On general kinetic equation for many particle systems with interaction, fragmentation and coagulation

V.P. Belavkin
School of Mathematics, Nottingham University,
Nottingham NG7 2RD, UK,
V.N. Kolokoltsov.
Department of Statistics, University of Warwick,
Coventry CV4 7AL, UK

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Abstract

We deduce the most general kinetic equation that describe the low density limit of general Feller processes for the systems of random number of particles with interaction, collisions, fragmentation and coagulation. This is done by studying the limiting as $\varepsilon \to 0$ evolution of Feller processes on $\cup_n^\infty X^n$ with $X = \mathbb{R}^d$ or $X = \mathbb{Z}^d$ described by the generators of the form $\varepsilon^{-1} \sum_{k=0}^K \varepsilon^k B^{(k)}$, $K \in \mathcal{N}$, where $B^{(k)}$ are the generators of k-arnary interaction, whose general structure is also described in the paper.

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1 Introduction

Exploiting the ideas that go back to [12] [10], [6], [7], we give a detailed exposition of the general procedure from [2], [3] and develop it further in order to derive general kinetic equations that describe the low density limit of the general Feller processes on systems of random number of particles with collision, fragmentation and coagulation. More precisely, we study the limiting as $\varepsilon \to 0$ evolution of Feller processes on $\mathcal{X} = \cup_{n=0}^{\infty} X^n$ with $X = \mathbb{R}^d$ or $X = \mathbb{Z}^d$ described by the generators of the form $\varepsilon^{-1} \sum_{k=0}^{K} \varepsilon^k B^{(k)}$, $K \in \mathcal{N}$, where $B^{(k)}$ are the most general generators of k-arnary interaction. In the simplest typical examples, interactions are allowed between small numbers of particles, i.e. K equals 2 or 3. However, interactions with an arbitrary K are also of interest. As is shown in the paper, these interactions (more precisely, its semiclassical limit) are described by (infinite-dimensional) Hamiltonian systems with Hamiltonian functions being polynomials of order K. Formally, our argumants work also for the case $K = \infty$, the corresponding limiting deterministic dynamics being described by infinite-dimensional analytical Hamiltonian systems.

The paper is organised as follows. In section 2, we give a structural theorem for the generators of general Feller processes on many particle systems (Theorem 1). In section 3, we formulate our main results (Theorems 2, 3 and 4) and indicate the main steps in their proofs. In particular, we write down the general kinetic equation and related hamiltonian systems and explain the method of their derivation. For the case of bounded Lindblad type generators, the general kinetic equation (24) and the Hamiltonian system (28), (29) were obtained in [2] in much more general framework of noncommutative probability which is applied also to quantum systems. On the classical level the equations from [2] correspond to the case of only Feller processes with bounded generators having duals acting on measures with densities with respect to the Lebesgue measure. In section 4 we discuss the main examples of our equation (24) that contains as particular cases the well-known Boltzman, Landau, Vlasov and Smoluchovskii equations, their generalisations and combinations. In section 5 we write down the evolution of generating functionals in terms of a (pseudo-differential) equation in functional derivatives (thus proving Theorem 4). In secton 6, we discuss the creation, annihilation and density number operators in Fock spaces descring Markov processes on systems with a random number of indistinguishable classical particles and give the canonical representation for general generators $B^{(k)}$ of k-arnary interactioins. In section 7 we prove Theorem 2 in a slightly more general framework than anticipated in section 3.

We shall list now the main (more or less standard) notations that will be used throughout the paper.

$\mathbb{N}$ (resp. $\mathbb{R}$)-the set of natural (resp. real) numbers, S^{d-1} - the unit sphere in $\mathbb{R}^d$

$\mathbf{1}$ denotes the function that equals identically 1 or the identity operator in a Banach space

C_n^k - binomial coefficients, $\delta_k^l = 1$ or 0 respectively for $k = l$ and $k \neq l$

if X is a localy compact space, then

$C_0(X)$ - the Banach space of continuous functions vanishing at infinity and equipped with sup-norm,

$\mathcal{M}(X)$ - its dual Banach space of finite (signed) measures on X,

$C(X)$ -the Banach space of all bounded continuous functions,

$C_0^+(X)$, $\mathcal{M}^+(X)$, $C^+(X)$ - the corresponding cones of non-negative elements. For $f \in C(X), \mu \in \mathcal{M}(X)$, the natural pairing is defined by the formula $(f, \mu) = \int f(x)\mu(dx)$.

if X is a smooth manifold, then $C^m(X)$ (respectively $C_k^m(X)$) denotes the space of m-times continuously differentiable functions on X (respectively its subspace of functions with a compact support)

By a symmetric function of n (vector) variables we shall understand a function, which is symmetric with respect to all permutations of these variables, and by a symmetric operator on the space of functions of n variables we shall understand an operator that preserves the set of symmetric functions.

If B is a linear operator in $C_0(X)$, $B^\star$ denotes its dual acting in $\mathcal{M}(X)$.

For a finite subset $I = \{i_1, ..., i_k\}$ of a countable set J, we denote by $|I|$ the number of elements in I, by $\bar{I}$ its complement $J \setminus I$, by x_I we denote the collection of variables $x_{i_1}, ..., x_{i_k}$ and similarly by dx_I the measure $dx_{i_1}...dx_{i_k}$

For a smooth manifold (possibly with a boundary) in Euclidean space $X = \mathbb{R}^d$, we denote by δ_M the delta-measure with support in M, in other words $\int_X f(x)\delta_M(x)\, dx = \int_M f(x) dv_M(x)$, where dv_M is the standard Riemannian volume-form on M.

A transitional kernel from X to Y means, as usual, a measurable function $\mu(x, .)$ from $x \in X$ to the cone of positive finite measures on Y.

For a function f of $n-1$ variables , we shall write $f(x_1, ..., \check{x}_j, ..., x_n)$ for $f(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$.

2 Preliminaries: Feller processes on disjoint unions of Euclidean spaces

Let a locally compact topological space $\mathcal{X}$ be defined as the disjoint union $\mathcal{X} = \cup_{j=0}^N \mathcal{X}_j$, $N \in \mathbb{N}$ or $N = \infty$, of finite or countable number of Euclidean spaces of different dimensions $\mathcal{X}_j = \mathbb{R}^{d_j}$ and let $T(t) = \{T_i^j(t)\}$, $T_i^j(t) : C_0(\mathcal{X}_j) \mapsto C_0(\mathcal{X}_i)$, be a Feller semigroup on $\mathcal{X}$, i.e. a semigroup of continuous linear operators in $C_0(\mathcal{X})$ such that for any $f \in C_0(\mathcal{X})$ (i) $0 \leq f \leq 1$ implies $0 \leq T(t)f \leq 1$ and (ii) $T(t)f \to f$ as $t \to 0$. Let $A = \{A_i^j\}$ be the generator of $T(t)$ with the (dense) domain $D(A) \subset C_0(\mathcal{X})$. In case $N = 1$, i.e. if $\mathcal{X} = X = \mathbb{R}^d$, the general form of A (under mild additional assumption on its domain) is given by the well known theorem of Courrège [8] that states that if $D(A)$ contains the space $C_k^2(X)$, then

$$Af(x) = A_{(G,d,\nu,\gamma)}f(x) = tr\,(G(x)\frac{\partial^2}{\partial x^2})f(x) + (d(x), \nabla)f(x)$$
$$+ \int \left(f(x+\xi) - f(x) - \frac{(\xi, \nabla)f(x)}{1+\xi^2} \right) \nu(x, d\xi) - \gamma(x)f(x) \qquad (1)$$

where G, d, ν are respectively a non-negative symmetric $d \times d$-matrix (of the diffusion coefficients), a vector (of the drift), and a Lévy measure (describing jumps) depending measurably on x, and $\gamma(x)$ is a non-negative measurable function. More precisely, it was noted in [8] that the generators of Feller processes satisfy the so called positive maximum principle (PMP) (i.e. they are such that whenever $f(x_0) = \max\{f(x)\} > 0$ one has necessarily $Af(x_0) \leq 0$), and then it was shown that (1) gives the general form of operators satisfying PMP. Notice however that the structural result of Courrège described above does not give any necessary and sufficient conditions on G, d, ν, γ that ensure that the operator (1) actually generates a Markov process. In fact, there exists an extensive literature devoted to the various methods of reconstruction of the Markov process from a given operator of the form (1) under various regularity assumptions on $G(x), d(x), \nu(x, .), \gamma(x)$ (see [9] or [11] and references therein).

We shall give now a simple generalisation of Courrège's theorem to a more general case with $N > 1$. We shall use the notation f^j for the natural projection of $C_0(\mathcal{X})$ onto $C_0(\mathcal{X}_j)$ defined as $f^j(x) = f(x^j)$ where $x = x^j \in \mathcal{X}_j \subset \mathcal{X}$

Theorem 1 *Suppose A is the generator of a Feller semigroup $T(t)$ on $\mathcal{X}$ and the domain $D(A)$ contains the space $C_k^2(\mathcal{X})$. Then*

$$A = A_{diag} + A_{jump} \tag{2}$$

where A_{diag} is pure diagonal operator, i.e. $A_{diag} = \{A^j_{diag}\}$, and each A^j_{diag} is the generator of a Feller semigroup in $\mathcal{X}_j$ (in particular it has form (1) with some $G^j, d^j, \nu^j, \gamma^j$ depending measurably on $x_j \in \mathcal{X}_j$), and where

$$(A_{jump}f)^j(x) = \sum_{i=1}^N \int_{\mathcal{X}_i} f^i(y)\mu_i^j(x,dy), \quad x \in \mathcal{X}_j \tag{3}$$

for $f = \{f^j\} \in C_0(\mathcal{X})$ with some family of transition kernels $\mu_i(x,dy) = \{\mu_i^j(x,dy)\}$ from $\mathcal{X}$ to $\mathcal{X}_i$, where $\mu_i^j(x,.) = \mu_i(x^j,.)$ for $x = x^j \in \mathcal{X}_j$. Moreover, for each j and $\forall x \in X_j$

$$\sum_i \|\mu_i^j(x,.)\| = \sum_i \int_{\mathcal{X}_i} \mu_i^j(x,dy) \leq \gamma(x) \tag{4}$$

with the equality if the semigroup $T(t)$ preserves constants and thus defines a Markov (a not just semi-Markov) process.

Proof. (i) Applying $T(t)$ to functions $f = \{f^m\}$ such that $f_m \neq 0$ only for $m = n$ we find (just as in case $N = 1$) that each A_n^n satisfies the PMP principle and thus the diagonal part of the generator has form (1) as required.

(ii) Since $T_m^n(t) \to 0$ as $t \to 0$ for $n \neq m$, it follows that $A_m^n = \lim_{t\to 0} T_m^n(t)/t$ and hence A_m^n preserves positivity of functions from its domain. To prove the boundedness, we note that since $T(t)\mathbf{1} \leq 1$ everywhere (where $\mathbf{1}$ denotes the function identically equal to 1), one has $\forall x \in \mathcal{X}_n$

$$\sum_{j\neq n} \frac{1}{t}(T_j^n(t)\mathbf{1})(x) \leq \frac{1}{t}(1 - (T_n^n\mathbf{1})(x))$$

and taking limit as $t \to 0$, we obtain

$$\sum_{j\neq n} A_j^n(\mathbf{1})(x) \leq \gamma^n(x).$$

From monotonicity of A_m^n we conclude that $(A_j^n(f))(x_n)$ is a bounded positive functional of f for any x_n and thus it can be given by an integral with respect to a certain measure, which completes the proof.

Remark. In our notation, the decomposition (2) of a given generator A is not unique, because the diagonal pure jump part of this generator can be included in both terms of this decomposition. Usually, we shall ensure the uniqueness by the requirement that the diagonal part of A_{jump} vanishes.

The dual equation for the evolution equation $\dot{f} = Af$ defining the semigroup $T(t)$ is an equation on measures. Since it is often more convenient to have an equation on densities, one is interested when the dual generator $A^\star$ preserves the class of measures that are absolutely continuous with respect to Lebesgue measure and hence can be defined as an operator on densities (which we shall again denote by $A^\star$ with the usual abuse of notations). Clearly the dual $A^\star_{jump}$ of the operator A_{jump} from (3) enjoys this property if all measures $\mu_i^j(x_j, .)$ are absolutely continuous. However, this assumption excludes lots of natural situations (it does not hold for the case of the classical Boltzmann equation). Therefore, we shall give now a more general criterion.

Proposition 1 *Suppose M is a smooth manifold in $X \times Y$ (may be with a boundary) that has a closed subset $N \subset M$ in M of Lebesgue measure zero such that the natural projections $M \setminus N \mapsto X$ and $M \setminus N \mapsto Y$ are both locally regular (i.e. surjective). Then for any bounded measurable function F on M*

$$F(x,y)\delta_M(x,y)\,dx\,dy = P(x,dy)\,dx = P'(y,dx)\,dy$$

with $P(x,dy) = F(x,y)\delta_{M_x}(y)\,dy$ and $P'(y,dx) = F(x,y)\delta_{M_y}(x)\,dx$ being transition kernels from X to Y and Y to X respectively, where $M_x = \{y : (x,y) \in M \setminus N\}$, $M_Y = \{x : (x,y) \in M \setminus N\}$.

In particular, if $A : C(Y) \mapsto C(X)$ acts as $Af(x) = \int f(y)P(x,dy)$, then $A^\star$ acts on the densities as $(A^\star\psi)(y) = \int \psi(x)P'(y,dx)$.

Proof. Proof is straightforward from standard measure theoretic arguments, since the condition of the theorem ensures that the subsets of positive measures in M will never be projected into zero measure sets under projections of M on X or Y.

Aa a direct corollary we get the following

Proposition 2 *Suppose $X = Y = \mathbb{R}^d$, and let Z be another smooth manifold. Let $M \subset X \times Y$ be given by $M = \{x, y(x,z)\}$ or $M = \{x(y,z), z\}$, where $x(y,z)$ and $y(x,z)$, $z \in Z$, are mutually inverse smooth functions,* *i.e.* $y = y(x(y,z),z))$ *holds for all* $x \in X$ *and* $y \in Y$*. Then, if* A *acts as* $(Af)(x) = \int f(y(x,z))F(x,z)\,dz$ *for some continuous* F*, then* $(A^\star\psi)(y) = \int \psi(x(y,z))\tilde{F}(y,z)\,dz$ *with*

$$\tilde{F}(y,z) = F(x(y,z),z))\det\frac{\partial x(y,z)}{\partial y}.$$

If additionally $\int F(x,z)\,dz = \int \tilde{F}(x,z)\,dz$*, then the adjoint to the operator*

$$Bf(x) = \int (f(y(x,z)) - f(x))F(x,z)\,dz$$

can be written as

$$B^\star\psi(y) = \int (\psi(x(y,z)) - \psi(y))\tilde{F}(x,z)\,dz.$$

Example. Let $X = Y = \mathbb{R}^d$. A pair of d-dimensional vectors $(v_1, v_2) \in X$ (respectively $(w_1, w_2) \in Y$) describes the velocities of two particles before (respectively after) a collision. Let M be defined by the laws of the conservation of momenta and kinetic energy, i.e.

$$M = \{(v_1, v_2, w_1, w_2) : \; v_1 + v_2 = w_1 + w_2, \quad \|v_1\|^2 + \|v_2\|^2 = \|w_1\|^2 + \|w_2\|^2\}.$$

Hence

$$M = \{(v_1, v_2, v_1 - n(v_1 - v_2, n), v_2 + n(v_1 - v_2, n)) : n \in S^{d-1}, (n, v_1 - v_2) \geq 0\},$$

$$= \{(w_1 - n(w_1 - w_2, n), w_2 + n(w_1 - w_2, n), w_1, w_2) : n \in S^{d-1}, (n, w_1 - w_2) \leq 0\}.$$

Hence, if

$$(Bf)(v_1, v_2) = \int_{n \in S^{d-1} : (n, v_1 - v_2) \geq 0} (f(v_1 - n(v_1 - v_2, n), v_2 + n(v_1 - v_2, n)) - f(v_1, v_2)) \times F(\|v_1 - v_2\|, n)\,dn,$$

then

$$(B^\star\psi)(w_1, w_2) = \int_{n \in S^{d-1} : (n, w_1 - w_2) \leq 0} (\psi(w_1 - n(w_1 - w_2, n), w_2 + n(w_1 - w_2, n)) - \psi(w_1, w_2))F(\|w_1 - w_2\|, n)\,dn. \tag{5}$$

3 Setting of the problem and main results.

1. States and observables. We are going to consider a special case of $\mathcal{X}$ from the previous section, where $\mathcal{X}_0 = X^0 = \mathbb{R}^0$ is a point, $\mathcal{X}_1 = X = \mathbb{R}^d$ with a fixed $d > 0$, and $\mathcal{X}_j = X^j = X \times ... \times X$ (j-times) for all j. In applications, X specifies the state space of one particle and $\mathcal{X} = \cup_{j=0}^{\infty} X^j$ stands for the state space of a random number of similar particles. Moreover, we are going to analyse the systems of indistinguishable particles, which means that we shall reduce the evolution to the set of symmetric functions $C_{sym}(\mathcal{X})$ and symmetric measures $\mathcal{M}_{sym}(\mathcal{X})$ and hence shall consider only symmetric generators (1), which clearly have the form SAS, where S is the symmetrisation operator and A has a general form (1). The elements of $\mathcal{M}^+_{sym}(\mathcal{X})$ and $C_{sym}(\mathcal{X})$ are called respectively the (mixed) states and observables for a Markov process on $\mathcal{X}$. The pairing between $C_{sym}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ is given by

$$(f, \rho) = \sum_{n=0}^{\infty} \int f(x_1, ..., x_n) \rho(dx_1 ... dx_n), \quad f \in C_{sym}(\mathcal{X}),\ \rho \in \mathcal{M}(\mathcal{X}),$$

so that $\|\rho\| = (\mathbf{1}, \rho)$ for $\rho \in \mathcal{M}^+(\mathcal{X})$.

A useful class of measures (and mixed states) on $\mathcal{X}$ is given by the decomposable measures of the form $\rho = Y^{\otimes}$, which are defined for an arbitrary finite measure $Y(dx)$ on X as

$$(Y^{\otimes})_n (dx_1 ... dx_n) = Y^{\otimes n}(dx_1 ... dx_n) = Y(dx_1) ... Y(dx_n).$$

Notice that $Y^{\otimes}$ need not be a finite measure on $\mathcal{X}$ even when Y is finite. Similarly the decomposable observables (or exponential vectors) are defined for an arbitrary $Q \in C(X)$ as

$$(Q^{\otimes})^n (x_1, ..., x_n) = Q^{\otimes n}(x_1, ..., x_n) = Q(x_1) ... Q(x_n). \qquad (6)$$

Remark. Instead of reducing the evolution to symmetric functions and measures on X^j, one could equivalently reduce the state space considering instead of powers X^j, either the spaces $\mathcal{X}_j$ of j point subsets of X as in [4] or the corresponding symplicies of ordered variables as in [5].

We shall denote by N the operator of the number of particles that acts on $C_{sym}(\mathcal{X})$ as $Nf(x_1, ..., x_n) = nf(x_1, .., x_n)$ and similarly on the dual space $\mathcal{M}_{sym}(\mathcal{X})$.

2. Generators of k-arnary interactions. For a symmetric operator

$$A^k = A_{(G_k, d_k, \nu_k, \gamma_k)} \tag{7}$$

of form (1) in $C(X^k)$ (in particular, A^0 is a constant) let us define a diagonal operator $B^{(k)}_{diag(A^k)}$ of k-arnary interaction on $C(\mathcal{X})$, which vanishes on the subspaces $C(X^m)$ for $m < k$, and for $m \geq k$ takes $C(X^m)$ in itsef according to the formula

$$(B^{(k)}_{diag(A^k)} f)(x_1, ..., x_n) = \sum_{I \subset \{1,...,n\}} (A^k_I f)(x_1, ..., x_n), \tag{8}$$

where the sum is over all subsets $I = \{i_1, ..., i_k\}$ of $\{1, ..., n\}$ of length $|I| = k$, and A^k_I acts on $f(x_1, ..., x_n)$ as a function of the variables $x_I = \{x_{i_1}, ..., x_{i_k}\}$, where other variables are considered as parameters. The dual operator $B^{(k)\star}_{diag(A^k)}$ clearly acts on $\mathcal{M}(\mathcal{X})$ in a similar way, where $A^{k\star}_I$ can be specified either by the duality

$$\int (A^{k\star}_I \rho)_n (dx_1 ... dx_n) f^n(x_1, ... x_n) = \int \rho_n(dx_1 ... dx_n)(A^k_I f)^n(x_1, ..., x_n),$$

or by the requirement that for the product sets $M_I \times M_{\bar{I}} \in X^n = X^{|I|} \times X^{|\bar{I}|}$

$$(A^{k\star}_I \rho)(M_I \times M_{\bar{I}}) = (A^{k\star} \rho)(M_I) \times \rho(M_{\bar{I}}).$$

Similarly, k-arnary jump interactions are defined by a family of transition kernels $P^k = \{P^k_l(x_1, ..., x_k; dy_1 ... dy_l)\}$ according to the formula

$$(B^{(k)}_{jump(P^k)} f)^n(x_1, ..., x_n) = \sum_{l=0}^{\infty} \sum_{I \in \{1,...,n\}, |I|=k} \int f^{l+(n-k)}(x_{\bar{I}}, y_1, ..., y_l) P^k_l(x_I; dy_1 ... dy_l) \tag{9}$$

for $n \geq k$ and vanishes otherwise. For the dual operator one gets:

$$(B^{(k)\star}_{jump(P^k)} \rho)_m (dx_1 ... dx_m) = \sum_{n \leq m+k} C^k_n (C^{n-k}_m)^{-1} \sum_{I \subset \{1,...,m\} : |I| = n-k}$$

$$\times \int_{z_1, ..., z_k} \rho_n(dx_I dz_1 ... dz_k) P^k_{m-(n-k)}(z_1, ..., z_k; dx_{\bar{I}}) \tag{10}$$

for $m > 0$ and vanishes for $m = 0$.

In this paper we are going to analyse the small ε asymptotics of the Markov evolutions given by the dual equations

$$\dot{\rho}(t) = B_\varepsilon^\star \rho(t), \quad \dot{f}(t) = B_\varepsilon f(t) \tag{11}$$

on $\mathcal{M}^+(\mathcal{X})$ and on $C(\mathcal{X})$ respectively, with

$$B_\varepsilon = \frac{1}{\varepsilon} \sum_{k=0}^{K} \varepsilon^k B^{(k)}_{A^k,P^k} = \frac{1}{\varepsilon} \sum_{k=0}^{K} \varepsilon^k (B^{(k)}_{diag(A^k)} + B^{(k)}_{jump(P^k)}) \tag{12}$$

where each operator $B^{(k)}_{diag(A^k)} + B^{(k)}_{jump(P^k)}$ specifying the k-arnary interaction in $\mathcal{X}$ is given by (7), (8), (9) and is the generator of a Feller semigroup in $C(\mathcal{X})$. In particular, as it follows from (4), the positive functions γ_k specifying the multiplication part of the operators (7), (1) satisfy the inequality

$$\gamma_k(x_1, ..., x_k) \geq \sum_{l=0}^{\infty} \int P_l^k(x_1, ..., x_k; dy_1...dy_l) \tag{13}$$

with transition kernels $\{P_l^k\}$ from (9), for

$$B^{(k)}_{jump(P^k)} \mathbf{1}(x_1, ..., x_n) = \sum_{l=0}^{\infty} \sum_{I \in \{1,...,n\}, |I|=k} \int P_l^k(x_I; dy_1...dy_l)$$

and

$$(B^{(k)}_{diag(A_{(0,0,0,\gamma_k)})} \mathbf{1})(x_1, ..., x_n) = - \sum_{I \subset \{1,...,n\}, |I|=k} \gamma_k(x_I),$$

In particular, if we are in the situation of conservation of probability (i.e. we have a genuine Markov, and not just a sub-Markov process) then for all k and all $x_1, ..., x_k$

$$\gamma_k(x_1, ..., x_k) = \sum_{l=0}^{\infty} \int P_l^k(x_1, ..., x_k; dy_1...dy_l). \tag{14}$$

3. Moment measures (correlation functions). In the analysis of many particle systems the states from $\mathcal{M}(\mathcal{X})$ are often characterised by their moment measures (or correlation functions). We shall introduce here the scaled moment measures $M_\varepsilon(\rho) = \mu = \{\mu_n\}$ of the state $\rho = \{\rho_n\}$ by the formula

$$\mu_n(dx_1...dx_n) = \varepsilon^n \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \int_{x_{n+1},...,x_{n+m}} \rho_{n+m}(dx_1...dx_{n+m}). \tag{15}$$

The moment measures have clear probabilistic meaning. For example, one easily sees that $\mu_0 = \|\rho\|$, and if $K \subset X$ and the system is in the state ρ, then $\mu_1(K)/\varepsilon$ determines the expectation of the number of particles in the domain K.

To analyse moment measures it is useful to consider the inverse dual mapping

$$f \mapsto g = (M_\varepsilon^{-1})^\star f : g(x_1, ..., x_n) = \frac{1}{n!}\varepsilon^{-n} \sum_{I\subset\{1,...,n\}} (-1)^{n-|I|} f(x_I) \qquad (16)$$

for $f \in C_{sym}(\mathcal{X})$ (surely $g^0 = f^0$ for $n = 0$). Clearly

$$(f, \rho) = (g, \mu) \qquad (17)$$

for μ, g given by (15), (16). This implies, in particular, that the duality condition

$$(f(t), \rho(0)) = (f(0), \rho(t)) \qquad (18)$$

for the solutions of the Cauchy problems of equations (11), is equivalent to the duality condition

$$(g(t), \mu(0)) = (g(0), \mu(t)) \qquad (19)$$

connecting the evolutions of g and μ.

4. Generating functionals (GFs). The GF for an observable $f = \{f^n\} \in C(\mathcal{X})$ and respectively for a state $\rho = \{\rho_n\} \in \mathcal{M}(\mathcal{X})$ are defined as the functionals Φ_f of $Y \in \mathcal{M}(X)$ and respectively $\tilde{\Phi}_\rho$ of $Q \in C(X)$ by the formulas

$$\Phi_f(Y) = (f, \frac{1}{N!}Y^\otimes), \quad \tilde{\Phi}_\rho(Q) = (Q^\otimes, \rho). \qquad (20)$$

Clearly this notion represents an infinite dimensional analogue of the familiar notion of the generating function of a discrete probability law. In terms of GFs, the transformation (16) can be described by the equation

$$\Phi_{N!g}(Y) = \exp\{-\int Y(dx)/\varepsilon\}\Phi_f(Y/\varepsilon). \qquad (21)$$

In fact,

$$\Phi_{N!g}(Y) = \sum_{n=0}^{\infty} \frac{1}{n!}\varepsilon^{-n} \int \sum_{l=0}^{n} (-1)^{n-l} C_n^l f(x_1, ..., x_l) Y(dx_1)...Y(dx_n)$$

$$= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{m!l!} \varepsilon^{-m-l} (-1)^m \int f(x_1, ..., x_l) Y(dx_1)...Y(dx_l) \int Y(dy_1)...Y(dy_m),$$

which yields the r.h.s. of (21).

5. Main results: kinetic equations and evolutions of GFs. In this paper we shall describe the limiting (as $\varepsilon \to 0$) evolution of the moment measures $\mu^\varepsilon(t)$, if the state $\rho^\varepsilon(t)$ evolves according to (11), (12). Namely we shall prove the following result.

Theorem 2 *If $\rho(t)$ and $f(t)$ satisfies* (11)*, and* (14) *holds, then the corresponding equations for μ and g given by* (15)*,* (16) *tend to the following limiting equations as $\varepsilon \to 0$:*

$$\dot{\mu}(dx_1...dx_l) = \sum_{k=0}^{K-1} \frac{1}{k!} \sum_{j=1}^{l} \int_{x_{l+1},...,x_{l+k}} (A^{(k+1)\star}_{j,l+1,...,l+k} \mu)(dx_1...dx_{l+k})$$

$$+ \sum_{k=0}^{K} \frac{1}{k!} \sum_{j=1}^{l} \int \mu_{l+k-1}(dx_1...d\check{x}_j...dx_l dz_1...dz_k) \sum_{m=0}^{\infty} (1+m) \int_{y_1,...,y_m} P^k_{1+m}(z_1, ..., z_k; dx_j dy_1...dy_m) \tag{22}$$

for $l \neq 0$ (surely $\dot{\mu}_0 = 0$ due to (14)*), and*

$$\dot{g}(x_1, ..., x_l) = \sum_{k=1}^{\min(k,l)} \frac{(l-k+1)!}{l!} \sum_{I \subset \{1,...,n\}, |I|=k} \sum_{j \in I} (A^k_I \tilde{g}_{I,j})(x_1, ..., x_l)$$

$$+ \sum_{k=0}^{\min(k,l)} \frac{(l-k+1)!}{l!} \sum_{I \subset \{1,...,l\}, |I|=k} g^{l+k-1}(x_{\bar{I}}, y) \sum_{m=0}^{\infty} (1+m) \int_{z_1,...,z_m} P^k_{1+m}(x_I, dy dz_1...dz_m), \tag{23}$$

where $(\tilde{g}_{I,j})(x_1, ..., x_l) = g(x_{\bar{I}}, x_j)$.

The (rather technical) proof of this theorem can be carried on by direct calculations. We shall give this proof in section 7 showing how one can appropriately organise these calculations using the concise notations of the theory of creation, annihilation and density number operators.

We are ready now to deduce the general kinetic equation.

Theorem 3 *(i) Equation* (22) *holds for a decomposable $\mu(t) = Y_t^{\otimes}$, if Y satisfies the following kinetic equation*

$$\dot{Y}_t(dx) = \sum_{k=0}^{K-1} \frac{1}{k!} \int_{z_1,...,z_k} (A^{(k+1)\star}(Y_t^{\otimes(k+1)}))(dx\, dz_1...dz_k)$$

$$+\sum_{k=0}^{K}\frac{1}{k!}\sum_{l=0}^{\infty}(l+1)\int_{z_1,\dots,z_k,y_1,\dots,y_l}P_{l+1}^k(z_1,...,z_k;dy_1...dy_l dx)Y_t(dz_1)...Y_t(dz_k). \tag{24}$$

(ii) Equation (24) *yields the characteristic equation for the evolution of the GF* Φ_{g_t} *of the observables* g_t *that satisfies* (23)*; namely, if* Y_t *solves* (24) *with initial condition* Y_0 *and* g_t *solves* (23) *with the initial condition* g_0*, then*

$$\Phi_{g_t}(Y_0)=\Phi_{g_0}(Y_t).$$

Proof. Substituting $Y^{\otimes}$ in (22) yields

$$\dot{Y}^{\otimes}(dx_1...dx_l)=\sum_{j=1}^{l}\prod_{i=1,i\neq j}^{l}Y(dx_i)(\sum_{k=0}^{K-1}\frac{1}{k!}\int_{z_1,\dots,z_k}(A^{(k+1)\star}Y^{\otimes(k+1)}(dx_j dz_1...dz_k)$$

$$+\sum_{k=0}^{K}\frac{1}{k!}\sum_{m=0}^{\infty}(1+m)\int_{z_1,\dots,z_k}\int_{y_1,\dots,y_m}P_{1+m}^k(z_1,...,z_k;dx_j dy_1...dy_m)Y(dz_1)...Y(dz_k)),$$

which clearly implies statement (i). Statement (ii) is a consequence of (i), equation (19) and the definition of GFs.

Next, we shall show how one can obtain (24) from another point of view. For an evolution of type (12) let us define the Hamiltonian function

$$H(Q,Y)=\sum_{k=0}^{K}\frac{1}{k!}\left(((B^{(k)}_{diag(A^k)}+B^{(k)}_{jump(P^k)})Q^{\otimes})^k,Y^{\otimes k}\right) \tag{25}$$

for $Q\in C(X)$, $Y\in\mathcal{M}(X)$, or more explicitly

$$H(Q,Y)=\sum_{k=0}^{K}\frac{1}{k!}\int(A^kQ^{\otimes k})(z_1,...,z_k)Y(dz_1)...Y(dz_k)$$

$$+\sum_{k=0}^{K}\sum_{l=0}^{\infty}\frac{1}{k!}\int_{z_1,\dots,z_k,y_1,\dots,y_l}P_l^k(z_1,...,z_k;dy_1...dy_l)Q(y_1)...Q(y_l)Y_t(dz_1)...Y_t(dz_k). \tag{26}$$

In section 5, we shall prove the following

Theorem 4 *In terms of GFs, the evolution* (11) *is given by the following equation in functional derivatives*

$$\varepsilon\frac{\partial}{\partial t}\tilde{\Phi}_{\rho_t}(Q)=H(Q,\varepsilon\frac{\delta}{\delta Q})\tilde{\Phi}_{\rho_t}(Q) \tag{27}$$

Notice that condition (14) is not necessary for the validity of (27).

6. Semiclassical expansion and canonical Hamiltonian system. Full expansion in $\varepsilon \to 0$ for the solutions of equation (27) can be obtained by an infinite-dimensional analogue of the classical WKB method. In particular, the Hamiltonian system that corresponds to Hamiltonian function (25) and which describes the semiclassical approximation for the evolution (27) clearly has the form

$$\dot{Y}(dx) = \frac{\delta H}{\delta Q} = \sum_{k=0}^{K-1} \frac{1}{k!} \int_{z_1,...,z_k} Q(z_1)...Q(z_k)(A^{(k+1)\star}(Y^{\otimes(k+1)}))(dx\, dz_1...dz_k)$$

$$+ \sum_{k=0}^{K} \frac{1}{k!} \sum_{l=0}^{\infty} (l+1) \int_{z_1,...,z_k,y_1,...,y_l} P^k_{l+1}(z_1,...,z_k; dy_1...dy_l dx) Q(y_1)...Q(y_l) Y_t(dz_1)...Y_t(dz_k), \tag{28}$$

$$\dot{Q}(z) = -\frac{\delta H}{\delta Y} = \sum_{k=0}^{K-1} \frac{1}{k!} \int_{z_1,...,z_k} (A^{(k+1)} Q^{\otimes(k+1)})(x, z_1,...,z_k) Y(dx) Y(dz_1)...Y(dz_k)$$

$$+ \sum_{k=0}^{K-1} \frac{1}{k!} \sum_{l=0}^{\infty} \int_{z_1,...,z_k,y_1,...,y_l} P^{k+1}_l(x, z_1,...,z_k; dy_1...dy_l) Q(y_1)...Q(y_l) Y_t(dz_1)...Y_t(dz_k), \tag{29}$$

Notice that

$$H(\mathbf{1}, Y) = -\sum_{k=0}^{K} \frac{1}{k!} \int \gamma^k(z_1,...,z_k) Y(dz_1)...Y(dz_k)$$

$$+ \sum_{k=0}^{K} \sum_{l=0}^{\infty} \frac{1}{k!} \int P^k_l(z_1,...,z_k; dy_1...dy_l) Y_t(dz_1)...Y_t(dz_k). \tag{30}$$

Hence if (14) holds, $H(\mathbf{1}, Y) = 0$ and system (28), (29) has a solution with $Q = 1$ identically, In that case, equation (28) coincides with the kinetic equation (24).

7. Propogation of chaos. The possibility of solving the limiting equations (22) in product form is a reflection of propogation of chaos (in the terminology of M.Kac, see [10], [15]) for large particle number mean-field limits – finite numbers of particles become asymptotically independent.

4 Examples.

1. Generalised Smoluchovskii equations. As an important class of interacting particle system, let us consider the case where (i) only pairwise interactions are allowed and there is no spontaneous births, i.e. P^k is not vasnishing only for $k=1$ and $k=2$ and (ii) the fragmentation and death occur independently of interaction, i.e. $P^2_l=0$ for $l>2$. Under these assumption, the kinetic equation (24) takes the form

$$\dot{Y}(dx)=(A^{1\star}Y)(dx)+\int_z(A^{2\star}Y^{\otimes 2})(dxdz)$$

$$+\sum_{l=1}^{\infty}\int_{z,y_1,\ldots y_l}(l+1)P^1_{l+1}(z;dx\,dy_1\ldots dy_l)Y(dz)$$

$$+\frac{1}{2}\int_{z_1,z_2}P^2_1(z_1,z_2;dx)Y(dz_1)Y(dz_2)-\gamma_1(x)Y(dx)-\int_z\gamma_2(x,z)Y(dx)Y(dz) \tag{31}$$

with

$$\gamma_1(x)=P^1_0(x)+\sum_{l=2}^{\infty}\int P^1_l(x;dy_1\ldots dy_l),\quad \gamma_2(x,z)=\int P^2_1(x,z;dy). \tag{32}$$

The first term in (31) stands for the underlying stochastic process describing the "free" motion of each particle, the second term stands for a pairwise interaction that does not change the number of particles, the third term stands for the (spontaneous) fragmentation and death, and the fourth term stands for the possible pairwise coagulations of particles. The last two terms come from the conservation of probability condition (14). Under additional assumptions (see Section 2), one can rewrite this equation in terms of functions (densities of Y with respect to Lebesgue measure). If this is not the case, in order to better visualise equation (31), one can rewrite it in a weak form

$$\frac{d}{dt}\int f(x)Y_t(dx)=\int(A^1f)(x)Y_t(dx)+\frac{1}{2}\int(A^2a_+f)(x,z)Y_t(dx)Y_t(dz)$$

$$+\sum_{l=2}^{\infty}\int(f(y_1)+\ldots+f(y_l)-f(z))P^1_l(z,dy_1\ldots dy_l)Y(dz)-\int P^1_0(z)f(z)Y(dz)$$

$$+\frac{1}{2}\int(f(x)-f(z_1)-f(z_2))P_1^2(z_1,z_2,dx)Y_t(dz_1)Y_t(dz_2) \qquad (33)$$

that must hold for all $f\in C(X)$. Here we used the standard notation $(a_+f)(x_1,x_2)=f(x_1)+f(x_2)$. In particular, if $P^1=0$ and $\gamma_1=0$, and if $x=(q,m)$ specifies the position q and the mass m of a particle, then equation (33) is the spatially non-trivial version of the generalised Smoluchovskii coagulation equation for continuous mass distribution in the weak form. (see e.g. [13], [1] for the discussion of some particular examples of this equation).

2. Processes preserving the number of particles. Suppose now that the process under consideration preserves the number of particles (no fragmentation or coagulation takes place). Then only the first two terms appear on the r.h.s. of (31). Moreover, suppose that $A^{1\star}$ and $A^{2\star}$ can be defined on densities, i.e. on measures of the form

$$\rho_n(dx_1...dx_n)=\psi(x_1,...,x_n)dx_1...dx_n$$

then the kinetic equation 31 takes its simplest form

$$\dot{\varphi}(x)=A^{1\star}\varphi(x)+\int_z(A^{2\star}\varphi^{\otimes 2})(x,z)\,dz. \qquad (34)$$

on measures of the form $Y(dx)=\varphi(x)\,dx$. Equation (34) contains as particular cases the well known equations of Landau, Boltzman and Vlasov. For instance, in order to see how the last two equations are obtained, let $X=\mathbb{R}^{2d}$, $x=(q,p)$, and the process on each X^n combines the deterministic motion given by the Hamiltonian equation

$$\dot{\psi}_n=\{H_n,\psi_n\}=\frac{\partial H_n}{\partial q}\frac{\partial\psi_n}{\partial p}-\frac{\partial H_n}{\partial p}\frac{\partial\psi_n}{\partial q}$$

with

$$H_n=\frac{1}{2}\sum_{j=1}^n p_j^2+\varepsilon\sum_{i\neq j}V(q_i,q_j),$$

where $V(q_i,q_j)$ (describing the potential of the interaction) is a symmetric function of two variables, and the process of collisions described by the Boltzman type operator (5), i.e.$A^{1\star}=-p\frac{\partial}{\partial q}$ and

$$(A^{2\star}\psi)(q_1,p_1,q_2,p_2)=\left(\frac{\partial\psi}{\partial p_1}\frac{\partial V}{\partial q_1}(q_1,q_2)+\frac{\partial\psi}{\partial p_2}\frac{\partial V}{\partial q_2}(q_1,q_2)\right)$$

$$+\int_{n\in S^{d-1}:(n,p_1-p_2)\le 0}(\psi(q_1,p_1-n(p_1-p_2,n),q_2,p_2+n(p_1-p_2,n))$$
$$-\psi(q_1,p_1,q_2,p_2))F(|p_1-p_2|,n)\chi(\|q_1-q_2\|)\,dn,$$

where the function χ descrides the characteristic distance between colliding particles (the limiting case $\chi(q_1-q_2)=\delta(q_1-q_2)$ yields zero-range collisions). Then the kinetic equation (34) takes the form

$$\dot\varphi(q,p)=-p\frac{\partial\varphi}{\partial q}(q,p)+\frac{\partial\varphi}{\partial p}(q,p)\int_{\mathbb{R}^{2d}}\frac{\partial V}{\partial q}(q,\tilde q)\varphi(\tilde q,\tilde p)\,d\tilde q d\tilde p$$

$$+\int_{(\tilde q,\tilde p)\in\mathbb{R}^{2}d,n\in S^{d-1}:(n,p_1-p_2)\le 0}(\varphi(q,p-n(p-\tilde p,n))\varphi(\tilde q,\tilde p-n(p-\tilde p,n))$$
$$-\varphi(q,p)\varphi(\tilde q,\tilde p))F(|p-\tilde p|,n)\chi(\|q-\tilde q\|)\,dnd\tilde q d\tilde p, \qquad (35)$$

which can be called the Boltzman-Vlasov equation, because it turns to the Vlasov equation for $F=0$ and turns to the (generalised) Boltzman equation for $V=0$ (the classical Boltzman equation corresponds to a particular choice of F and to the case of χ being the δ-function $\delta(q_1-q_2)$). We refer to [14] for the modern discussion of the Boltzman equation and the various ways of its derivation.

3. Discrete case. In case X is a discrete countable set (e.g. $X=\mathbb{Z}$ or $X=\mathbb{N}$), the functions and measures on X are represented by families of real numbers, say Q_j and Y_j, $j\in X$. Moreover, the diagonal parts A^k of $B^{(k)}$ contain only pure jump component (that can be included in $B^{(k)}_{jump}$) and a multiplication operator. Hence, the general Hamiltonian (26) can be written in the form

$$H(Q,Y)=\sum_{k=0}^{K}\sum_{m=0}^{\infty}\frac{1}{k!}\sum_{i_1,\dots,i_k;j_1,\dots,j_m}P^k_m(i_1,\dots i_k;j_1,\dots,j_m)Y_{i_1}\dots Y_{i_k}Q_{j_1}\dots Q_{j_m}$$
$$-\sum_{k=0}^{K}\frac{1}{k!}\sum_{i_1,\dots,i_k}\gamma_k(i_1,\dots,i_k)Y_{i_1}\dots Y_{i_k}Q_{i_1}\dots Q_{i_k} \qquad (36)$$

with arbitrary non-negative P^k_m and γ_k such that

$$\gamma_k(i_1,\dots,i_k)\ge\sum_{l=0}^{\infty}\sum_{j_1,\dots,j_l}P^k_l(i_1,\dots,i_k;j_1,\dots,j_l). \qquad (37)$$

In case of Markov process, i.e. if one has the equality in (37), Hamiltonian (36) takes simpler form

$$H(Q,Y) = \sum_{k=0}^{K}\sum_{m=0}^{\infty}\frac{1}{k!}\sum_{i_1,...,i_k;j_1,...,j_m} P_m^k(i_1,...i_k;j_1,...,j_m)Y_{i_1}...Y_{i_k}(Q_{j_1}...Q_{j_m}-Q_{i_1}...Q_{i_k}). \tag{38}$$

Hence the general discrete kinetic equation (24) is

$$\dot{Y}_i = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!}\sum_{i_1,...,i_k}\sum_{j_1,...,j_l} Y_{i_1}...Y_{i_k}$$

$$\times[(l+1)P_{l+1}^k(i_1,...i_k;j_1,...,j_l,i) - P_l^{k+1}(i_1,...,i_k,i;j_1,...,j_l)Y_i]. \tag{39}$$

More specifically, consider a model of pure coagulation and fragmentation in case of discrete mass parameter $M = X = \{1,2,...\}$, where a particle of mass $m > 1$ can be fragmented into several particles (of total mass m), any $k \leq K$ particles can coagulate in a single particle, and no other transitions occur. Thus only P_k^1, $k > 1$, and P_1^k, $1 < k \leq K$, are allowed to be positive in (38). Moreover, due to the conservation of mass, $P_1^k(i_1,...,i_k;i)$ and $P_k^1(i;i_1,...,i_k)$ both vanish whenever $j \neq i_1 + ...i_k$. Under these assumptions, denoting for brevity $P^k(i_1,...,i_k) = P_1^k(i_1,...,i_k;i_1+...+i_k)$ one can write the Hamiltonian (38) as

$$H(Q,Y) = \sum_{i=1}^{\infty}\sum_{l=2}^{i}\sum_{i_1+...+i_l=i, i_j>0} P_l^1(i;i_1,...,i_l)(Q_{i_1}...Q_{i_l} - Q_i)Y_i$$

$$+\sum_{k=2}^{K}\frac{1}{k!}\sum_{i_1,...,i_k=1}^{\infty} P^k(i_1,...i_k)Y_{i_1}...Y_{i_k}(Q_{i_1+...+i_k} - Q_{i_1}...Q_{i_k}). \tag{40}$$

Classical Smoluchovskii coagualation equation for discrete mass distribution is clearly a particular example of the corresponding kinetic equation (39) (with $K = 2, P_l^1 = 0$ and a particular expression for P^2).

5 Evolution of generating functionals and proof of theorem 4

Here we shall obtain the equation (27) in variational derivatives for the evolution of GFs of states and observables that evolve according to (11).

Having in mind that the iterated derivatives of a functional Ψ on a linear topological space S can be considered either as polylinear forms on S or as linear functonals on the tensor powers of S, we shall use the following notations for these derivatives:

$$\frac{\delta^n}{\delta Z^n}\Psi(Z)(v_1,...,v_n) = (v_1 \otimes ... \otimes v_n, (\frac{\delta}{\delta Z})^n)\Psi(Z)$$

$$= \frac{d}{dt_1}|_{t_1=0} ... \frac{d}{dt_n}|_{t_n=0}\Psi(z + t_1 v_1 + ... + t_n v_n).$$

This clearly defines the derivative $(V_n, (\frac{\delta}{\delta Z})^n)\Psi(Z)$ (if it exists of course) for an arbitrary element $V_n \in S^{\otimes n}$. In particular, for a symmetric continuous function f^k of k variables and the GF $\tilde{\Phi}_\rho(Q)$ of a state $\rho \in \mathcal{M}(\mathcal{X})$, we get the following basic formula

$$\frac{1}{k!}(f^k, (\frac{\delta}{\delta Q})^k)\tilde{\Phi}_\rho(Q) = \sum_{n=k}^{\infty} \sum_{I \subset \{1,...,n\}, |I|=k} \int f(x_I) \prod_{i \notin I} Q(x_i)\rho_n(dx_1...dx_n). \quad (41)$$

The folloing identities are of importance.

Proposition 3 *For an operator A^k of form* (7) *and $Q \in C(X)$*

$$\frac{1}{k!}(A^k Q^{\otimes k}, (\frac{\delta}{\delta Q})^k)(Q^{\otimes}, \rho) = (B^{(k)}_{diag(A^k)} Q^{\otimes}, \rho). \quad (42)$$

and

$$\frac{1}{k!}(B^{(k)}_{jump(P^k)} Q^{\otimes})^k, (\frac{\delta}{\delta Q})^k)(Q^{\otimes}, \rho) = (B^{(k)}_{jump(P^k)} Q^{\otimes}, \rho). \quad (43)$$

Proof. Equation (42) follows from (41) and (8). To prove (43) notice that for $n \geq k$

$$(B^{(k)}_{jump(P^k)} Q^{\otimes})(x_1,...,x_n)$$

$$= \sum_{l=0}^{\infty} \sum_{I \subset \{1,...,n\}, |I|=k} \int \prod_{j \in \bar{I}} Q(x_j) Q(y_1)...Q(y_l) P^k_l(x_I; dy_1...dy_l).$$

Hence the l.h.s. of (43) and the r.h.s. of (43) are both equal to

$$\sum_{n=k}^{\infty} \sum_{I \subset \{1,...,n\}, |I|=k} \int \sum_{l=0}^{\infty} P^k_l(x_I; dy_1...dy_l) Q(y_1)...Q(y_l) \prod_{j \in \bar{I}} Q(x_j)\rho(dx_1...dx_n).$$

Clearly (42), (43) imply that for Hamiltonian function (25)

$$\frac{1}{\varepsilon}H(Q,\varepsilon\frac{\delta}{\delta Q})(Q^{\otimes},\rho)=\frac{1}{\varepsilon}\sum_{k=0}^{K}\varepsilon^k\left((B^{(k)}_{diag(A^k)}+B^{(k)}_{jump(P^k)})Q^{\otimes},\rho\right), \qquad (44)$$

which in turn implies the statement of theorem 4.

6 Canonical representation of the generators of k-arnary interaction

1. Creation and annihilation operators in $\mathcal{M}(\mathcal{X})$ and $C(\mathcal{X})$. Let us recall that for an arbitrary $Y\in\mathcal{M}(X)$ the annihilation operator $a_-(Y)$ acts as

$$(a_-(Y)f)(x_1,...,x_n)=\int_{x_{n+1}}f(x_1,...,x_n,x_{n+1})Y(dx_{n+1})$$

on $C_{sym}(\mathcal{X})$ and for an arbitrary $h\in C(X)$ the creation operator $a_+(h)$ acts on $C_{sym}(\mathcal{X})$ as

$$(a_+(h)f)(x_1,...,x_n)=\sum_{i=1}^{n}f(x_1,...,\check{x}_i,...,x_n)h(x_i), n\neq 0,$$

$(a_+(h)f)^0=0$. Their duals are the creation and annihilation operators on $\mathcal{M}_{sym}(\mathcal{X})$ defined as

$$(a^\star_-(Y)\rho)(dx_1...dx_n)=\frac{1}{n}\sum_{i=1}^{n}\rho(dx_1...\check{dx}_i...dx_n)Y(dx_i),$$

$$(a^\star_+(h)\rho)(dx_1...dx_n)=(n+1)\int_{x_{n+1}}\rho_{n+1}(dx_1...dx_{n+1})h(x_{n+1}).$$

These operators satisfy the canonical commutation relations

$$[a_-(Y),a_+(h)]=\int h(x)Y(dx)\mathbf{1}, \quad [a^\star_+(h),a^\star_-(Y)]=\int h(x)Y(dx)\mathbf{1}.$$

We shall write shortly a_+ and $a^\star_+$ for the operators $a_+(\mathbf{1})$ and $a^\star_+(\mathbf{1})$ respectively.

The powers of $a_+(h)$ and its dual are obviously given by the formulas

$$((a_+(h))^m f)^n(x_1,...,x_n) = m! \sum_{I\subset\{1,...,n\},|I|=m} f^{n-m}(x_{\bar{I}})) \prod_{i\in I} h(x_i),$$

$$((a_+^\star(h))^m \rho)_n(dx_1...dx_n) = \frac{(n+m)!}{n!} \int_{x_{n+1},...,x_{n+m}} \rho_{n+m}(dx_1...dx_{n+m}) \prod_{j=1}^m h(x_{n+j})$$

(the first formula holds only for $m \le n$, and for $m > n$ the power $((a_+(h))^m f)^n$ vanishes).

2. Moment measures (revisited). In terms of canonical creation and annihilation operators, the transformations (15), (16) are given by formulas

$$\rho \mapsto \mu = N!\varepsilon^N e^{a_+^\star}\rho. \tag{45}$$

and

$$f = \{f^j\} \mapsto g = \{g_j\} = (N!)^{-1}\varepsilon^{-N} e^{-a_+} f. \tag{46}$$

3. Tensor powers of a_-, a_+ and the canonical representation of the jump part of the generators of k-arnary interactions. Tensor power $a_-^\otimes$ is defined as a linear mapping from $\mathcal{M}_{sym}(\mathcal{X})$ to operators in $C_{sym}(\mathcal{X})$ such that

$$(a_-^\otimes(\sigma)f)(x_1,...,x_n) = \sum_{l=0}^\infty (a_-^{\otimes l}(\sigma_l)f)(x_1,...,x_n)$$

$$= \sum_{l=0}^\infty \int f(x_1,...,x_n,y_1,...,y_n)\sigma(dy_1...dy_l)$$

Similarly one defines $a_+^\otimes$ as a linear mapping from $C_{sym}(X)$ to operators in $C_{sym}(\mathcal{X})$, where

$$((a_+^{\otimes m}h)f)(x_1,...,x_l) = m! \sum_{I\subset\{1,...,n\},|I|=m} f(x_{\bar{I}})h(x_I), h \in C_{sym}(X^m).$$

Notice that $a_-^{\otimes m}$ and $a_+^{\otimes m}$ are uniquely specified by linearity and the property that

$$a_-^{\otimes m}(Y^{\otimes m}) = ((a_-(Y))^m, \quad a_+^{\otimes m}(h^{\otimes m}) = ((a_+(h))^m$$

for $Y \in \mathcal{M}(X)$ and $h \in C(X)$ respectively.

From these definitions it follows that the jump part (9) of the k-arnary interaction generator can be expressed canonically as

$$B^{(k)}_{jump(P^k)} = \frac{1}{k!} a_+^{\otimes k} P^k a_-^{\otimes},$$

if one inderstands here (as we shall do) that $a_-^{\otimes}$ acts on P^k from the right. However, the non-integral diagonal part (8) of the generator can not be expressed in this form, and we need to introduce yet another basic operator in Fock space.

4. Density number operator. Let us recall first that for an operator D on $C(X)$, the density number operator $n(D)$ (or gauge operator, or the second quantisation of D) in $C_{sym}(\mathcal{X})$ is defined by the formula

$$n(D)f(x_1,...,x_n) = \sum_{i=1}^n D_i f(x_1,...,x_n), \tag{47}$$

where D_i acts on $f(x_1,...,x_n)$ as a function of x_i. One easily checks that

$$[n(A), n(D)] = n([A,D])$$

and

$$[n(D), a_+(h)] = a_+(Dh), \quad [a_-(Y), n(D)] = a_-(D^\star Y).$$

In particular, if D is a finite-dimensional operator of form

$$D = \sum_{j=1}^l h_j \otimes Y_j, \quad h_j \in C(X),\, Y \in \mathcal{M}(X), \tag{48}$$

that acts as $Df = \sum_{j=1}^l (f, Y_j) h_j$, then $n(D) = \sum_{j=1}^l a_+(h_j) a_-(Y_j)$, which can be also written as $a_+ D a_- = a_+(\sum_{j=1}^l h_j \otimes Y_j) a_-$, if one inderstands here that a_- acts from the right. Hence for finite dimensional D, the functor n acts as the dressing by a_+ and a_- from the left and right respectively. In general, one can not express $n(D)$ in terms of a_+ and a_-.

Now let us generalise definition (47). For a (possibly unbounded) operator $L^k : C_{sym}(\mathcal{X}) \mapsto C_{sym}(X^k)$ we define the operator $n^{\otimes k}(L^k)$ in $C_{sym}(\mathcal{X})$ by the formula

$$(n^{\otimes k}(L^k)f)(x_1,...,x_n) = \sum_{I \subset \{1,...,n\}, |I|=k} (L^k f_{x_{\bar{I}}})(x_I), \tag{49}$$

where we defined

$$f_{x_1,...,x_n}(y_1,...,y_m) = f(x_1,...,x_n,y_1,...,y_m)$$

for an arbitrary $f \in C_{sym}(\mathcal{X})$.

It follows that the generator of the k-arnary interacton from (12) can be written in the form

$$B^{(k)}_{A^k,P^k} = n^{\otimes k}(L^k), \quad L^k = A^k + P^k a_-^{\otimes}. \tag{50}$$

Remark. Notation $n^{\otimes k}$ comes from the observation that for a finite-dimensional D of form (48)

$$(D^{\otimes m}f)(x_1,...,x_n) = \delta^n_m \sum_{j_1,...,j_m=1}^{l} h_{j_1}(x_1)...h_{j_m}(x_m) \int f(y_1,...,y_m) Y_{j_1}(dy_1)...Y_{j_m}(dy_m)$$

and hence

$$n^{\otimes m}(D^{\otimes m}) =: (n(D))^m := \sum_{j_1,...,j_m=1}^{l} a_+(h_{j_1})...a_+(h_{j_m})(x_m)a_-(Y_{j_1})...a_-(Y_{j_m}),$$

where $: L :$ denote the normal (or Wick) ordering of the operator L.

7 Proof of Theorem 2

Proposition 4 *The following commutation relations hold:*

$$n^{\otimes k}(L^k)a_+^m = \sum_{l=0}^{m} C^l_m a_+^{m-l} n^{\otimes k}(L^k a_+^l), \tag{51}$$

$$a_+^m n^{\otimes k}(L^k) = \sum_{l=0}^{m} C^l_m (-1)^l n^{\otimes k}(L^k a_+^l) a_+^{m-l}, \tag{52}$$

and for an analytic function φ:

$$n^{\otimes k}(L^k) f(a_+) = \sum_{l=0}^{\infty} \frac{1}{l!} f^{(l)}(a_+) n^{\otimes k}(L^k a_+^l), \tag{53}$$

$$f(a_+)n^{\otimes k}(L^k) = \sum_{l=0}^{\infty} \frac{1}{l!}(-1)^l n^{\otimes k}(L^k a_+^l) f^{(l)}(a_+). \tag{54}$$

In particular,

$$n^{\otimes k}(L^k)e^{\alpha a_+} = e^{\alpha a_+} n^{\otimes k}(L^k e^{\alpha a_+}), \quad e^{\alpha a_+} n^{\otimes k}(L^k) = n^{\otimes k}(L^k e^{-\alpha a_+}))e^{\alpha a_+}. \tag{55}$$

for an arbitrary real α.

Proof. In fact,

$$(n^{\otimes k}(L^k)a_+ f)(x_1, ..., x_n) = \sum_{I \subset \{1,...,n\}, |I|=k} L^k(a_+ f)_{x_{\bar{I}}}(x_I)$$

$$= \sum_{j \in \bar{I}} (L^k f)_{\bar{I} \setminus j}(x_I) + (L^k(a_+ f_{x_{\bar{I}}}))(x_I) = a_+ n^{\otimes k}(L^k) f + n^{\otimes k}(L^k a_+) f,$$

which proves (51) for $m = 1$. In general, equations (51), (52) are then obtained by simple induction, Other formulas repersent direct consequences of (51) and (52).

We shall denote by pr_l the projection on the l-particle subspace that acts on $\mathcal{M}_{sym}(\mathcal{X})$ as

$$\rho = \{\rho_n\} \mapsto pr_l(\rho) = \{\rho_n \delta_l^n\}$$

and similarly on $C(\mathcal{X})$. In particular, for an arbitrary operator $L^k : C_{sym}(\mathcal{X}) \mapsto C_{sym}(X^k)$ one has an obvious identity

$$L^k = \sum_{l=0}^{\infty} L^k \, pr_l. \tag{56}$$

The following statement follows from definitions.

Proposition 5 *For an arbitrary operator $L^k : C_{sym}(\mathcal{X}) \mapsto C_{sym}(X^k)$ and for an arbitrary real function φ on $\mathbb{N} \cup \{0\}$*

$$\varphi(N - k + l) n^{\otimes k}(L^k \, pr_l) = n^{\otimes k}(L^k \, pr_l) \varphi(N). \tag{57}$$

Now we can prove the main result of this section.

Proposition 6 *Let $\rho_\varepsilon \in \mathcal{M}(\mathcal{X})$ and $f_\varepsilon \in C_{sym}(\mathcal{X})$ satisfy the dual equations*

$$\dot{\rho}_\varepsilon(t) = \frac{1}{\varepsilon}\sum_{k=0}^{K} \varepsilon^k (n^{\otimes k}(L^k))^\star \rho_\varepsilon(t), \quad \dot{f}_\varepsilon(t) = \frac{1}{\varepsilon}\sum_{k=0}^{K} \varepsilon^k n^{\otimes k}(L^k) f_\varepsilon(t) \tag{58}$$

where L^k is a family of operators $C_{sym}(\mathcal{X}) \mapsto C_{sym}(X^k)$, $k = 0, 1, ..., K$. Then in terms of moment measures and functions given by (45) *and* (46) *equations* (58) *take the form*

$$\dot{\mu} = \frac{1}{\varepsilon}\frac{N!}{(N+k-l)!}\sum_{k=0}^{K}\sum_{l=0}^{\infty} \varepsilon^l (n^{\otimes k}(L^k e^{a_+} pr_l))^\star \mu \tag{59}$$

and

$$\dot{g} = \frac{1}{\varepsilon}\frac{(N-k+l)!}{N!}\sum_{k=0}^{K}\sum_{l=0}^{\infty} \varepsilon^l n^{\otimes k}(L^k e^{a_+} pr_l) g \tag{60}$$

respectively.

Proof. From (45), (55)

$$\dot{\mu} = N!\varepsilon^N e^{a_+^\star} \frac{1}{\varepsilon}\sum_{k=0}^{K} \varepsilon^k (n^{\otimes k}(L^k))^\star \rho$$

$$= N!\varepsilon^N \frac{1}{\varepsilon}\sum_{k=0}^{K} \varepsilon^k (n^{\otimes k}(L^k e^{a_+}))^\star e^{a_+^\star} \rho,$$

which by (56) and (57) with $\varphi(n) = \varepsilon^n$ and $\varphi(n) = n!$ yields the r.h.s. of (59). Similarly, one obtains (60).

Proposition 7 *If all L^k are such that $L^k\mathbf{1} = 0$, then as $\varepsilon \to 0$, equations* (59),(60) *turn to the limiting equations*

$$\dot{\mu} = \sum_{k=0}^{K} \frac{N!}{(N+k-1)!}(n^{\otimes k}(L^k e^{a_+} pr_1))^\star \mu \tag{61}$$

and

$$\dot{g} = \sum_{k=0}^{K} \frac{(N-k+1)!}{N!} n^{\otimes k}(L^k e^{a_+} pr_1) g. \tag{62}$$

Proof. It follows from Proposition 6 and the observation that $L^k e^{a_+} pr_0$ vanishes, because it equals to the result of the application of L^k to a constant, which vanishes by the assumption.

To show the validity of Theorem 2 it remains to note that equations (61), (62) coincide with equations (22), (23) respectively for L^k from (50).

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