)$ any more, because the existence of all integrals involved is already known from (3.6). As by Proposition A1

$$Lf_t(z)\chi_{a|v|}(|z|) \ge C(\beta)c_1A(d-1)\sigma\frac{M_0^2}{\|f_0\|_{1,2}}\chi_{a|v|}(|z|)(1+|v|^\beta), \tag{3.10}$$

we conclude again by Proposition A3 that

$$\int f_t(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz \le \max\left(C(d,\beta,\gamma)FM_0(1+|v|)^{-(s+\gamma)}, \int f_0(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz\right).$$

This implies (3.8), because $\|f_0\|_{1,s} \le FM_0$ (up to a constant) and by Proposition A2

$$\int f_0(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz \le C(d,\gamma,s,r)(\|f_0\|_{1,s}+\|f_0\|_{\infty,r})(1+|v|)^{-(s+\gamma(r-s)/d)}.$$

Suppose $\beta < \gamma \le 2\beta$. As above, but using the statement (ii) of Proposition A4 instead of (i), one gets

$$\frac{d}{dt}\int f_t(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz + \int f_t(z)Lf_t(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz$$

$$\le C(d,\gamma)c_0(1+|v|)^{-s}\sup_\tau\|f_\tau\|_{1,s}\sup_{|z|\ge a|v|/\sqrt{2}}(f_t\star P_{-(\gamma-\beta)})(z),$$

which by the previous step, i.e. by (3.7) with $\gamma-\beta \le \beta$ instead of γ, is estimated from above by

$$C(d,\gamma,s,r)(FM_0+\|f_0\|_{\infty,r})c_0\sup_\tau\|f_\tau\|_{1,s}(1+|v|)^{-s-\min(\gamma-\beta,\,s+(\gamma-\beta)(r-s)/d)}.$$

Hence, again by Propositions A2 and A3, the integral $\int f_t(z)\chi_{a|v|}(|z|)|v-z|^{-\gamma}\,dz$ does not exceed (up to a constant $C(d,\gamma,s,r)$) the maximum of

$$(FM_0+\|f_0\|_{\infty,r})F(1+|v|)^{-s-\min(\gamma,\,s+\beta+(\gamma-\beta)(r-s)/d)}$$

and

$$(\|f_0\|_{1,s}+\|f_0\|_{\infty,r})(1+|v|)^{-(s+\gamma(r-s)/d)},$$

which implies (3.8), because (as we assumed $r \le s+d$)

$$\gamma(r-s)/d \le \min(\gamma, s+\beta+(\gamma-\beta)(r-s)/d)$$

and because $FM_0 \ge \|f_0\|_{1,s}$ (up to a constant).

At last, a simple induction completes the proof of (3.8) for all $\gamma < d-1$.

Lemma 3.2. *Under the assumptions of Lemma 3.1 suppose additionally that $\beta \leq s+1$, and if $\beta \in (0,1)$ let $n = 1, 2, ...$ be such that $(n-1)\beta < (1-\beta) \leq n\beta$. Then for arbitrary $\lambda \in [0, \pi/2]$, $z \in \mathbf{R}^d$ and almost all directions $(\bar{z}-z)/|\bar{z}-z|$ one has:*

(i) if $\beta \geq 1$, then

$$\int_{E_{z,\bar{z},\lambda}} f_t(u)\, dE_{z,\bar{z}}(u) \leq \max\left(C(d,\gamma) \frac{c_0}{c_1\sigma} \sup_\tau \|f_\tau\|_{1,\beta-1}, \int_{E_{z,\bar{z},\lambda}} f_0(u)\, dE_{z,\bar{z}}(u) \right), \tag{3.11}$$

and if $\beta \in (0,1)$ the same estimate holds but with

$$\left[\left(\frac{c_0}{c_1\sigma} \right)^n \sup_{\tau \geq 0} \|f_\tau\|_{1, n\beta-(1-\beta)} + \chi_1(n) \left(\frac{c_0}{c_1\sigma} \right)^{n-1} \|f_0\|_{\infty,0} \right]$$

instead of $\sup_{\tau\geq 0} \|f_\tau\|_{1,\beta-1}$;

(ii) if $\beta \geq 1$, then

$$\int_{E_{z,\bar{z},\lambda}} \chi_{a|v|}(|u|) f_t(u)\, dE_{z,\bar{z}}(u)$$

$$\leq \max\left(C(d,s,\beta) F \sup_{\tau\geq 0} \|f_\tau\|_{1,\beta-1} (1+|v|)^{-(s+1)}, \int_{E_{z,\bar{z},\lambda}} \chi_{a|v|}(|u|) f_0(u)\, dE_{z,\bar{z}}(u) \right), \tag{3.12}$$

and if $\beta \in (0,1)$ the same estimate holds but with $[F^n M_0 + \chi_1(n) F^{n-1} \|f_0\|_{\infty,r}]$ instead of the value $\sup_{\tau\geq 0} \|f_\tau\|_{1,\beta-1}$.

Proof. The notation $E_{z,w,\lambda}$ and $E_{z,w}$ is explained in Section 1 after formula (1.20). Let $r(u) = r_{z,\bar{z}}(u)$ be the distance from u to $Ez, \bar{z}$ and let

$$\phi_j(u) = \phi_j(u; z, \bar{z}) = (j\pi^{-1})^{d/2} \exp\{-jr(u)^2\}.$$

Next, let $\phi_{j,\lambda}(u)$ equals $\phi_j(u)$ for u from the cylinder

$$\{v + \sigma(\bar{z}-z) : v \in E_{z,\bar{z},\lambda}, \sigma \in \mathbf{R}\}$$

and vanishes otherwise. Then for an integrable function g and almost all directions $(\bar{z}-z)/|\bar{z}-z|$

$$\int_{E_{z,\bar{z},\lambda}} \chi_{a|v|}(|u|) g(u)\, dE_{z,\bar{z}}(u) = \lim_{j\to\infty} \int_{\mathbf{R}^d} \phi_{j,\lambda}(u) \chi_{a|v|}(u) g(u)\, du. \tag{3.13}$$

Multiplying the Boltzmann equation by $\phi_{j,\lambda}\chi_{a|v|}$ and integrating yields

$$\frac{d}{dt} \int_{\mathbf{R}^d} \phi_{j,\lambda}(u) \chi_{a|v|}(|u|) f_t(u)\, du + \int_{\mathbf{R}^d} \phi_{j,\lambda}(u) \chi_{a|v|}(|u|) L f_t(u) f_t(u)\, du$$

$$\leq \int_{\mathbf{R}^d} \phi_{j,\lambda}(u) \chi_{a|v|}(|u|) G(f_t, f_t)(u)\, du.$$

By the last inequality of Proposition A5, the r.h.s. can be estimated by

$$C(d,s) c_0 \|f_t\|_{1,s} (1+|v|)^{-s} \sup_{\{|u|\geq a|v|/\sqrt{2}\}} (f_t \star P_{-(1-\beta)})(u). \tag{3.14}$$

If $\beta \geq 1$, this is bounded by

$$C(d,s) c_0 \sup_\tau \|f_\tau\|_{1,\beta-1} \sup_\tau \|f_\tau\|_{1,s} (1+|v|)^{-s-(1-\beta)},$$

and by (3.10) and Proposition A3 we obtain (3.12). If $\beta \in (0,1)$, we use (3.7) with $\gamma = 1-\beta$ (and the obvious observation that $1-\beta \le s+(1-\beta)(r-s)/d$) to estimate (3.14) by

$$C(d,s)c_0 \sup_\tau \|f_\tau\|_{1,s}[F^n M_0 + \chi_1(n)F^{n-1}\|f_0\|_{\infty,r}](1+|v|)^{-s-(1-\beta)},$$

where $(n-1)\beta < 1-\beta \le n\beta$. Hence the required modification of (3.12) follows from Proposition A3 and (3.10).

Similarly to get (3.11) one writes

$$\frac{d}{dt}\int_{\mathbf{R}^d} \phi_{j,\lambda}(u) f_t(u)\,du + \int_{\mathbf{R}^d} \phi_{j,\lambda}(u) L f_t(u) f_t(u)\,du \le \int_{\mathbf{R}^d} \phi_{j,\lambda}(u) G(f_t,f_t)(u)\,du,$$

and by Proposition A5 the r.h.s. does not exceed $C(d)c_0 M_0 \sup_\tau \|f_\tau\|_{1,\beta-1}$ or

$$C(d)c_0M_0\|f_t \star P_{-(1-\beta)}\|_{\infty,0} \le C(d,\beta)c_0M_0\left[\left(\frac{c_0}{c_1\sigma}\right)^n \sup_\tau \|f_\tau\|_{1,n\beta-(1-\beta)} + \chi_1(n)\left(\frac{c_0}{c_1\sigma}\right)^{n-1}\|f_0\|_{\infty,0}\right]$$

respectively for $\beta \ge 1$ and $\beta < 1$ (where we used (3.6) for $\beta \in (0,1)$). Using (3.9) and Proposition A3 yields (3.11).

For proving the statement (iii) of Theorem 3.1 we shall need a general estimate of $G(f, g^{v,ext})$, where $g^{v,ext}$ is defined in (1.21).

Lemma 3.3. *Suppose (3.3) holds. Let $s \ge 2$, $r \in [0, s+d]$ and let $m = 0,1,2,...$ be such that $(m-1)\beta < d-1-\beta \le m\beta$. Let $f_0 \in L_{1,s} \cap L_{\infty,r}$ and be non-negative. Let g_t be another time-depending family of non-negative functions. Set*

$$m_q(R) = m_q(R;g) = ess \sup_{t\ge 0, |v|>R} g_t(v)|v|^q, \quad m_q^0(R) = m_q^0(R;g) = ess \sup_{|v|>R} g_0(v)|v|^q. \tag{3.15}$$

Then for arbitrary $\rho > 0$, $\delta \in (0,1/4)$, $q \in (d-1, r)$, and for all $|v| \ge \rho/\delta$

$$G(f_t, g_t^{v,ext})(v) \le 2c_0 \frac{A(d-2)}{q-(d-1)} m_q(|v|/\sqrt{2})|v|^{-(q-\beta)}\big[M_0(1-\delta)^{-q}$$

$$+2^{(d-1+q-2\beta)/2}E_0\rho^{-2} + 2^{q/2}C(d,\beta,s,r)\rho^{-\kappa}(F^m M_0 + F^{m-1}\|f_0\|_{\infty,r})\big] \tag{3.16}$$

with $\kappa = s + (d-1-\beta)((r-s)/d-1)$, whenever $0 < \beta < d-1$. If $\beta \ge d-1$, the same estimate holds but with $\sup_\tau \|f_\tau\|_{1,\beta-(d-1)}$ instead of $(F^m M_0 + F^{m-1}\|f_0\|_{\infty,r})$ and with $\kappa = s+d-1-\beta$.

demoProof For $\rho > 0$ set $f = f^\rho + \tilde{f}^\rho$, where $\tilde{f}^\rho(v) = \chi_\rho(v) f(v)$. Suppose $m_q(.)$ is finite for $q < r$ (otherwise the estimate is obvious). Applying (1.22) yields

$$G(f_t, g_t^{v,ext})(v) \le c_0 \int_{\mathbf{R}^d} f_t^\rho(v')|v-v'|^{-(d-1-\beta)}\,dv' \int_{E_{v,v'}} g_t^{v,ext}(w)dE_{v,v'}(w)$$

$$+c_0 \int_{\mathbf{R}^d} \tilde{f}_t^\rho(v')|v-v'|^{-(d-1-\beta)}\,dv' \int_{E_{v,v'}} g_t^{v,ext}(w)dE_{v,v'}(w). \tag{3.17}$$

By elementary computation

$$\int_{\{w\in E_{v,v'}:|w|>R\}} g_t(w)dE_{v,v'}(w) \le \frac{A(d-2)}{q-(d-1)} m_q(R)R^{d-1-q} \tag{3.18}$$

for $q > d-1$ and consequently,

(3.19)
$$\int_{E_{v,v'}} g_t^{v,ext}(w)dE_{v,v'}(w) = \int_{\{w\in E_{v,v'}:|w|>|v|/\sqrt{2}\}} g_t(w)dE_{v,v'}(w) \le \frac{A(d-2)}{q-(d-1)} m_q(|v|/\sqrt{2})2^{(q-d+1)/2}|v|^{d-1-q}.$$

If $|v'| < \rho$ and $|v| > \rho$, then $|v'-v|^{-(d-1-\beta)} \le (|v|-\rho)^{-(d-1-\beta)}$. By conservation of energy (or equivalently, because vectors $w-v$ and $v'-v$ are perpendicular), $|w|^2 > |v|^2 - |v'|^2 \ge (|v|-\rho)^2$ and hence $|w| > |v| - \rho$. Consequently, again using (3.18) we conclude that the first term in (3.17) does not exceed

(3.20)
$$c_0 \int_{\mathbf{R}^d} f_t^\rho(v')||v|-\rho|^{-(d-1-\beta)}\, dv' \int_{E_{v,v'}} g_t(w)dE_{v,v'}(w) \le c_0 \frac{A(d-2)}{q-(d-1)} M_0 m_q(|v|-\rho)(|v|-\rho)^{-(q-\beta)}$$

for $|v| > \rho$. Next, if $|v| > \rho$, we conclude by (3.8) and the conservation of energy that

$$\int_{\mathbf{R}^d} \tilde{f}_t^\rho(v')|v-v'|^{-(d-1-\beta)}\, dv' \le \int_{\{|v'|<|v|/2\}} (|v'|/\rho)^2 \tilde{f}_t^\rho(v')|v-v'|^{-(d-1-\beta)}\, dv' + \int_{\{|v'|\ge|v|/2\}} \tilde{f}_t^\rho(v')|v-v'|^{-(d-1-\beta)}\, dv'$$

$$\le \left(2^{d-1-\beta}E_0\rho^{-2} + C(d,\beta,s,r)\rho^{-\kappa}[F^m M_0 + F^{m-1}\|f_0\|_{\infty,r}]\right)|v|^{-(d-1-\beta)} \tag{3.21}$$

with $\kappa = s + (d-1-\beta)((r-s)/d-1)$, whenever $\beta < d-1$. If $\beta \ge d-1$, the same estimate holds but with $\sup_\tau \|f_\tau\|_{1,\beta-(d-1)}$ instead of $(F^m M_0 + F^{m-1}\|f_0\|_{\infty,r})$ and with $\kappa = s + d - 1 - \beta$.

Let us consider further only the case $\beta < d-1$. By (3.19) and (3.21), the second term of (3.17) does not exceed

(3.22)
$$2c_0 \frac{A(d-2)}{q-(d-1)} m_q(\frac{|v|}{\sqrt{2}})2^{(q-d+1)/2}|v|^{-(q-\beta)} \left(2^{d-1-\beta}E_0\rho^{-2} + C(d,\beta,s,r)\rho^{-\kappa}[F^m M_0 + F^{m-1}\|f_0\|_{\infty,r}]\right).$$

Next, if $|v| \ge \rho/\delta$ with an arbitrary $\delta \in (0,1/4)$, then $|v|-\rho \ge (1-\delta)|v| > |v|/\sqrt{2}$ and $(|v|-\rho)^{-(q-\beta)} \le |v|^{-(q-\beta)}(1-\delta)^{-q}$. Hence (3.16) follows from (3.17), (3.21) and (3.22).

As a direct consequence of Lemma 3.3 one gets the following estimate.

Corollary. *Suppose the assumptions of Lemma 3.3 hold. If* $0 < \beta < d-1$, *then*

$$G(f_t, g_t^{v,ext})(v) \le 2c_0 \frac{A(d-2)}{q-(d-1)} m_q(|v|/\sqrt{2})|v|^{-(q-\beta)} M_0 \frac{1+\delta}{(1-\delta)^q} \tag{3.23}$$

whenever $|v| \ge \rho/\delta$ *and*

$$2^{(d-1+r-2\beta)/2}E_0\rho^{-2} \le M_0\delta/2, \qquad 2^{r/2}C(d,\beta,s,r)\rho^{-\kappa}(F^m M_0 + F^{m-1}\|f_0\|_{\infty,r}) \le M_0\delta/2,$$

or equivalently

$$\rho \ge \sqrt{E_0/(M_0\delta)}\, 2^{(d+1+r-2\beta)/4}, \tag{3.24}$$

$$\rho \ge 2^{r/2\kappa} \left(\frac{C(d,\beta,s,r)}{M_0\delta}(F^m M_0 + F^{m-1}\|f_0\|_{\infty,r})\right)^{1/\kappa}. \tag{3.25}$$

If $\beta \ge d-1$, *the same holds with the changes indicated above in Lemma 3.3.*

Proof of Theorem 3.1. (i) Consider only the case with $\beta \geq 1$ (the case $\beta \in (0,1)$ differs only by using a different constant in Lemma 3.2). The reduced Gustafsson-Carleman representation of the gain term and Lemma 3.2 imply

$$G(f_t, f_t)(v) \leq c_0 C(\beta) \int_{\mathbf{R}^d} f_t(v')|v-v'|^{-(d-1-\beta)}\, dv'$$

$$\times \max\left[C(d,s,\beta) F \sup_{\tau\geq 0} \|f_\tau\|_{1,\beta-1}(1+|v|)^{-(s+1)}, \int_{E_{v,v',\pi/4}} f_0^{v,ext}(w')dE_{v,v'}(w')\right]$$

$$+c_0 C(\beta) \int_{\mathbf{R}^d} f_t^{v,ext}(v')|v-v'|^{-(d-1-\beta)}\, dv'$$

$$\times \max\left[\frac{c_0}{c_1\sigma} C(d,\beta) \sup_{\tau\geq 0}\|f_\tau\|_{1,\beta-1}, \int_{E_{v,v',\pi/4}} f_0(w')dE_{v,v'}(w')\right].$$

Suppose $1 \leq \beta < d-1$. Noticing that

$$s + (d-1-\beta)(r-s)/d \leq 1 + s + \min(d-1-\beta,\ s + (d-1-\beta)(r-s)/d),$$

that $F \geq c_0/c_1\sigma$ (up to a constant), and changing max to + in the previous estimate, we get by Lemma 3.1 with $\gamma = d-1-\beta$ that $G(f_t,f_t)(v)$ does not exceed

$$C(d,s,r,\beta) \sup_{\tau\geq 0}\|f_\tau\|_{1,\beta-1} c_0 (F^{m+1}M_0 + \chi_1(m)F^m\|f_0\|_{\infty,r})(1+|v|)^{-[s+(d-1-\beta)(r-s)/d]}$$

$$+c_0 C(\beta)\int_{\mathbf{R}^d} f_t(v')|v-v'|^{-(d-1-\beta)}\, dv' \int_{E_{v,v',\pi/4}} f_0^{v,ext}(w')dE_{v,v'}(w')$$

$$+c_0 C(\beta)\int_{\mathbf{R}^d} f_t^{v,ext}(v')|v-v'|^{-(d-1-\beta)}\, dv' \int_{E_{v,v',\pi/4}} f_0(w')dE_{v,v'}(w'). \tag{3.26}$$

By elementary calculations

$$\int_{E_{v,v',\pi/4}} f_0^{v,ext}(w')dE_{v,v'}(w') \leq C(r,d)\|f_0\|_{\infty,r}(1+|v|)^{-r}|v-v'|^{d-1},$$

$$\int_{E_{v,v',\pi/4}} f_0(w')dE_{v,v'}(w') \leq C(d)\|f_0\|_{\infty,0}|v-v'|^{d-1}.$$

Hence the last two terms in (3.26) can be estimated from above by

$$C(r,d,\beta)c_0\|f_0\|_{\infty,r}\left[(1+|v|)^{-r}\int_{\mathbf{R}^d} f_t(v')|v-v'|^{\beta}\, dv' + \int_{\mathbf{R}^d} f_t^{v,ext}(v')|v-v'|^{\beta}\, dv'\right]$$

$$\leq C(r,d,\beta)c_0\|f_0\|_{\infty,r}\left[\sup_{\tau\geq 0}\|f_\tau\|_{1,\beta}(1+|v|)^{-(r-\beta)} \sup_{\tau\geq 0}\|f_\tau\|_{1,s}(1+|v|)^{-(s-\beta)}\right]$$

$$\leq C(r,d,\beta)c_0\|f_0\|_{\infty,r}\sup_{\tau}\|f_\tau\|_{1,s}(1+|v|)^{-(r-\beta)}$$

(where we used that $\beta \leq s$ and $r \leq s$). Since $r-\beta \leq s+(d-1-\beta)(r-s)/d$, this leads to the estimate

$$G(f_t,f_t) \leq c_0 C(r,d,\beta)\|f_0\|_{\infty,r} + \sup_{\tau}\|f_\tau\|_{1,s}$$

$$+C(d,s,r,\beta)\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}c_0(F^{m+1}M_0+F^m\|f_0\|_{\infty,r})](1+|v|)^{-(r-\beta)}$$

$$\le C(d,s,r,\beta)c_0\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}(F^{m+1}M_0+F^m\|f_0\|_{\infty,r}),$$

because $\sup_\tau \|f_\tau\|_{1,s}\le FM_0$ (up to a constant). Using this estimate and (3.10) one completes the proof of statement (i) for $1\le\beta<d-1$ by applying Proposition A3 to the Boltzmann equation. In case $\beta\ge d-1$ the first term in (3.26) changes to

$$C(d,s,r,\beta)\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}c_0F\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-(d-1)}(1+|v|)^{-(s+d-1-\beta)},$$

and the rest remains the same leading to the corresponding change in a final constant.

(ii) Again consider only $\beta\ge 1$. Using the reduced Carleman representation of the gain term and Lemma 3.2 gives

$$G(f_t,f_t)(v)\le c_0C(\beta)\int_{\mathbf{R}^d}f_t(v')|v-v'|^{-(d-1-\beta)}\,dv'$$

$$\times\max\left[C(d,s,\beta)F\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}(1+|v|)^{-(s+1)},\int_{E_{v,v'}}f_0^{v,ext}(w')dE_{v,v'}(w')\right].$$

Suppose $\beta<d-1$. Using (3.7) with $\gamma=d-1-\beta$ yields the estimate

$$G(f_t,f_t)\le C(d,s,r,\beta)c_0\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}(F^{m+1}M_0+F^m\|f_0\|_{\infty,r})(1+|v|)^{-\kappa}$$

$$+2c_0\int_{\mathbf{R}^d}f_t(v')|v-v'|^{-(d-1-\beta)}\,dv'\int_{E_{v,v'}}f_0^{v,ext}(w')dE_{v,v'}(w') \tag{3.27}$$

with $\kappa=1+s+\min(d-1-\beta,s+(d-1-\beta)(r-s)/d)$. (Of course, for $\beta<1$ we would have the same estimate but with $(F^nM_0+F^{n-1}\|f_0\|_{\infty,0})$ instead of $\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}$.) By elementary calculations

$$\int_{E_{v,v'}}f_0^{v,ext}(w')dE_{v,v'}(w')\le C(r,d)\int_{|w'|>a|v|}f_0(w')(1+|w'|)^r(1+|w'|)^{-r}dE_{v,v'}(w')$$

$$\le C(r,d)\|f_0\|_{\infty,r}(1+|v|)^{-(r-(d-1))},$$

as $r>d-1$. Hence, again by (3.7) with $\gamma=d-1-\beta$, the last term in (3.27) does not exceed

$$C(r,d)c_0\|f_0\|_{\infty,r}(F^mM_0+F^{m-1}\|f_0\|_{\infty,r})(1+|v|)^{-(r-\beta)}. \tag{3.28}$$

It is easy to see that $r-\beta\le\kappa$ for $s\le r\le s+d$ and therefore it follows now from (3.27) that

$$G(f_t,f_t)\le C(d,s,r,\beta)c_0(F^mM_0+F^{m-1}\|f_0\|_{\infty,r})[F\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}+\|f_0\|_{\infty,r}](1+|v|)^{-(r-\beta)}$$

(and it would be the same but with $(F^nM_0+F^{n-1}\|f_0\|_{\infty,r})$ instead of $\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-1}$ for $\beta\in(0,1)$). Applying now Proposition A3 to the Boltzmann equation and using (3.10) completes the proof of statement (ii) for $\beta<d-1$. In case $\beta\ge d-1$ one gets instead the same estimates (3.27), (3.28) but with $\sup_{\tau\ge 0}\|f_\tau\|_{1,\beta-(d-1)}$ instead of $(F^mM_0+F^{m-1}\|f_0\|_{\infty,r})$, which implies the required change in the final upper bound.

(iii) Since

$$Lf_t(v)\ge A(d-2)c_1\int f_t(w)|v-w|^\beta\,dw\ge A(d-2)c_1(M_0\min(1,2^{1-\beta})|v|^\beta-E_0)$$

$$\geq A(d-2)c_1 M_0 |v|^\beta \left(\min(1, 2^{1-\beta}) - \frac{E_0}{\rho^\beta M_0}\right).$$

it follows that $Lf_t(v) \geq A(d-2)c_1 M_0 |v|^\beta \left(\min(1, 2^{1-\beta}) - \delta\right)$ whenever

$$\rho \geq (E_0/(M_0\delta))^{1/\beta}. \tag{3.29}$$

We shall use now Lemma 3.3 and its Corollary with $g_t = f_t$. This Lemma, the modified Garleman representation and Proposition A3 imply that

$$m_q\left(\frac{\rho}{\delta}\right) \leq m_q^0\left(\frac{\rho}{\delta}\right) + \frac{2c_0}{c_1(q-(d-1))} \frac{1+\delta}{(1-\delta)^r(\min(1,\, 2^{1-\beta}) - \delta)} m_q\left(\frac{\rho}{\delta\sqrt{2}}\right) \tag{3.30}$$

for all $q < r$ whenever ρ satisfies (3.29), (3.24), (3.25). Notice that by (ii), $m_q(.)$ is finite for all $q < r$.

Consider further only the case $1 \leq \beta < d-1$ (for other β usual modifications do the job). There are two possibilities: either (1) $m_q(R) = m_q(0)$ for all $R \in [0, \rho/\delta]$ or (2) $m_q(\rho/\delta) < m_q(0)$. In the first case, it follows from (3.30) and the assumption of statement (iii) of the Theorem that one can choose a small $\delta = \delta(c_0, c_1, r, d, \beta)$ such that $m_q(0) \leq m_q^0(0) + (1-\epsilon/2)m_q(0)$ with

$$\epsilon = 1 - \frac{2c_0}{c_1(r-(d-1))} \max(1,\, 2^{\beta-1}) > 0,$$

and hence $m_q(0) \leq 2m_q^0(0)/\epsilon$. In the second case one clearly has

$$m_q(0) = ess \sup_{|v| \leq \rho/\delta, t \geq 0} f_t(v)|v|^q,$$

and consequently, estimating $\sup_\tau \|f_\tau\|_{\infty,s}$ by (i) with $r' = s$, yields

$$m_q(0) \leq (\rho/\delta)^{r-s} C(d,s,\beta) \frac{c_0}{c_1\sigma} \sup_\tau \|f_\tau\|_{1,\beta-1}(F^{m+1}M_0 + F^m\|f_0\|_{\infty,r}).$$

As both estimates for $m_q(0)$ do not depend on $q < r$, we conclude that

$$m_r(0) \leq C(d,s,r,\beta,c_0,c_1)\rho^{r-s} \frac{\|f_0\|_{1,2}}{\sigma M_0^2} \sup_\tau \|f_\tau\|_{1,\beta-1}(F^{m+1} + F^m\|f_0\|_{\infty,r}).$$

Choosing ρ to be the maximum of the r.h.s. of (3.24), (3.25), (3.29), completes the proof of Theorem 3.1.

4. Propagation of smoothness

We make here the same assumptions on the collision kernel as given at the beginning of the previous section, i.e. assume (1.16), (1.19), (3.1), (3.2). We start with a result on continuous dependence on initial data.

Theorem 4.1. *Let $f_0, g_0 \in L_{1,s+\beta}$ with $s \geq \max(2, 2\beta)$. Let f_t and g_t be the corresponding energy preserving solutions of the Boltzmann equation. Then*

$$\|f_t - g_t\|_{1,s} \leq \|f_0 - g_0\|_{1,s} \exp\{C(d,s)t \sup_{\tau\in[0,T]} (\|f_\tau\|_{1,s+\beta} + \|g_\tau\|_{1,s+\beta})\} \tag{4.1}$$

and

$$\|f_t - g_t\|_{\infty,0} \leq \|f_0 - g_0\|_{\infty,0} + C([f],[g]) \sup_{\tau\in[0,T]} \|f_\tau - g_\tau\|_{1,\beta} \tag{4.2}$$

for all $t \le T$ and an arbitrary T, where $C([f],[g])$ here and below depends on lower bounds for the quantities $M(f_0), M(g_0)$ and on upper bounds to the entropies of f_0, g_0, and $\sup_{\tau\in[0,T]} \|f_\tau + g_\tau\|_{\infty,\beta}$, $\sup_{\tau\in[0,T]} \|f_\tau + g_\tau\|_{1,\max(2,2\beta)}$.

Remark. (i) Inequality (4.1) is well known for $\beta \le 2$, see [Gu1]; in this case it holds even with $s = 2$. (ii) We do not make the constant $C([f],[g])$ explicit to shorten the formulas, but they can be easily found from the proof and the estimates of the previous section.

Proof. Using (3.4) yields

$$\frac{d}{dt}\|f_t - g_t\|_{1,s} = \int |v-w|^\beta \, dvdw \int_0^{\pi/2} \sin^{d-2}(2\theta) h(\theta)\, d\theta$$

$$\times \int_{S^{d-2}} dm[(\sigma K)(w') + (\sigma K)(v') - (\sigma K)(w) - (\sigma K)(v)](f_t(v)f_t(w) - g_t(v)g_t(w)),$$

where $K(v) = 1 + |v|^s$ and $\sigma = \sigma_t(v)$ is the sign of the difference $f_t(v) - g_t(v)$. Consequently

$$\frac{d}{dt}\|f_t - g_t\|_{1,s} = \int |v-w|^\beta \, dvdw \int_0^{\pi/2} \sin^{d-2}(2\theta) h(\theta)\, d\theta$$

$$\times \int_{S^{d-2}} dm[(\sigma K)(w') + (\sigma K)(v') - (\sigma K)(w) - (\sigma K)(v)](f_t(v) - g_t(v))f_t(w)$$

$$+ \int |v-w|^\beta \, dvdw \int_0^{\pi/2} \sin^{d-2}(2\theta) h(\theta)\, d\theta$$

$$\times \int_{S^{d-2}} dm[(\sigma K)(w') + (\sigma K)(v') - (\sigma K)(w) - (\sigma K)(v)](f_t(w) - g_t(w))g_t(v).$$

Let us estimate the first term I_1 in this sum. One has

$$I_1 = \int |v-w|^\beta \, dvdw \int_0^{\pi/2} \sin^{d-2}(2\theta) h(\theta)\, d\theta$$

$$\times \int_{S^{d-2}} dm[\sigma(v)(\sigma K)(w') + \sigma(v)(\sigma K)(v') - \sigma(v)(\sigma K)(w) - K(v)]|f_t(v) - g_t(v)|f_t(w)$$

$$\le \int |v-w|^\beta \, dvdw \int_0^{\pi/2} \sin^{d-2}(2\theta) h(\theta)\, d\theta$$

$$\times \int_{S^{d-2}} dm[|w'|^s + |v'|^s - |w|^s - |v|^s + 2K(w)]|f_t(v) - g_t(v)|f_t(w).$$

Using a modified Povzner inequality as obtained e.g. in [MW] or [Lu] (which is valid for any $s \ge 2$) and estimating h by (3.2) yields

$$I_1 \le C(d,s)c_0 \int |v-w|^\beta [|v|^{s/2}|w|^{s/2} + 2K(w)]|f_t(v) - g_t(v)|f_t(w)\, dvdw,$$

which does not exceed $C(d,s)\|f_t - g_t\|_{1,s}\|f_t\|_{s+\beta}$, because $s \ge 2\beta$. Estimating I_2 in the same way leads to the estimate

$$\frac{d}{dt}\|f_t - g_t\|_{1,s} \le C(d,s)\|f_t - g_t\|_{1,s}(\|f_t\|_{s+\beta} + \|g_t\|_{s+\beta}).$$

Gronwall's lemma completes the proof of (4.1).

The main ingredient in the proof of (4.2) is the following estimate

$$\int_{E_{z,\bar z,\pi/4}} |f_t - g_t|(u)\, dE_{z,\bar z}(u)$$

$$\le \max\left(\int_{E_{z,\bar z,\pi/4}} |f_0 - g_0|(u)\, dE_{z,\bar z}(u),\; C([f],[g])(1+|\bar z - z|^{d-1}) \sup_{\tau\le t} \|f_\tau - g_\tau\|_{1,\beta}\right). \tag{4.3}$$

To prove (4.3), we get from the Boltzmann equation

$$\frac{d}{dt}\int_{E_{z,\bar z,\lambda}} |f_t - g_t|(u)\, dE_{z,\bar z}(u) + \int_{E_{z,\bar z,\lambda}} |f_t - g_t|(u) Lf_t(u)\, dE_{z,\bar z}(u)$$

$$\le \int_{E_{z,\bar z,\lambda}} g_t(u)|Lf_t - Lg_t|(u)\, dE_{z,\bar z}(u) + \int_{E_{z,\bar z,\lambda}} (|G(f_t - g_t, f_t)| + |G(g_t, f_t - g_t)|)(u)\, dE_{z,\bar z}(u). \tag{4.4}$$

By Proposition A5 the second term on the r.h.s. does not exceed $C([f],[g])\|f_t - g_t\|_{1,\beta}$ From Proposition A6 one deduces by the same arguments as in Lemma 3.2 that

$$\int_{E_{z,\bar z,\lambda}} g_t(u)|u|^\beta\, dE_{z,\bar z}(u) \le \max\left(\int_{E_{z,\bar z,\lambda}} g_0(u)|u|^\beta\, dE_{z,\bar z}(u), C([f],[g]) \sup_{\tau\le t}\|f_\tau - g_\tau\|_{1,\beta}\right).$$

Consequently, the first term on the r.h.s. of (4.4) does not exceed

$$\|g_0\|_{\infty,\beta}|\bar z - z|^{d-1} + C([f],[g])\sup_{\tau\le t}\|f_\tau - g_\tau\|_{1,\beta}.$$

Hence, (4.3) follows from Proposition A3.

At last, from the Boltzmann equation we have

$$\frac{d}{dt}|f_t - g_t|(v) + |f_t - g_t|(v)Lf_t(v) \le g_t(v)|Lf_t - Lg_t|(v) + |G(f_t - g_t, f_t) + G(g_t, f_t - g_t)|(v). \tag{4.5}$$

The first term on the r.h.s. does not exceed $(1 + |v|^\beta)\|f_t - g_t\|_{1,\beta}\|g_t\|_{\infty,0}$. The second term can be estimated (up to a constant) from the Gustafsson-Carleman representation as

$$\int |f_t - g_t|(v')|v - v'|^{-(d-1-\beta)} \int_{E_{v,v',\pi/4}} (f_t + g_t)(w')dE(w')$$

$$+\int |f_t + g_t|(v')|v - v'|^{-(d-1-\beta)} \int_{E_{v,v',\pi/4}} |f_t - g_t|(w')dE(w'),$$

which (by (4.3)) does not exceed

$$\int |f_t - g_t|(v')|v - v'|^\beta C([f],[g]) + \int (f_t + g_t)(v')|v - v'|^\beta |f_0 - g_0\|_{\infty,0}$$

$$+C([f],[g])\int (f_t + g_t)(v')(|v - v'|^\beta + |v - v'|^{-(d-1-\beta)}) \sup_{\tau\le t}|f_\tau - g_\tau\|_{1,\beta},$$

and which does not exceed $C([f],[g])(1+|v|^\beta)\sup_{\tau\le t}|f_\tau - g_\tau\|_{1,\beta}$ (by Lemma 3.1 if $\beta < d-1$). Hence (4.2) follows by Proposition A3 and (3.10).

Corollary. *For any* $f_0 \in L_{1,\beta+s}$ *with* $s \geq \max(2, 2\beta)$ *there exists a unique solution* f_t *satisfying the condition* $\sup_{t\in[0,T]} \|f_t\|_{1,\beta+s} < \infty$ *for any* $T > 0$.

As a consequence of (4.2) we give the following results on propagation of smoothness of solutions with respect to initial data and the velocity v.

Theorem 4.2. *Let* $s \geq \max(2, 2\beta)$ *and* $f_0 = f_0(x, .)$ *be a family of non-negative functions from* $L_{1,s+\beta}$ *that depends on a real parameter* $x \in \mathbf{R}$ *in such a way that*

(i) $\sup_x \|f_0(x, .)\|_{1,s+\beta} < \infty$ *and* $\inf_x M(f_0(x, .)) > 0$,

(ii) $f_0(x, .) \ln f_0(x, .) \in L_1$ *for all* x *and* $H_0 = \sup_x \int f_0(x, v) \ln f_0(x, v)\, dv < \infty$,

(iii) $\sup_x \|f_0(x, .)\|_{\infty,\beta} < \infty$,

(iv) for each $v \in \mathbf{R}^d$ *the derivative* $\nabla_x f_0(x, v)$ *exists for almost all* x *and both* $\sup_{x,v} |\nabla_x f_0(x, v)|$ *and* $\sup_x \|\nabla_x f_0(x, .)\|_{1,s}$ *are bounded.*

Then for all $t > 0$ *and* $v \in \mathbf{R}^d$, *the derivative* $\nabla_x f_t(x, v)$ *exists for almost all* x *and both the quantities* $\sup_{t\leq T,x,v} |\nabla_x f_t(x, v)|$ *and* $\sup_{t\leq T,x} \|\nabla_x f_0(x, .)\|_{1,s}$ *are finite for arbitrary* $T > 0$.

Proof. Follows directly from Theorem 3.1 with $r = \beta$ and Theorem 4.1.

Remark. Of course, one can provide more or less straightforward extension to higher derivatives as in Theorem 2.4.

Theorem 4.3. *Let* $s \geq \max(2, 2\beta)$, *a non-negative* $f_0 \in L_{1,s+\beta}$ *be such that*

(i) $f_0 \ln f_0 \in L_1$ *and* $\|f_0\|_{\infty,\beta} < \infty$,

(ii) the derivative $\nabla f_0 = \{\nabla_i f_0\}_{i=1}^d$ *exists for almost all* v *and both* $\|\nabla f_0\|_{\infty,0}$ *and* $\|\nabla f_0\|_{1,s}$ *are finite.*

Then for all $t \in [0, T]$ *with arbitrary* $T > 0$ *and almost all* $v \in \mathbf{R}^d$, *the derivative* $\nabla f_t(v)$ *exists and*

$$\|\nabla f_t\|_{1,s} \leq \|\nabla f_0\|_{1,s} \exp\{C(d, s)t \sup_{\tau\in[0,T]} \|f_\tau\|_{1,s+\beta}\} \tag{4.6}$$

$$\|\nabla f_t\|_{\infty,0} \leq \|\nabla f_0\|_{\infty,0} + C([f]) \sup_{\tau\in[0,T]} \|\nabla f_\tau\|_{1,\beta}, \tag{4.7}$$

where $C([f]$ *depends on a low bound to* $M(f_0)$ *and on upper bounds to the entropy of* f_0, *and* $\sup_{\tau\in[0,T]} \|f_\tau\|_{\infty,\beta}$, $\sup_{\tau\in[0,T]} \|f_\tau\|_{1,\max(2,2\beta)}$. *In particular, due to the time independent estimates for* $\|\nabla f_t\|_{1,\beta}$ *obtained in [MW], inequality (4.7) implies a time independent estimate to* $\|\nabla f_t\|_{\infty,0}$.

Proof. Inequalities (4.6), (4.7) are obtained in the same way as inequalities (4.1), (4.2) of Theorem 4.1.

One can get now various results on the propagation of moments $\|\nabla f_t\|_{\infty,r}$ in the spirit of Theorem 3.1. Let us give a result for large r based on the Carleman method (Lemma 3.3). The main technique is incorporated in the following

Theorem 4.4. *Let (3.3) hold,* $r > s \geq \max(2, 2\beta)$ *and* $r > d - 1 + 4\max(1, 2^{\beta-1})c_0/c_1$. *Let* $f_0 \in L_{1,s+\beta} \cap L_{\infty,r}$ *and there exist the derivative* $\nabla f_0 \in L_{1,s} \cap L_{\infty,r}$. *Then* f_t *is differentiable for all* t *and* $\sup_{t\leq T} \|\nabla f_t\|_{\infty,r} < \infty$ *for all* $T > 0$ *with a bound depending on* T *through the bound for* $\sup_{t\leq T} \|\nabla f_t\|_{1,s}$.

Proof. **Step 1.** Let us show that for $\gamma \in [0, d - 1)$

$$\||\nabla f_t| \star P_{-\gamma}\|_{\infty,0} \leq C(d, \beta, \gamma)\|\nabla f_0\|_{\infty,0} + C([f]) \sup_{\tau\leq t} \|\nabla f_\tau\|_{1,\beta}. \tag{4.8}$$

The proof is similar to the proof of Lemma 3.1. Multiplying $\nabla f_t(z)$ by its sign $\sigma_t(z)$ and by $|v - z|^{-\gamma}$, integrating and using the differentiated Boltzmann equation (2.4) yields

$$\frac{d}{dt} \int |\nabla f_t|(z)|v - z|^{-\gamma}\, dz + \int |\nabla f_t|(z) L f_t |v - z|^{-\gamma}\, dz$$

$$= -\int f_t(z)\sigma_t(z)L\nabla f_t(z)|v-z|^{-\gamma}\,dz + \int (G(\nabla f_t, f_t) + G(f_t, \nabla f_t))(z)\sigma_t(z)|v-z|^{-\gamma}\,dz.$$

By Proposition A2, the first term on the r.h.s. can be estimated in magnitude by

$$\|\nabla f_t\|_{1,\beta}(\|f_t\|_{1,\beta} + \|f_t\|_{\infty,\beta}),$$

and by Proposition A4, the second term can be estimated (up to a constant) by $\|\nabla f_t\|_{1,\beta-\gamma}\|f_t\|_{1,\beta-\gamma}$ if $\beta \geq \gamma$ and by $\|\nabla f_t\|_{1,0}\sup_{\tau\leq t}(\|f_\tau\|_{1,\beta} + \|f_\tau\|_{\infty,0})$ in case $\beta < \gamma$ (where Lemma 3.1 was used in the latter case). By Proposition A3 this implies that

$$\||\nabla f_t| \star P_{-\gamma}\|_{\infty,0} \leq \||\nabla f_0| \star P_{-\gamma}\|_{\infty,0} + C([f])\sup_{\tau\leq t}\|\nabla f_\tau\|_{1,\beta},$$

and hence the required estimate by Proposition A2.

Step 2. By a similar modification of the proof of Lemma 3.1 one shows that for $r \in (0, s+d]$ and $a \geq 1/2$

(4.9)
$$\int \chi_{a|v|}(|z|)\nabla f_t(z)|v-z|^{-\gamma}\,dz \leq (C(d,\beta,\gamma)\|\nabla f_0\|_{\infty,r} + C([f])\sup_{\tau\leq t}\|\nabla f_\tau\|_{1,s})(1+|v|)^{-(s+\gamma(r-s)/d)}.$$

Like above, one deduces (4.9) from Propositions A3 and A4 and the inequality

$$\frac{d}{dt}\int \chi_{a|v|}(|z|)|\nabla f_t|(z)|v-z|^{-\gamma}\,dz + \int \chi_{a|v|}(|z|)|\nabla f_t|(z)Lf_t|v-z|^{-\gamma}\,dz$$
$$\leq \int f_t(z)\chi_{a|v|}(|z|)L\nabla f_t(z)|v-z|^{-\gamma}\,dz + 2\int \chi_{a|v|}(|z|)G(|\nabla f_t|, f_t)(z)|v-z|^{-\gamma}\,dz$$

that follows from the Boltzmann equation.

Step 3. From (2.4) and (1.22) one obtains for each $i = 1, ..., d$

$$\frac{d}{dt}\nabla_i f_t(v) + \nabla_i f_t L f_t(v)$$
$$= -f_t(v)L\nabla_i f_t(v) + 2G(\nabla_i f_t^{v,int}, f_t^{v,ext})(v) + 2G(f_t, \nabla_i f_t^{v,ext})(v). \tag{4.10}$$

We shall follow now the same arguments as in the proof of Theorem 3.1 (iii). Denoting

$$m_q^T(R; g) = ess\sup_{t\in[0,T],|v|\geq R}|g_t||v|^q$$

for a family $g = g_t$ of functions depending on $t \geq 0$, one deduces from (3.23) that

$$G(f_t, |\nabla_i f_t^{v,ext}|) \leq 2c_0\frac{A(d-2)}{q-(d-1)}m_q^T(|v|/\sqrt{2}; \nabla f)|v|^{-(q-\beta)}M_0\frac{1+\delta}{(1-\delta)^q}$$

for $|v| \geq \rho/\delta$ with ρ large enough. Using the same arguments as in the proof of Lemma 3.3 (where (4.9) is needed) one deduces similarly that

$$G(|\nabla_i f_t|, f_t^{v,ext}) \leq 2c_0\frac{A(d-2)}{q-(d-1)}m_q^T(|v|/\sqrt{2}; f_t)|v|^{-(q-\beta)}\sup_{\tau\leq T}\|\nabla f_\tau\|_{1,0}\frac{1+\delta}{(1-\delta)^q}$$

for $|v| \geq \rho/\delta$ with $\rho = \rho(T)$ large enough. The first term on the r.h.s. does not exceed in magnitude

$$\sup_{\tau\leq T}\|\nabla f_\tau\|_{1,\beta}\sup_{\tau\leq T}\|f_\tau\|_{\infty,r}|v|^{-(q-\beta)}.$$

We conclude by Proposition A3 that

$$m_q^T\left(\frac{\rho}{\delta}, \nabla_i f\right) \leq m_q^0\left(\frac{\rho}{\delta}, \nabla_i f\right) + (1-\epsilon)m_q^T\left(\frac{\rho}{\delta\sqrt{2}}, \nabla_i f\right) + C([f])\rho^{-q}\sup_{\tau\leq T}\|\nabla f_\tau\|_{1,\beta},$$

and complete the proof as in Theorem 3.1 (iii) noting that $\sup_{\tau\leq T}\|\nabla f_\tau\|_{\infty,0} < \infty$ by Theorem 4.3.

We can now prove the following result.

Theorem 4.5. *Let (3.3) hold, $r > d + \beta + \max(2, 2\beta)$, $r > d - 1 + 4\max(1, 2^{\beta-1})c_0/c_1$. Let f_0 and all its derivatives till the order l (with an arbitrary integer l) belong to $L_{\infty,r}$. Then f_t is differentiable for all t and $L_{\infty,r}$-norms of f_t and all its derivatives up to order l are uniformly bounded in time.*

Proof. Arbitrary l are obtained by induction. Consider just $l = 1$. By assumptions on r one can choose s such that $\max(2, 2\beta) \le s < r - d - \beta$ and hence $f_0 \in L_{1,s+\beta} \cap L_{\infty,r}$ and $\nabla f_0 \in L_{1,s} \cap L_{\infty,r}$. By Proposition 4.4 we get that $\nabla f_t \in L_{\infty,r}$ with bounds depending on the bounds for $L_{1,s}$-norms of ∇f_t. Now choosing ω in such a way that $2s + d < \omega < 2r - d$, we see from the upper bound for ω that $|\nabla f_0|^2 \in L_{1,\omega}$ and hence by the results of [MV] $|\nabla f_t|^2 \in L_{1,\omega}$ with uniform bounds in time. Hence (by Cauchy inequality) $\nabla f_t \in L_{1,s}$ for all t with uniform bounds, which implies, as we mentioned above, the uniform bounds for $L_{\infty,r}$-norms of ∇f_t. The proof is complete.

Remark. Strictly speaking, the results of [MV] we used in the proof of Theorem 4.5 are proved only for $\beta \in (0, 2)$. However, it seems that when the existence and uniqueness is obtained for arbitrary positive β (Corollary to Theorem 4.1), it is not difficult to generalise the results of [MV] to this case as well.

Appendix

We collect here some auxiliary estimates omitting all proofs as they are essentially known (see e.g. [Ca], [Ar2], [Gu1]). We only make a straightforward extension of the range of validity of some parameters (e.g. the dimension), make more precise some constants, and in Proposition A4-A6 we extend to $G(f, g)$ and $G(f, f)(u)|u|^l$ the usual estimates for $G(f, f)$ (that are consequences of representation (1.17)).

For an arbitrary real λ let $P_\lambda(v) = |v|^\lambda$ denote the corresponding power function on $\mathbf{R}^d$ and χ_λ denote the indicator of the half-line $[\lambda, \infty)$, i.e. $\chi_\lambda(x)$ equals 1 for $x \ge \lambda$ and vanishes otherwise. Convolution is defined as usual by $(f \star g)(v) = \int f(z)g(v - z)\,dz$, and $M(f)$, $E(f)$ denote the total mass $\int f(v)\,dv$ and the energy $\int |v|^2 f(v)\,dv$ of the state f. Our first Proposition makes precise a trivial observation that the convolution $(f \star P_\lambda)(v)$ should behave like $|v|^\lambda$ for large λ and for f that decrease rapidly enough at infinity. By C we shall denote various constants indicating in brackets the parameters on which they depend.

Proposition A1. *If $f \in L_{1,\lambda}$ is non-negative and such that $\int f(z)\ln^+ f(z)\,dz \le h^+$ with some $h^+ \ge 0$, then for any $\lambda \ge 0$*

$$(f \star P_\lambda)(v) \ge \sigma M(f) \tag{A1}$$

and

$$\frac{1}{3}\min(1,\, 2^{1-\lambda})\sigma\frac{(M(f))^2}{\|f\|_{1,\lambda}}(1 + |v|^\lambda) \le (f \star P_\lambda)(v) \le \|f\|_{1,\lambda}(1 + |v|^\lambda) \tag{A2}$$

with

$$\sigma = \sigma(\lambda, d, h^+, M(f)) = \frac{1}{2}\min\left(1, \left(\frac{M(f)}{4V(d)}\right)^{\lambda/d}\exp\left\{-\frac{4h^+\lambda}{dM(f)}\right\}\right). \tag{A3}$$

The usual characteristics of f related to Boltzmann equation are the mass $M(f)$, the energy $E(f)$, and the entropy $H(f) = \int f(z)\ln f(z)\,dz$ and hence it is desirable to have estimates in terms of $H(f)$ and not h^+ as above.

Corollary. *If $f \in L_{1,2}$ is non-negative and such that $f\ln f \in L_{1,0}$, $H(f) \le h$, $E(f) \le E$ with some constants $E > 0$, h, then (A1), (A2) hold for any $\lambda \in [0, 2]$ with*

$$\sigma = \sigma(\lambda, d, h, M(f), E) = \frac{1}{2}\min\left(1, \left(\frac{M(f)}{4V(d)}\right)^{\lambda/d}\exp\left\{-\frac{4(h + E + (d+1)\pi^{d/2})\lambda}{dM(f)}\right\}\right), \tag{A4}$$

We shall now estimate the convolutions with $P_{-\lambda}$, $\lambda > 0$.

Proposition A2. *Let $s \geq 0$, $r \geq 0$, $a \geq 1/2$, $\lambda \in (0,d)$, and let f be a non-negative measurable function. Then*

$$\int_{\mathbf{R}^d} f(z)\chi_{a|v|}(|z|)|v-z|^{-\lambda}\,dz \leq C(d,\lambda,s,r)(\|f\|_{1,s}+\|f\|_{\infty,r})(1+|v|)^{-s-\lambda\min(1,(r-s)/d)}$$

and

$$(f \star P_{-\lambda})(v) \leq C(d,\lambda,s,r)(\|f\|_{1,s}+\|f\|_{\infty,r})(1+|v|)^{-b},$$

where $b=\min(\lambda, s+\lambda\min(1,(r-s)/d))$.

The next statement is the main tool for proving the uniform in time boundedness of the norms $\|f_t\|_{\infty,r}$ of the solutions to the Boltzmann equation.

Proposition A3. *Suppose h_1, h_2 are continuous functions of $t \in \mathbf{R}_+$ with $h_1>0$. If $f(t) \geq 0$ is differentiable and $\frac{d}{dt}f + h_1 f \leq h_2$, then*

$$\sup_{t\geq 0} f(t) \leq \max\left(f(0), \sup_{t>0}\frac{h_2(t)}{h_1(t)}\right).$$

For the next two results assume (3.1), (3.2) hold with some $\beta \geq 0$.

Proposition A4. *Suppose $a>1/2$, $\gamma \in [0,d-1)$, and f, g are measurable non-negative functions.*
(i) If $0 \leq \gamma \leq \beta$, $s \geq \beta-\gamma$, then

$$\|G(f,g)\star P_{-\gamma}\|_{\infty,0} \leq C(d,\gamma)c_0(M(f)\|g\|_{1,\beta-\gamma}+M(g)\|f\|_{1,\beta-\gamma})$$

$$\int \chi_{a|v|}(|u|)G(f,g)(u)|v-u|^{-\gamma}\,du$$

$$\leq 2^{\beta-\gamma}C(d,\gamma)c_0\|g\|_{1,s}M(f)(1+|v|)^{-(s-\beta+\gamma)}+2^{\beta-\gamma}C(d,\gamma)c_0\|f\|_{1,s}M(g)(1+|v|)^{-(s-\beta+\gamma)}.$$

(ii) If $\beta \leq \gamma$, $s \geq 0$, then

$$\|G(f,g)\star P_{-\gamma}\|_{\infty,0} \leq C(d,\gamma)c_0M(g)\|f\star P_{-(\gamma-\beta)}\|_{\infty,0},$$

$$\int \chi_{a|v|}(|u|)G(f,g)(u)|v-u|^{-\gamma}\,du \leq C(d,\gamma)c_0(1+|v|)^{-s}$$

$$\times(\|f\|_{1,s}\sup_{|u|\geq a|v|/\sqrt{2}}\int_{\mathbf{R}^d}|u-w|^{\beta-\gamma}g(w)\,dw+\|g\|_{1,s}\sup_{|u|\geq a|v|/\sqrt{2}}\int_{\mathbf{R}^d}|u-w|^{\beta-\gamma}f(w)\,dw).$$

Moreover, in all these formulas the coefficients $C(d,\gamma)$ are non-decreasing in γ (and tend to infinity as $\gamma \to d-1$).

Proposition A5. *For arbitrary $a>0$, $s\geq 0$, $z \in \mathbf{R}^d$ and for almost all directions $(\bar{z}-z)/|\bar{z}-z|$ (see Carleman's representation in Section 1 for the notation $E_{z,\bar{z}}$)*

$$\int_{E_{z,\bar{z}}}|G(f,g)(u)|dE_{z,\bar{z}}(u) \leq C(d)c_0M(g)\|f\star P_{-(1-\beta)}\|_{\infty,0}, \quad \beta\leq 1,$$

$$\int_{E_{z,\bar{z}}}|G(f,g)(u)|dE_{z,\bar{z}}(u) \leq C(d,\beta)c_0(M(g)\|f\|_{1,\beta-1}+M(f)\|g\|_{1,\beta-1}), \quad \beta \geq 1,$$

$$\int_{E_{z,\bar{z}}}\chi_{a|v|}(|u|)|G(f,g)(u)|dE_{z,\bar{z}}(u) \leq C(d,s)c_0(1+|v|)^{-s}$$

$$\times(\|g\|_{1,s}\sup_{\{|u|\geq a|v|/\sqrt{2}\}}(f\star P_{-(1-\beta)})(u)+\|f\|_{1,s}\sup_{\{|u|\geq a|v|/\sqrt{2}\}}(g\star P_{-(1-\beta)})(u)).$$

The last statement is just a slight modification of the previous one.

Proposition A6. *For arbitrary $l \geq 0$, $z \in \mathbf{R}^d$ and for almost all directions $(\bar{z}-z)/|\bar{z}-z|$*

$$\int_{E_{z,\bar{z}}} |G(f,f)|(u)|u|^l dE_{z,\bar{z}}(u) \leq C(d)c_0 \|f\|_{1,l}(M(f)+\|f\|_{\infty,0}) \quad \beta < 1,$$

$$\int_{E_{z,\bar{z}}} |G(f,f)(u)||u|^l dE_{z,\bar{z}}(u) \leq C(d,\beta)c_0 M(f)\|f\|_{1,l+\beta-1}, \quad \beta \geq 1.$$

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